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REDUCTION OF SETS OF MATRICES TO A TRIANGULAR FORM

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A set Ω of $n \times n$ matrices is said to have Property T if the following two conditions are satisfied: (i) If Ω is looked upon as a set of linear transformations of a vector space V of dimension n then V has an Ω-decomposition into primary components; i.e. $V = V_1 \oplus \cdots \oplus V_t$, where the restrictions of the elements of Ω to any V_i are primary linear transformations; and (ii) V has an Ω-composition series with 1-dimensional composition-factors.

Our aim in this paper will be to characterize sets of nonsingular linear transformations having Property T.

The latter condition (ii) has been called Property P for Ω. It is known that Property P is equivalent to simultaneous triangularisation of the elements of Ω, and also to the existence of common characteristic vectors for all of Ω and to the fact that the additive commutators $AB - BA$ of pairs A, B from Ω belong to the radical of the enveloping associative algebra generated by Ω ([5], page 592–600).

It is also known that Ω has Property T if it is a commutative set of matrices ([1], page 41). Also for a Lie algebra of linear transformations of a finite dimensional vector space, is known that Property T is equivalent to the nilpotency (in the Lie-sense) of the Lie algebra ([2], page 878–879).

Throughout we shall identify Ω with a set of nonsingular linear transformations of a finite dimensional vectorspace V over an algebraically closed field F of characteristic zero.

2. DEFINITION. Let A be any nonsingular linear transformation. Then it is known that A can be factorized uniquely as $A = SU$, where U is a unipotent linear transformation and S is semi-simple, and $SU = US$, ([1], page 41). U is called the unipotent part of A and S is called the semi-simple part of A. This will be referred to as the Jordan-multiplicative decomposition of A.

S can also be characterized by the fact that the module determined by it is completely reducible; and hence, over an algebraically closed field, S is representable as an $n \times n$ diagonal matrix.

We shall let Ω_s be the set of the semi-simple parts of the elements

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of Ω, and Ω_t be the set of their unipotent parts.

We prove,

THEOREM 1. *Let V have an Ω-composition-series with 1-dimensional composition-factors. Then a necessary and sufficient condition for Ω to have Property T is that Ω_s commutes with Ω elementwise.*

Proof. For the necessity part of the theorem, we observe that if Ω has Property T, then the matrices in Ω can be assumed to be a direct sum of triangular blocks, each of which is in the triangular form having a single characteristic root along the diagonal. Thus any element A in Ω can be supposed to be of the form,

$$A = \begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix},$$

where * denotes possible nonzero entries. Then $A = (\lambda \cdot I) \cdot (\lambda^{-1} \cdot A) = A_s \cdot A_u$, where $A_s = \lambda \cdot I$ and $A_u = \lambda^{-1} A$, so that $A_s A_u = A_u A_s$, and A_s is semi-simple while A_u is unipotent. Then from the uniqueness of the Jordan multiplicative decomposition, we conclude that A_s is in Ω_s and A_u is in Ω_u. Thus Ω_s consists of scaler matrices only and hence commutes with Ω elementwise.

For the sufficiency part of the theorem, let A be any element of Ω, and $A = A_s A_u$ be its Jordan-multiplicative decomposition, so that A_s is in Ω_s and A_u is in Ω_u. Let $V = V_{\lambda_1, A} \oplus \cdots \oplus V_{\lambda_t, A}$, where λ_i are the distinct characteristic roots of A_s and hence of A, be a decomposition of V into primary components with respect to A_s. Since A_s is semi-simple, so for any vector $\mathscr{U}$ in $V_{\lambda_k, A}$, we have,

$$\mathscr{U} \cdot (A_s - \lambda_k I) = 0.$$

If B is an arbitrary element of Ω, then $(\mathscr{U}B)(A_s - \lambda_k I) = \mathscr{U}(A_s - \lambda_k I)B = 0$, since Ω_s is assumed to commute with Ω elementwise. Thus each component $V_{\lambda_k, A}$ is invariant with respect to the whole of Ω. Also the restriction of A_u to any $V_{\lambda_k, A}$ is itself unipotent. Therefore the restriction of A to each $V_{\lambda_k, A}$ is primary.

If some C in Ω is not primary on any of the $V_{\lambda_k, A}$, we repeat the process with C in place of A, so that we can conclude that V has an Ω-decomposition $V = V_1 \oplus \cdots \oplus V_t$, such that the restrictions of the elements of Ω to any V_i are primary.

Combined with the hypothesis on the existence of Ω-composition series with 1-dimensional composition factors, the above conclusion gives Property T for Ω.

The following analogue of McCoy's result in ([5], page 593) can be easily verified.

LEMMA 1. *Ω has Property P if and only if for each pair A, B of elements in Ω, $ABA^{-1}B^{-1} - I$ lies in the radical of the enveloping associative algebra $\bar{\Omega}$ generated by Ω.*

Using this we conclude at once,

THEOREM 2. *A set of necessary and sufficient conditions for Ω to have property T is that,*

(i) Ω_s commute with Ω elementwise, and

(ii) for every pair A, B of elements in Ω, $ABA^{-1}B^{-1} - I$ lies in the radical of the enveloping associative algebra $\bar{\Omega}$ generated by Ω.

3. In this section we limit Ω to be an algebraic group ([1], page 29). The following results are well-known and the proofs are omitted here.

LEMMA 2 (Lie-Kolchin). *A connected algebraic group Ω has Property P if and only if it is solvable:* ([3], page 30).

LEMMA 3. *If Ω is a connected nilpotent algebraic group, then Ω_s is contained in the centre* ([1], page *Theorem* **11.1**).

LEMMA 4. *If N is an invariant commutative algebraic subgroup of a connected algebraic group Ω, and consists of semi-simple elements only, then N is contained in the centre of Ω* ([1], page 45, *Proposition* 7·9).

It may be relevant to recall that connectivity is taken here in the sense of the Zariski-Topology in Ω([3], page 26).

THEOREM 3. *A necessary and sufficient condition for a connected algebraic group Ω to have Property T is that Ω be nilpotent.*

Proof. For the sufficiency we observe that by Lemma 3 Ω_s commutes with Ω elementwise. Then by Lemma 2, Ω has Property P. Thus, Theorem 1 implies that Ω has Property T.

For the necessity, let Ω have Property T. Again we can assume that any element A of Ω has the form,

$$A = \begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} = (\lambda \cdot I) \cdot (\lambda^{-1} A) = A_s \cdot A_u \cdot$$

If F^* denotes the multiplicative group of the nonzero elements of the ground field F, then Ω is isomorphic to the external direct product, $F^* \times U$, where U is the group of unipotent matrices $(\lambda^{-1} \cdot A)$.

U is a group of unipotent matrices in triangular form, and such groups are known to be nilpotent, so Ω being a product of two nilpotent groups, is itself nilpotent.

Another characterization of Property T can be obtained in,

THEOREM 4. *A necessary and sufficient condition for a connected algebraic group Ω to have Property T is that Ω_s be an algebraic subgroup contained in the centre of Ω.*

Proof. If Ω has Property T, then by Theorem 3, Ω is nilpotent, so that Ω_s is an algebraic subgroup of the centre, ([1], page 53, Theorem 11·1).

Conversely, let Ω_s be an algebraic subgroup of the centre of Ω. Then it can be shown that Ω is equal to the internal direct product $\Omega_s \times \Omega_u$ ([1], page 53, Theorem 11·1). Therefore, we at once have that $\Omega/\Omega_u \cong \Omega_s$ is Abelian and hence $\Omega_u \supseteq$ the commutator-subgroup of Ω. From this it follows at once that Ω has Property P. (See for example, the proof of Theorem 4·11 in [3], page 31).

Now Ω_s commutes with Ω elementwise, and Ω has Property P. Therefore, by virtue of Theorem 1, Ω has Property T.

The converse part of the above theorem has an interesting generalization to arbitrary subgroups of the general linear group $GL(n, F)$. In order to exhibit it, we shall use the notation $\langle \Omega_s \rangle$ for the group generated by Ω_s in $GL(n, F)$. This is necessitated by the fact that we now drop the restriction of algebraic connectivity for Ω, so that Ω_s may no longer be a part of Ω. We now state.

THEOREM 5. *Let Ω be a subgroup of $GL(n, F)$ such that Ω_s commutes with Ω elementwise. Then Ω has Property T and is nilpotent of class at most $(n-1)$.*[1]

Proof. We divide the proof in four parts.

(i) First observe that if the underlying vector space V is irreducible under $\Omega \cup \Omega_s$, then V is irreducible under Ω. For, suppose to the contrary that V_1 is a proper minimal invariant Ω-subspace of V. Then

[1] The author is highly indebted to the referee for conveying this theorem and its proof to him.

for each u in $\langle \Omega_s \rangle$, $V_1 u$ is a minimal invariant Ω-subspace, because u commutes with every element of Ω. Now $\sum V_1 u$, where the summation is over all u in $\langle \Omega_s \rangle$, is invariant under $\Omega \cup \Omega_s$ and is therefore the whole of V in view of the irreducibility of V with respect to $\Omega \cup \Omega_s$. Therefore, $V = W_1 \oplus \cdots \oplus W_k$, where $W_i = V_1 u_i$, (u_i in $\langle \Omega_s \rangle$), and the W_i are Ω-invariant.

Corresponding to this decomposition, there is a basis for V such that the matrix X in Ω has the block-decomposition,

$$X = \begin{bmatrix} X_1 & & & 0 \\ & X_2 & & \\ & & \ddots & \\ 0 & & & X_k \end{bmatrix},$$

where the X_i are square-blocks of dimension n/k. The corresponding matrix for X_s in Ω_s is clearly,

$$X_s = \begin{bmatrix} (X_1)_s & & & 0 \\ & (X_2)_s & & \\ & & \ddots & \\ 0 & & & (X_h)_s \end{bmatrix},$$

where, as usual, $(X_i)_s$ denotes the semisimple part of X_i.

However, this implies that V is reducible with respect to $\Omega \cup \Omega_s$, contrary to the hypothesis.

(ii) Next, we assert that if V is irreducible with respect $\Omega \cup \Omega_s$, then V has dimension 1. For, by (i) we can assume that V is Ω-irreducible. Since Ω_s commutes with Ω elementwise and F is algebraically closed, it follows by Schur's Lemma, ([6]; Theorem 27·3), that Ω_s is a set of scalars; i.e., $\Omega_s \subseteq F^*$. I, where I is the identity matrix, and F^* denotes the multiplicative group of nonzero elements of the base field F.

Let $\Omega_1 = \{X \text{ in } \Omega \cdot F^* \mid \det X = 1\}$. As F is algebraically closed, so $\Omega \subseteq \Omega_1 \cdot F^*$, and therefore Ω_1 is also an irreducible group. Since $(\Omega \cdot F^*)_s = \Omega_s \cdot F^* = F^* \cdot I$, so each X in Ω_1 has a unique characteristic root of multiplicity n. For every X in Ω_1, we have trace $X =$ trace X_s, so that the set $\{\text{trace } X \mid X \text{ in } \Omega_1\} \subseteq \{\lambda \text{ in } F^* \mid \lambda^n = 1\}$, and so is finite. But, by an argument of Burnside, an irreducible group with only a finite set of trace-values is finite ([6], Theorem 36·1). Thus Ω_1 is a finite irreducible group. Since characteristic of F is zero, so every element of Ω_1 is semi-simple and $\Omega_1 = (\Omega_1)_s \subseteq (\Omega \cdot F^*)_s = F^* \cdot I$. So Ω_1 is an irreducible group of scalars which is possible only when the dimension $n = 1$.

(iii) Now we prove that Ω has Property P. For, let V have a basis with respect to which $\Omega \cup \Omega_s$ has the form,

$$X = \begin{bmatrix} X_1 & & & * \\ & X_2 & & \\ & & \ddots & \\ 0 & & & X_n \end{bmatrix},$$

with diagonal-blocks X_i, and possible nonzero entries only above these diagonal blocks. Suppose X_i's cannot be reduced any further. Then the mapping $X \to X_i$ defines a homomorphism of $\langle \Omega \cup \Omega_s \rangle$ for each i, such that the images of Ω and Ω_s are, say, $\Omega^{(i)}$ and $\Omega_s^{(i)}$ respectively, for a fixed i. Clearly, $(X_i)_s = (X_s)_i$, and so $\Omega_s^{(i)} = (\Omega^{(i)})_s$. Since $\Omega \cup \Omega_s$ cannot be further reduced, $\Omega^{(i)} \cup \Omega_s^{(i)}$ is irreducible. Also $\Omega^{(i)}$ is a group and $\Omega_s^{(i)}$ commutes with $\Omega^{(i)}$ elementwise. Hence by (ii), each block X_i must be of dimension 1.

(iv) Finally, by Theorem 1, combined with (iii), we immediately conclude that Ω has Property T. Also from the proof of Theorem 3, it then follows that Ω is nilpotent of class at-most $(n-1)$, for $n \geq 2$, since the group of all upper triangular unipotent matrices is known to be nilpotent of class $(n-1)$.

We remark that Theorem 5 shows that a nilpotent connected algebraic group has nilpotency class $\leq (n-1)$. On the other hand, matrix groups of degree n and arbitrary nilpotency-class $k \geq 1$ are known to exist, ([7], page 57). Thus we observe that such groups cannot have Property T for $k \geq n$. Thus, for a general matrix-group nilpotency does not imply Property T.

COROLLARY 1. *If Ω is a connected algebraic group, then a necessary and sufficient condition for Ω to have Property T is that Ω_s be an invariant commutative algebraic subgroup of Ω.*

This follows at once from Lemma 4 and Theorem 4.

Combining the above results we deduce the following equivalence of propostions.

COROLLARY 2. *For a connected algebraic group Ω, the followings are equivalent,*

(i) *Ω is nilpotent,*

(ii) *Ω has Property T,*

(iii) *Ω_s is an invariant commutative algebraic subgroup of Ω,*

(iv) *Ω_s is an algebraic subgroup in the centre of Ω.*

Lastly we note that connectivity is an essential part in our hypothesis as can be seen by taking Ω to be a non abelian finite nilpotent group. Then Ω can have Property P or Property T if and only if it is commutative. This follows at once from the Theorem of Maschke about the complete reducibility of finite groups: ([4]).

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