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ON NON-CONVEX POLYHEDRAL SURFACES IN E^3

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In this note only the simplest type of non-convex polyhedral surface will be examined. These surfaces will be characterized as the only non-convex polyhedral surfaces which have a convex polar.

Proofs will depend upon the possibility of associating convex surfaces to the non-convex ones. Thus, by comparing a non-convex surface with its convex "second polar", it will be possible to discover how bending affects the angles of the non-convex surface. In a similar way the Gauss-Bonnet relationship will be verified.

The tools of vector analysis provide simple and efficient means for keeping track of the sizes of angles on one side of a non-convex polyhedral surface. The triple scalar product, together with the rules for its manipulation and expansion, will be used a great deal. Thus Lemma 1 expresses known facts in terms of these products. In this connection, $\operatorname{sgn}[t_1t_2t_3]$ is to mean the algebraic sign of the triple scalar product $[t_1t_2t_3]$ of the vectors t_1, t_2, and t_3. Furthermore $\operatorname{sgn}[t_1t_2t_3]$ will only be written when $[t_1t_2t_3] \neq 0$.

Polyhedral corners. Let $t_1, \cdots, t_k$ be an ordered set of vectors in E^3 where $k \geqq 3$ and any three consecutive vectors t_{i-1}, t_i, and t_{i+1} are linearly independent. (All indices will always be reduced modulo k.) The set of all vectors which are linear combinations of t_i and t_{i+1} with nonnegative coefficients will be denoted by $\prod_{i,i+1}$. Furthermore let the intersection of $\prod_{i-1,i}$ and $\prod_{j,j+1}$ be no more than the origin unless $i = j$, $i - 1 = j + 1$, or $i - 1 = j$. When these conditions are satisfied the collection of all vectors in the $\prod_{i,i+1}$'s will be called a *polyhedral corner*. The origin is the *vertex*, the t_i's are the *edges*, and the $\prod_{i,i+1}$'s are the *faces* of the polyhedral corner. The angle between t_i and t_{i+1} is the *face angle* $\varphi_{i,i+1}$ of $\prod_{i,i+1}$. The *normal* to the face $\prod_{i,i+1}$ is the vector $n_{i,i+1} = t_i \times t_{i+1}$. The *exterior angle* e_i formed by $\prod_{i-1,i}$ and $\prod_{i,i+1}$ will have the magnitude of the angle between $n_{i-1,i}$ and $n_{i,i+1}$, and the sign of $[t_{i-1}t_it_{i+1}]$. The *dihedral angle* δ_i formed by $\prod_{i-1,i}$ and $\prod_{i,i+1}$ is $180° - e_i$. A polyhedral corner Σ will be called *convex* if for each i, the plane of $\prod_{i,i+1}$ is a plane of support for Σ.

LEMMA 1. *Let $\Sigma = \Sigma(t_i)$ be a polyhedral corner. Then Σ is convex*

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if and only if $\operatorname{sgn}[t_{i-1}t_it_{i+1}]$ *is constant.*

Proof. Suppose Σ is convex. It is sufficient to show that

$$\operatorname{sgn}[t_{i-1}t_it_{i+1}] = \operatorname{sgn}[t_it_{i+1}t_{i+2}]$$

for any i. However t_{i-1} and t_{i+2} must lie on the same side of the plane of $\prod_{i,i+1}$, so this equality holds.

The "if" part of this lemma will not be used in what follows and is only mentioned for completeness. For this reason the proof, which is not trivial, will be omitted.

LEMMA 2. *An ordered set of vectors* $\{t_1, \cdots, t_k\}$ *determines a convex polyhedral corner if and only if* $\operatorname{sgn}[t_{i-1}t_it_j]$ *is constant for all* i *and all* j *different from* $i-1$ *and* i.

Proof. If $\Sigma = \Sigma(t_i)$ is a convex corner, then $\operatorname{sgn}[t_{i-1}t_it_{i+1}]$ is constant for all i. For any particular i, the plane of $\prod_{i-1,i}$ supports Σ so that $\operatorname{sgn}[t_{i-1}t_it_j] = \operatorname{sgn}[t_{i-1}t_it_{i+1}]$ for all j different from $i-1$ and i.

To prove the converse we must first show that the t_i's determine a polyhedral corner. Since $[t_{i-1}t_it_{i+1}] \neq 0$ for all i, any three consecutive vectors are linearly independent. Now suppose the faces $\prod_{i-1,i}$ and $\prod_{j,j+1}$ are distinct, and $i \neq j$ and $i-1 \neq j+1$. Any vector common to both of them is of the form $\alpha t_{i-1} + \beta t_i = \gamma t_j + \delta t_{j+1}$, where $\alpha, \beta, \gamma, \delta$ are nonnegative. By taking the inner product of both sides of this equation with $t_{i-1} \times t_i$ we get

$$0 = \gamma[t_{i-1}t_it_j] + \delta[t_{i-1}t_it_{j+1}] ,$$

so that $\gamma = \delta = 0$ and $\prod_{i-1,i} \cap \prod_{j,j+1} = 0$. Thus we have a polyhedral corner $\Sigma(t_i)$. The hypothesis now implies that $\Sigma(t_i)$ is convex.

Corners and polars. Starting with some given polyhedral corner $\Sigma = \Sigma(t_i)$ we may ask whether or not the vectors $n_{k,1}, n_{1,2}, \cdots, n_{k-1,k}$ form the edges of a new polyhedral corner. For this to occur we must first of all have every consecutive set of three of them linearly independent. However

$$[n_{i-1,i}n_{i,i+1}n_{i+1,i+2}] = [t_it_{i+1}t_{i+2}][t_{i-1}t_it_{i+1}] \neq 0 ,$$

so that this is automatically satisfied. The second condition, that nonadjacent faces intersect in exactly the origin, need not be satisfied in every instance. When the normals do form a new polyhedral corner we call it the *polar polyhedral corner*, or just the *polar* of Σ, and denote it by Σ^p.

LEMMA 3. *A polyhedral corner $\Sigma = \Sigma(t_i)$ is convex if and only if it has a polar Σ^p which is convex, and has* $\operatorname{sgn}[n_{i-1,i}n_{i,i+1}n_{j,j+1}]$ *positive for all i and for all j different from $i-1$ and i.*

Proof. Suppose Σ is convex. Using Lemma 2, the existence and convexity of Σ^p will follow from the positivity of $\operatorname{sgn}[n_{i-1,i}n_{i,i+1}n_{j,j+1}]$ for all i and all j different from $i-1$ and i. This in turn follows by expanding

$$\begin{aligned}[n_{i-1,i}n_{i,i+1}n_{j,j+1}] &= [(t_{i-1} \times t_i) \times (t_i \times t_{i+1})]\cdot(t_j \times t_{j+1}) \\ &= [t_{i-1}t_it_{i+1}]\cdot[t_it_jt_{j+1}]\end{aligned}$$

and noting that, by Lemma 2, it is positive.

To prove the converse let $\operatorname{sgn}[n_{i-1,i}n_{i,i\cdot 1}n_{j,j+1}]$ be positive for all i and all j different from $i-1$ and i. The preceding equation tells us that $\operatorname{sgn}[t_{i-1}t_it_{i+1}] = \operatorname{sgn}[t_it_jt_{j+1}]$ for those same i's and j's. One more application of Lemma 2 shows that $\Sigma(t_i)$ is convex.

There is still another way for a polyhedral corner to have a convex polar. This would occur if the sign in Lemma 3 were taken to be negative.

LEMMA 4. *Let $\Sigma = \Sigma(t_i)$ be a polyhedral corner with at least four edges. Then* $\operatorname{sgn}[t_it_jt_{j+1}]$ *is constant as a function of j, where j is different from $i-1$ and i, and alternates as a function of i if and only if* $\operatorname{sgn}[n_{i-1,i}n_{i,i+1}n_{j,j+1}]$ *is negative for all i and all j different from $i-1$ and i. When this is the case Σ has a convex polar. In fact Σ has exactly four edges.*

Proof. The equivalence can be shown, in the following way, to be a consequence of the identity

$$[n_{i-1,i}n_{i,i+1}n_{j,j+1}] = [t_{i-1}t_it_{i+1}]\cdot[t_it_jt_{j+1}] \,.$$

First suppose $\operatorname{sgn}[t_it_jt_{j+1}]$ satisfies the hypothesis. Then

$$\begin{aligned}\operatorname{sgn}[t_{i-1}t_it_{i+1}] &= \operatorname{sgn}[t_{i-1}t_{i+1}t_{i+2}] \\ &= -\operatorname{sgn}[t_it_{i+1}t_{i+2}] = -\operatorname{sgn}[t_it_jt_{j+1}]\end{aligned}$$

and $\operatorname{sgn}[n_{i-1,i}n_{i,i+1}n_{j,j+1}]$ is negative. Conversely the negativity of $\operatorname{sgn}[n_{i-1,i}n_{i,i+1}n_{j,j+1}]$ shows that $\operatorname{sgn}[t_it_jt_{j+1}]$ is constant as j varies, and also that $\operatorname{sgn}[t_it_jt_{j+1}] = -\operatorname{sgn}[t_{i-1}t_it_{i+1}] = -\operatorname{sgn}[t_{i-1}t_jt_{j+1}]$.

When these hypotheses are satisfied, Lemma 2 shows that Σ^p exists and is convex.

The expansion of $(t_1 \times t_2) \times (t_j \times t_{j+1})$ can be performed in two ways to yield the identity

$$[t_1t_2t_{j+1}]t_j - [t_1t_2t_j]t_{j+1} = [t_1t_jt_{j+1}]t_2 - [t_2t_jt_{j+1}]t_1 \, .$$

By hypothesis $\operatorname{sgn}[t_1t_2t_{j+1}] = \operatorname{sgn}(-[t_1t_2t_j])$ and

$$\operatorname{sgn}[t_1t_jt_{j+1}] = \operatorname{sgn}(-[t_2t_jt_{j+1}]) \, .$$

Now should $\operatorname{sgn}[t_1t_2t_{j+1}] = \operatorname{sgn}[t_1t_jt_{j+1}]$, then the faces $\Pi_{1,2}$ and $\Pi_{j,j+1}$ would intersect in points besides the origin. We conclude that

$$\operatorname{sgn}[t_1t_2t_{j+1}] = -\operatorname{sgn}[t_1t_jt_{j+1}]$$

when neither j nor $j+1$ is equal to 1 or 2. Now suppose Σ has five or more edges. Then

$$\operatorname{sgn}[t_1t_2t_5] = \operatorname{sgn}[t_2t_3t_5] = \operatorname{sgn}[t_3t_4t_5] = \operatorname{sgn}[t_1t_4t_5]$$

which is impossible. (It is easy to give examples of four-edged polyhedral corners which do satisfy the hypothesis.)

We shall call such non-convex polyhedral corners, which have convex polars, *saddle corners.*

Bending a saddle corner. Starting with a saddle corner $\Sigma = \Sigma(t_i)$ we shall form Σ^{pp}, the polar of the polar of Σ, and see how it is related to Σ. An edge of Σ^p is $n_{i,i+1} = t_i \times t_{i+1}$ and a corresponding edge of Σ^{pp} is

$$m_i = n_{i-1,i} \times n_{i,i+1} = (t_{i-1} \times t_i) \times (t_i \times t_{i+1}) = [t_{i-1}t_it_{i+1}]t_i \, .$$

Corresponding to the face angle $\varphi_{i,i+1} = \sphericalangle(t_i, t_{i+1})$ of Σ is the face angle

$$\varphi'_{i,i+1} = \sphericalangle(m_i, m_{i+1}) = \sphericalangle([t_{i-1}t_it_{i+1}]t_i, [t_it_{i+1}t_{i+2}]t_{i+1})$$

of Σ^{pp}. Since

$$\operatorname{sgn}[t_{i-1}t_it_{i+1}] = -\operatorname{sgn}[t_it_{i+1}t_{i+2}] \, ,$$

$\varphi'_{i,i+1} = 180° - \varphi_{i,i+1}$. Corresponding to the dihedral angle

$$\delta_i = 180° - \operatorname{sgn}[t_{i-1}t_it_{i+1}] \sphericalangle (n_{i-1,i}, n_{i,i+1})$$

of Σ, the dihedral angle of Σ^{pp} is

$$\delta'_i = 180° - \operatorname{sgn}[m_{i-1}m_im_{i+1}] \sphericalangle (m_{i-1} \times m_i, m_i \times m_{i+1}) \, .$$

Now

$$\begin{aligned}\operatorname{sgn}&[m_{i-1}m_im_{i+1}]\\ &= \operatorname{sgn}[[t_{i-2}t_{i-1}t_i]t_{i-1}, [t_{i-1}t_it_{i+1}]t_i, [t_it_{i+1}t_{i+2}]t_{i+1}]\\ &= \operatorname{sgn}([t_{i-2}t_{i-1}t_i]\cdot[t_{i-1}t_it_{i+1}]^2[t_it_{i+1}t_{i+2}]) = +1 \, ,\end{aligned}$$

and

$$\begin{aligned}
&\sphericalangle(m_{i-1} \times m_i, m_i \times m_{i+1}) \\
&\quad = \sphericalangle([t_{i-2}t_{i-1}t_i]t_{i-1} \times [t_{i-1}t_it_{i+1}]t_i, [t_{i-1}t_it_{i+1}]t_i \times [t_it_{i+1}t_{i+2}]t_{i+1}) \\
&\quad = \sphericalangle(-n_{i-1,i}, -n_{i,i+1}) = \sphericalangle(n_{i-1,i}, n_{i,i+1}) .
\end{aligned}$$

Thus

$$\delta_i' = 180° - \sphericalangle(n_{i-1,i}, n_{i,i+1}) .$$

The second relation will be used to obtain the following result. The first will be used later.

BENDING THEOREM. *Let $\Sigma = \Sigma(t_i)$ and $\Gamma = \Gamma(r_i)$ be two saddle corners whose corresponding face angles are equal. Let δ_i be the dihedral angle of Σ at t_i and γ_i be the dihedral angle of Γ at r_i. Then* $\operatorname{sgn}(\delta_i - \gamma_i)$ *alternates as a function of i, in the case where for all i,* $\operatorname{sgn}[r_{i-1}r_ir_{i+1}] = -\operatorname{sgn}[t_{i-1}t_it_{i+1}]$. *In the case where for all i,* $\operatorname{sgn}[r_{i-1}r_ir_{i+1}] = \operatorname{sgn}[t_{i-1}t_it_{i+1}]$, *either all $\delta_i - \gamma_i$ have the same sign or they are all zero.*

Proof. The normal $r_i \times r_{i+1}$ to a face of Γ will be called $s_{i,i+1}$. In the case where $\operatorname{sgn}[r_{i-1}r_ir_{i+1}] = -\operatorname{sgn}[t_{i-1}t_it_{i+1}]$,

$$\begin{aligned}
\delta_i - \gamma_i &= [180° - \operatorname{sgn}[t_{i-1}t_it_{i+1}] \sphericalangle(n_{i-1,i}, n_{i,i+1})] \\
&\qquad - [180° - \operatorname{sgn}[r_{i-1}r_ir_{i+1}] \sphericalangle(s_{i-1,i}, s_{i,i+1})] \\
&= \operatorname{sgn}[r_{i-1}r_ir_{i+1}][\sphericalangle(s_{i-1,i}, s_{1,i+1}) + \sphericalangle(n_{i-1,i}, n_{i,i+1})]
\end{aligned}$$

which alternates as a function of i. In the case where

$$\operatorname{sgn}[r_{i-1}r_ir_{i+1}] = \operatorname{sgn}[t_{i-1}t_it_{i+1}] ,$$

$$\begin{aligned}
\delta_i - \gamma_i &= \operatorname{sgn}[r_{i-1}r_ir_{i+1}][\sphericalangle(s_{i-1,i}, s_{i,i+1}) - \sphericalangle(n_{i-1,i}, n_{i,i+1})] \\
&= \operatorname{sgn}[r_{i-1}r_ir_{i+1}]\{[180° - \sphericalangle(n_{i-1,i}, n_{i,i+1})] \\
&\qquad\qquad - [180° - \sphericalangle(s_{i-1,i}, s_{i,i+1})]\} \\
&= \operatorname{sgn}[r_{i-1}r_ir_{i+1}]\cdot(\delta_i' - \gamma_i') ,
\end{aligned}$$

where δ_i' and γ_i' are dihedral angles of Σ^{pp} and Γ^{pp} respectively, corresponding to δ_i and γ_i respectively. We are now able to apply a well-known "four vertex" theorem [1, Chapt. II, p. 12] to the two convex polyhedral corners Σ^{pp} and Γ^{pp}. For the situation we are considering, this theorem states that either $\delta_i' - \gamma_i'$ is zero for all i, or is zero for no i and alternates in sign as a function of i. Since $[r_{i-1}r_ir_{i+1}]$ also alternates in sign, the assertion is proved.

This theorem has the following interpretation when we think of the saddle corner as having hinged edges. Picture the corner being bent and thereby having its dihedral angles altered. Then all of its dihedral angles will be altered in the same direction provided it remains a saddle corner throughout the process.

The case where the signs alternate arises in the following way. Take a saddle corner, form a mirror image of it, and "bend" this mirror image. Then compare this last corner with the original one.

The Gauss-Bonnet result. In [2] Polya discussed and proved a version of the Gauss-Bonnet theorem for convex polyhedral surfaces. We shall extend his Lemma II to saddle corners. (His Lemma II is just the statement of the $G-B$ theorem for convex corners.) Other methods will probably be needed to extend this lemma to general non-convex polyhedral angles.

Let Σ be a saddle corner so Σ^p is convex. Call $-K$ the (negative) total curvature of Σ which is, in magnitude, equal to the solid angle included by Σ^p. Since $(\Sigma^{pp})^p$ is Σ^p (or its mirror image), the total curvature of Σ^{pp} is just K. The total geodesic curvature can be found by taking a polygonal path around a polyhedral corner and computing the total change in direction along this path. This turns out to be exactly the sum of the face angles for that corner. The $G-B$ theorem is valid for Σ^{pp} so $K+\varphi'_{1,2}+\cdots+\varphi'_{4,1}=360°$. The sum of the face angles of Σ is

$$\begin{aligned}\varphi_{1,2}+\cdots+\varphi_{4,1}&=(180°-\varphi'_{1,2})+\cdots+(180°-\varphi'_{4,1})\\&=720°-(\varphi'_{1,2}+\cdots+\varphi'_{4,1})\,.\end{aligned}$$

Therefore

$$\begin{aligned}&(-K)+(\varphi_{1,2}+\cdots+\varphi_{4,1})\\&\quad=-K+720°-(\varphi'_{1,2}+\cdots+\varphi'_{4,1})\\&\quad=720°-360°=360°\,,\end{aligned}$$

and the $G-B$ theorem is valid for Σ.

References

1. H. Hopf, *Geometry*, Mimeographed lecture notes presented to the Institute of Mathematics and Mechanics, New York, N.Y. (1952).
2. G. Polya, *An elementary analogue to the Gauss-Bonnet theorem*, Amer. Math. Monthly, **61** (1954), 601-603.

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