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THE COLLINEATION GROUPS OF DIVISION RING PLANES II: JORDAN DIVISION RINGS

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In this paper the authors continue their study of the collineation groups of division ring planes (The collineation groups of division ring planes I. Jordan division algebras, J. Reine and Angew. Math. vol. 216, 1964). Some of the results obtained for finite dimensional Jordan division algebras are extended to a special class of infinite dimensional algebras.

As is well-known the study of the collineation group of a projective plane π coordinatized by an algebra $\mathscr{R}$ can be reduced to the study of the autotopism group of $\mathscr{R}$ or the group of autotopic collineations of π, $\mathscr{H}(\pi)$. The pair (a, b), $a, b \in \mathscr{R}$, is defined to be admissible if and only if there exists an element α in $\mathscr{H}(\pi)$ with $(1, 1)\alpha = (a, b)$. Modulo the automorphism group of $\mathscr{R}$, the determination of $\mathscr{H}(\pi)$ is equivalent to the determination of all admissible pairs (a, b) and coset representatives $\varphi_{a,b} \in \mathscr{H}(\pi)$ such that $(1, 1)\varphi_{a,b} = (a, b)$. With either the assumption $\mathscr{R}$ algebraic over its center, or the assumptions characteristic of $\mathscr{R}$ not equal to 0 and the centers of $\mathscr{R}$ and $\mathscr{R}'$ (the algebra of all elements of $\mathscr{R}$ algebraic over the center of $\mathscr{R}$) equal, the admissible pairs (a, b) are determined. Use is made of Kleinfeld's result on the middle nucleus of Jordan rings (Middle nucleus = center in a simple Jordan ring, to appear.) We also prove and use the result that the algebra $\mathscr{E}$ consisting of all right multiplications $\mathbf{R}_f$ is commutative, where f is in the subalgebra generated by a and a^{-1} over the base field.

Let $\mathfrak{R}$ be any nonalternative division ring (i.e., $(\mathfrak{R} - \{0\}, \cdot)$ is a loop), and let $\pi(\mathfrak{R})$ be the projective plane coordinatized by $\mathfrak{R}$. Then, as is well known, the study of the collineation group of π, $G(\pi)$, can be reduced to the study of the autotopism group of $\mathfrak{R}$, or the group of autotopic collineations of π, $H(\pi)$. If α is a collineation of π, then $\alpha \in H(\pi)$ if and only if $(\infty)\alpha = (\infty)$, $(0)\alpha = (0)$, $(0, 0)\alpha = (0, 0)$. Now, in [3], the pair (a, b) was defined to be admissible if and only if there exists an element $\alpha \in H(\pi)$ with $(1, 1)\alpha = (a, b)$, and it was shown that, modulo the automorphism group of $\mathfrak{R}$, $H_1(\mathfrak{R})$, the determination of $H(\pi)$ is equivalent to the determination of all admissible pairs (a, b) and coset representatives $\varphi_{a,b} \in H(\pi)$:

$$(1, 1)\varphi_{a,b} = (a, b) . \tag{1}$$

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The second part of [3] was concerned with planes coordinatized by finite dimensional Jordan division algebras, and it was proved that if $\mathfrak{R}$ is a finite dimensional Jordan division algebra of characteristic $\neq 2, 3$, then (a, b) is admissible if and only if a and b are elements in the center of $\mathfrak{R}$, and coset representatives of $\varphi_{a,b}$ were obtained for a, b in the center of $\mathfrak{R}$. In this paper, we shall prove the following theorem:

THEOREM A. *If $\mathfrak{R}$ is a Jordan division algebra of characteristic $\neq 2, 3$, and if either*

(*a*) *$\mathfrak{R}$ is algebraic over its center, Z; or,*

(*b*) *$\mathfrak{R}$ has characteristic $\neq 0$, and the center of $\mathfrak{R}$ is equal to the center of $\mathfrak{R}'$—the algebra of all elements algebraic over Z; then (a, b) is admissible if and only if a and b are both in Z.*

We shall need a recent result of Kleinfeld in the proof of Theorem A, and quote it here:

THEOREM 1 [2]. *If $\mathfrak{R}$ is a simple Jordan ring of characteristic $\neq 2$, the three nuclei of $\mathfrak{R}$ are equal.*

This generalizes Theorem 15 of [3] and is useful in that with this result we need only show that an element, a, is in any one of the nuclei of $\mathfrak{R}$ in order to prove that a is in the *center* of $\mathfrak{R}$.

Our first step will be to prove some results about Jordan division rings which are analogous to known theorems about finite dimensional Jordan algebras, and which are necessary tools for this paper. Recall that the linearized form of the Jordan identity can be written [1],

$$
\begin{aligned}
& R_x R_{zw} + R_w R_{xz} + R_z R_{xw} \\
(2) \qquad & = R_{zw} R_x + R_{xz} R_w + R_{xw} R_z \\
& = R_{x(zw)} + R_z R_x R_w + R_w R_x R_z \\
& = R_{z(xw)} + R_x R_z R_w + R_w R_z R_x ,
\end{aligned}
$$

where R_x is the linear transformation corresponding to multiplication in $\mathfrak{R}$ by the element x.

We now prove

THEOREM 2. *Let a be an element of a Jordan division algebra, $\mathfrak{R}$. Then a and a^{-1} generate an associative subalgebra of $\mathfrak{R}$.*

Proof. If $\mathfrak{R}$ is finite dimensional over Z, this result is a trivial consequence of the well known result [1] that any Jordan algebra is power-associative. For the infinite-dimensional case, it suffices to prove

$$(3) \qquad a^{-i}a^k = a^{k-i} \qquad \text{for all} \quad i,\ k \geqq 1.$$

For $i = 1$, set $x = a^k$, $z = w = a$ in (2), and apply the last two resulting transformations to a^{-1} to obtain

$$a^{-1}(R_{a^{k+2}} + 2R_aR_{a^k}R_a) = a^{-1}(R_{a^{k+2}} + R_{a^k}R_aR_a + R_aR_aR_{a^k}),$$

or

$$a^{-1}a^{k+2} + 2a^{k+1} = a^{-1}a^{k+2} + [(a^{-1}a^k)a]a + a^{k+1},$$

which implies $a^{k+1} = [(a^{-1}a^k)a]a = a^kR_a = [(a^{-1}a^k)a]R_a$. Since $\Re$ is a division ring, R_a is nonsingular, and the last equation implies $a^k = (a^{-1}a^k)a = (a^{k-1})R_a = (a^{-1}a^k)R_a$, which, in turn, implies $a^{k-1} = a^{-1}a^k$. Thus, (3) is verified for $i = 1$. For $i > 1$, set $x = a^{-1}$, $y = a$, $z = a^k$ in (2) and apply the first two resulting transformations to a^{-i}:

$$\begin{aligned} &a^{-i}(R_{a^{-1}}R_{a^{k+1}} + R_{a^k} + R_aR_{a^{k-1}}) \\ &= a^{-i}(R_{a^{k+1}}R_{a^{-1}} + R_{a^k} + R_{a^{k-1}}R_a)\ , \end{aligned}$$

or,

$$(4) \qquad \begin{aligned} &a^{-(i+1)}a^{k+1} + a^{-i}a^k + [(a^{-i})a]a^{k-1} \\ &= (a^{-i}a^{k+1})a^{-1} + a^{-i}a^k + [a^{-i}a^{k-1}]a\ . \end{aligned}$$

If we assume $a^{-j}a^k = a^{k-j}$ for all k and all $j < i + 1$, (4) becomes

$$a^{-(i+1)}a^{k+1} + 2a^{k-i} = 3a^{k-i}\ ,$$

which implies $a^{-(i+1)}a^{k+1} = a^{k-i}$, which together with the truth of (3) for $i = 1$ and all k, completes the inductive proof of the theorem.

Another result which is analogous to a well-known theorem for finite-dimensional Jordan algebras [1] is:

THEOREM 3. *If $\Re$ is a Jordan division algebra over a field $\mathfrak{F}$ of characteristic $\neq 2$, and if a is any element of $\Re$, then the algebra $\mathfrak{S}$ generated by all R_x, for $x \in \mathfrak{F}[a, a^{-1}]$, is commutative.*

Proof. In (2), set $x = a^{-1}$, $w = a$, $z = a^i$, and get

$$(5) \qquad R_{a^{-1}}R_{a^{i+1}} + R_aR_{a^{i-1}} + R_{a^i} = R_{a^{i+1}}R_{a^{-1}} + R_{a^{i-1}}R_a + R_{a^i}\ .$$

In [1], it is shown that the Jordan identity implies that R_{x^i}, R_{x^j} commute for any x, and all $i,\ j \geqq 0$. Thus, for $i \geqq 1$, (5) can be simplified to

$$(6) \qquad R_{a^{-1}}R_{a^{i+1}} = R_{a^{i+1}}R_{a^{-1}}\ .$$

Next, let $x = a^{-2}$, $w = a^i$, $z = a$ in (2), and get

$$\begin{aligned} (7) \qquad & R_{a^{-2}}R_{a^{i+1}} + R_{a^i}R_{a^{-1}} + R_aR_{a^{i-2}} \\ & = R_{a^{i+1}}R_{a^{-2}} + R_{a^{-1}}R_{a^i} + R_{a^{i-2}}R_a \ . \end{aligned}$$

If $i \geqq 2$, we can use (6) and the fact that $R_aR_{a^{i-2}} = R_{a^{i-2}}R_a$ to simplify (7) to

$$(8) \qquad R_{a^{-2}}R_{a^{i+1}} = R_{a^{i+1}}R_{a^{-2}} \ .$$

Thus, for all $i \geqq 3$, R_{a^i} commutes with all elements in $\mathfrak{S}$ generated by $R_{a^{-1}}$ and $R_{a^{-2}}$. Since the set of all R_f, $f \in \mathfrak{F}[a^{-1}]$, is generated by $R_{a^{-1}}$ and $R_{a^{-2}}$ [1], we can conclude that for $i \geqq 3$, R_{a^i} is in the center of $\mathfrak{S}$. Similarly, we can show that for $i \geqq 3$, $R_{a^{-i}}$ is in the center of $\mathfrak{S}$. Next, we substitute in (2), $x = z = a^2$, $w = a^{-4}$, and, using the fact that $R_{a^{-4}}$ is in the center of $\mathfrak{S}$, we conclude

$$(9) \qquad R_{a^2}R_{a^{-2}} = R_{a^{-2}}R_{a^2} \ .$$

Finally, substituting $x = z = a$, $w = a^{-2}$, and using (9), we can deduce

$$(10) \qquad R_aR_{a^{-1}} = R_{a^{-1}}R_a \ .$$

But from (6), we know that $R_{a^{-1}}R_{a^2} = R_{a^2}R_{a^{-1}}$. Thus, we see that $R_{a^{-1}}$ commutes with R_a, R_{a^2}, $R_{a^{-1}}$, and $R_{a^{-2}}$, and hence that $R_{a^{-1}}$ is in the center of $\mathfrak{S}$. Similarly, we can conclude that R_a, R_{a^2}, $R_{a^{-2}}$ are also in the center of $\mathfrak{S}$, and thus that $\mathfrak{S}$ is commutative.

We now turn to the proof of Theorem A. Recall that in [3] the admissibility of (a, b) for any nonalternative $\mathfrak{R}$ was seen to be equivalent with the isomorphism of $\mathfrak{R}$ and a certain isotope of $\mathfrak{R}$, $\mathfrak{S}_{a,b}$, obtained by recoordinatizing π with the new coordinate points $(\infty)' = (\infty)$, $(0)' = (0)$, $(0, 0)' = (0, 0)$, and $(1, 1)' = (a, b)$. Now, in [3], (Sec. 9) the following theorem was proved but not stated explicitly.

THEOREM 4. *If $\mathfrak{R}$ is commutative, and if the middle nucleus of $\mathfrak{R}$ is equal to the center of $\mathfrak{R}'$ and if $\mathfrak{F}[R_{x^i}]$ is commutative for all $x \in R$, then if $\mathfrak{S}_{a,b}$ is commutative, $\mathfrak{S}_{a,b}$ can be represented as follows: $\mathfrak{S}_{a,b} \approx (\mathfrak{R}, \oplus, *)$, where*

$$(11) \qquad x \oplus y = x + y \ ,$$

and multiplication in $\mathfrak{S}_{a,b}$ is given in terms of multiplication in $\mathfrak{R}$:

$$(12) \qquad (y * x) = [(yR_{a^{-1}}^{-1})(xR_{a^{-1}}^{-1})]R_a^{-1}R_{a^{-1}} \ .$$

Also, $a^2b^{-1} \in Z$.

Notice that (12) is equivalent to

$$(13) \qquad \tilde{R}_x = R_{a^{-1}}^{-1}R_xR_{a^{-1}}^{-1}R_a^{-1}R_{a^{-1}} \ .$$

Since if $\mathfrak{S}_{a,b}$ is to be isomorphic with $\mathfrak{R}$, $\mathfrak{S}_{a,b}$ must be commutative and using Theorems 1 and 3, we have the validity of Theorem 4 in our present study. From this point on, then we assume that $\mathfrak{S}_{a,b}$ is of the form given by Theorem 4 and wish to determine under what conditions on the element a, $\mathfrak{R}$ and $\mathfrak{S}_{a,b}$ are isomorphic.

We begin with

THEOREM 5. *Let $\mathfrak{R}$ be a Jordan division ring, of characteristic $\neq 2, 3$, and let $\mathfrak{S}_{a,b}$ be as in Theorem 4. Then if $\mathfrak{S}_{a,b}$ satisfies the Jordan identity, we have*

$$R_{a^i} = R_a^i\,[q_i(T - I) + I] \qquad \textit{for any } i \geqq 0\,, \tag{14}$$

where

$$T = R_a R_{a^{-1}}\,, \tag{15}$$

and

$$q_i = \frac{i(i-1)}{2}\,. \tag{16}$$

Proof. This theorem for $\mathfrak{R}$ finite dimensional is contained in [3] (Sec. 10, Lemma 2). The proof for the infinite-dimensional case is exactly the same, keeping in mind that Theorem 3 must be invoked to let us permute elements of the form R_{a^i} and $R_{a^{-j}}$.

Assume now that $\mathfrak{R}$ has characteristic $p \neq 0$. We shall prove that Theorem 5, together with the Jordan identity for $\mathfrak{R}$ imply that a is algebraic over Z if $\mathfrak{S}_{a,b}$ satisfies the Jordan identity. To see this, observe first that for any k, (14) implies the equalities

$$R_{a^{kp}} = R_a^{kp} = R_{(a^p)^k} = (R_a^p)^k\,, \tag{17}$$

since $q_{kp} = 0$.

Thus, if $c = a^p$, we have

$$R_{c^k} = R_c^k, \qquad \text{for all } k \geqq 0\,. \tag{18}$$

Next, recall [3] that the linearized form of the Jordan identity for $\mathfrak{S}_{a,b}$ can be written

$$\begin{aligned} &R_x R_a^{-1} R_{(zw)} R_a^{-1} + R_w R_a^{-1} R_{(zx)} R_a^{-1} + R_z R_a^{-1} R_{(xw)} R_a^{-1} \\ &= R_{(zw)} R_a^{-1} R_x + R_{(zx)} R_a^{-1} R_a^{-1} R_w + R_{(xw)} R_a^{-1} R_a^{-1} R_z \\ &= R_{(zw)} R_a^{-1} R_x R_a^{-1} + R_z R_a^{-1} R_x R_a^{-1} R_w + R_w R_a^{-1} R_x R_a^{-1} R_z\,. \end{aligned} \tag{19}$$

We wish to prove a commutator identity:

THEOREM 6. *If $\mathfrak{S}_{a,b}$ satisfies the Jordan identity, then*

(20) $$[R_x, R_{c^{i+1}}] = (i+1)[R_{xc^i}, R_c],$$

for $i \geqq 0$, $c = a^p$, *and for all* $x \in \mathfrak{R}$.

Proof. In (19), let $w = c$, $z = c^i$. Then we have

$$R_x R_{c^{i+1}} + R_c R_{xc^i} + R_{c^i} R_{xc}$$
$$= R_{c^{i+1}} R_x + R_{xc^i} R_c + R_{xc} R_{c^i},$$

or,

(21) $$[R_x, R_{c^{i+1}}] = [R_{xc^i}, R_c] + [R_{xc}, R_{c^i}].$$

Thus (20), for $i = 1$ is verified. If $i > 1$, we apply our induction hypothesis to the right hand side of (21), and write

(22) $$[R_{xc}, R_{c^i}] = i[R_{(xc)c^{i-1}}, R_c] = i[R_x R_c R_{c^{i-1}}, R_c].$$

But by (18), we can write $R_c R_{c^{i-1}} = R_c R_c^{i-1} = R_c^i = R_{c^i}$, so (22) becomes

(23) $$[R_{xc}, R_{c^i}] = i[R_{xc^i}, R_c],$$

which allows us to write (21) as

(24) $$[R_x, R_{c^{i+1}}] = (i+1)[R_{xc^i}, R_c],$$

and complete the inductive proof of Theorem 6.

Now in (20), if we set $i = p - 1$, we obtain

(25) $$[R_x, R_{c^i}] = 0, \qquad \text{for all } x \in \mathfrak{R},$$

but this is equivalent, in our case, to asserting that $c = a^p$ *is in the center* of $\mathfrak{R}$. Thus, as promised, we demonstrated that if $\mathfrak{S}_{a,b}$ is a Jordan ring, then $a^p = c \in Z$, and a *is algebraic over* Z.

The completion of the proof of Theorem A is quite simple. If $\mathfrak{S}_{a,b}$ satisfies the Jordan identity, and if a is algebraic over Z, then $a \in \mathfrak{R}'$—the algebra of all algebraic elements. Now, let a', a'' be any two elements of $\mathfrak{R}'$, and consider $\mathfrak{R}[a, a', a'']$, the subalgebra of $\mathfrak{R}'$ generated by a, a', a''. Since (19) holds for all $x, y, z \in \mathfrak{R}$, it certainly also holds for all $x, y, z \in \mathfrak{R}[a, a', a'']$. But in [3], it was shown that if (2) and (19) hold for any commutative algebra, then a is in the center of that algebra. Thus, a is in the center of $\mathfrak{R}[a, a', a'']$ for any $a', a'' \in \mathfrak{R}'$, which completes the proof of Theorem A.

As a final remark, we observe that [3] (Sec. 14), a coset representative $\varphi_{a,b}$ for $H(\pi)/H_i(\mathfrak{R})$ was explicitly determined for every admissible pair (a, b) for which both a and b were in the center. Thus, for those algebras satisfying the conditions of Theorem A, all the coset representatives are actually known, and the collineation group for such a plane is thus *completely determined* modulo the automorphism group of the algebra.

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