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PROPERTIES OF SOLUTIONS OF N^{th} ORDER LINEAR DIFFERENTIAL EQUATIONS

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The purpose of this paper is to establish zero properties for solutions of the general n^{th} order linear differential equation

$$(1) \qquad L_n y = \sum_{i=0}^{n} r_i(x) y^{(i)} = 0 ,$$

where $r_n(x) \neq 0$ and all the coefficients are continuous.

This work was stimulated by the recent investigations of fourth-order equations by Leighton and Nehari [11], Barrett [3, 4, 5] and Howard [8]. For example, if $\eta_1(a)$ is defined to be the first point $b > a$ for which there exist a solution of

$$(2) \qquad (r(x)y'')'' - p(x)y = 0 \quad (r, p > 0, r \in C_2, p \in C)$$

with four zeros in $[a, b]$ (counting multiplicities), then it is known [11] that there is a solution which vanishes with double zeros at a and $\eta_1(a)$ ($\eta_1(a)$ is termed the first (right) conjugate point of a). If $\mu_1(a)$ is defined to be the first point $b > a$ for which there exists a non-trivial solution of (2) satisfying the conditions $y(a) = y'(a) = (ry'')(b) = (ry'')'(b) = 0$, then it is known [3] that $a < \mu_1(a) < \eta_1(a)$ (a is called the first (left) focal point of $\mu_1(a)$). These results were subsequently extended to general even-order self-adjoint equations by Reid [13] and Hunt [9].

In § 1 we derive some basic results which are used in later sections.

In § 2 we extend the definition of conjugate point found in [11]. Using this definition we obtain generalizations (in a direction different from that of [9] and [13]) of results of [11].

In § 3 we define a notion of focal point for (1) and extend the discussions of [3, 4, 5] to obtain results similar to those of § 2.

In § 4 we develop an eigenvalue relationship from which we can easily obtain a generalization of the following theorem of Leighton and Nehari:

THEOREM. *Equation* (2) *is disconjugate* (*i.e.*, $\eta_1(a)$ *does not exist*)

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on $[a, \infty)$ *if and only if the minimum eigenvalue of the associated problem*

$$(ry'')'' - py = \lambda y\ , \quad y(a) = y'(a) = y(b) = y'(b) = 0\ ,$$

is positive for all $b > a$.

1. **Preliminaries.** We begin our discussion with a number of theorems concerning properties of real functions.

For reference we state the following basic lemma, which was proven by Leighton and Nehari [11].

LEMMA 1. *Let* $u(x)$ *and* $v(x)$ *be of class* C_1 *in* (a, b) *and let* $v(x)$ *be of constant sign* $(\neq 0)$ *in the interval. If* $x = \alpha$ *and* $x = \beta$ $(a < \alpha < \beta < b)$ *are consecutive zeros of* $u(x)$, *then there exists a constant* μ *such that the function* $u(x) - \mu v(x)$ *has at least a double zero in* (α, β).

This lemma has been extended by Azbelev and Chaluk [1] to the case where β is allowed to be a double zero of $u(x)$ and a simple zero of $v(x)$. We shall use the method of the latter together with Lemma 1 to prove

THEOREM 1. *Let* $u(x)$ *be a function such that* $u(x)$ *has a zero of order* $n \geqq 1$ *at* $x = a$ *and a zero of order* $m \geqq 1$ *at* $x = b$, *and* $u(x)$ *is of constant sign* $(\neq 0)$ *in* (a, b). *Let* $v(x)$ *be a function such that* $v(x)$ *has a zero of order* $n_1 < n$ *at* $x = a$ *and a zero of order* $m_1 < m$ *at* $x = b$, *and* $v(x)$ *is of constant sign* $(\neq 0)$ *in* (a, b). *Further suppose* $u(x)$ *and* $v(x)$ *are both of class* $C_M[a, b]$, *where* $M = \max(n_1, m_1)$. *Then there exists a linear combination* $z(x)$ *of* $u(x)$ *and* $v(x)$ *such that* $z(x)$ *has a double zero in* (a, b).

Proof. We may as well suppose that $u(x)$ and $v(x)$ are both positive on (a, b). Let $c \in (a, b)$ and let

$$w(x) = v(x) - \frac{v(c) + 1}{u(c)} u(x)\ .$$

There exists an $h_1 \in (a, c)$ such that $w(h_1) > 0$, since, by the zero properties of $u(x)$ and $v(x)$, $w^{(n_1)}(a) = v^{(n_1)}(a) > 0$ and $w^{(i)}(a) = 0$ $(i = 0, 1, \cdots, n_1 - 1)$. Similarly there is an $h_2 \in (c, b)$ such that $w(h_2) > 0$, since $(-1)^{m_1} w^{(m_1)}(b) > 0$ and $w^{(i)}(b) = 0$ $(i = 0, 1, \cdots, m_1 - 1)$. Also $w(c) = -1 < 0$; thus $w(x)$ has a zero in (a, c) and a zero in (c, b). Since $w(x)$ is continuous, there exists α and β $(a < \alpha < c < \beta < b)$ such that $w(\alpha) = w(\beta) = 0$ and $w(x) < 0$ on (α, β). Applying Lemma 1 to the functions $w(x)$ and $v(x)$ on (α, β), have our result.

THEOREM 2. *Suppose* $r(x)f(x) \in C_n$ *and* $f(x) \in C_n$ *in a neighborhood of the point* $x = a$. *If* $f^{(i)}(a) = 0$ $(i = 0, 1, \cdots, n)$ *and if* $r(x)$ *is a bounded function in a neighborhood of* a, *then* $(rf)^{(i)}(a) = 0$ $(i = 0, 1, \cdots, n)$.

Proof. The theorem is trivial for $n = 0$. For $n = 1$,

$$\begin{aligned} |(rf)'(a)| &= \left| \lim_{h \to 0} \frac{(rf)(a+h) - (rf)(a)}{h} \right| \\ &= \left| \lim_{h \to 0} \frac{r(a+h)f(a+h) - r(a)f(a)}{h} \right| \\ &\leqq \overline{\lim_{h \to 0}} |r(a+h)| \left| \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right| \\ &= M |f'(a)| = 0 , \end{aligned}$$

where $M = \overline{\lim}_{x \to a} |r(x)|$. Now using the formula

$$(rf)^{(k)}(a) = \lim_{h \to 0} \frac{\sum_{i=0}^{k} (-1)^i C(k, i)(rf)(a + (k-i)h)}{h^k}$$

we shall show that $(rf)^{(k)}(a) = 0$ $(1 \leqq k \leqq n)$. Applying L'Hospital's rule, and using the zero properties of $f(x)$ at $x = a$ and the boundedness of r at $x = a$, we have

$$\begin{aligned} |(rf)^{(k)}(a)| &= \left| \lim_{h \to 0} \sum_{i=0}^{k} \frac{(-1)^i C(k, i) r(a + (k-i)h) f(a + (k-i)h)}{h^k} \right| \\ &\leqq M \sum_{i=0}^{k-1} \left| \frac{(k-i)^k}{i!\,(k-i)!} \lim_{h \to 0} \frac{f^{(k-1)}(a + (k-i)h) - f^{(k-1)}(a)}{(k-i)h} \right| \\ &= M |f^{(k)}(a)| \sum_{i=0}^{k-1} \frac{(k-i)^k}{i!\,(k-i)!} . \end{aligned}$$

Since $f^{(k)}(a) = 0$, the result follows.

COROLLARY. *Suppose* $f(x) \in C_n$, $r(x)f(x) \in C_n$, *and* $r(x) \in C$ *in a neighborhood of* $x = a$. *If* $f^{(i)}(a) = 0$ $(i = 0, 1, \cdots, n-1)$, $(rf)^{(n)}(a) = 0$, *and* $r(a) \neq 0$, *then* $f^{(n)}(a) = 0$.

Proof. Since $r(x)$ is continuous at $x = a$, we can replace M by $r(a)$, "$\leqq$" by "$=$", and drop the absolute value signs in the inequality in the proof of Theorem 2, and obtain a valid equality; the result is then immediate.

In the next theorem and in the remainder of the paper we shall deal with the equation

$$(1) \qquad L_n y = \sum_{i=0}^{n} r_i(x) y^{(i)}(x) = 0 ,$$

where $r_n(x) \neq 0$ and the coefficients $r_i (i = 0, 1, \cdots, n)$ are real valued continuous functions.

The following basic existence theorem is apparently sufficiently well-known so that references to it in the literature are scarce; we include it here for the sake of completeness.

THEOREM 3. *If* $a_i \in [a, b]$ $(i = 1, \cdots, k \leqq n-1)$ *and* $a_1 < a_2 < \cdots < a_k$, *then there exists a nontrivial solution of* (1) *which satisfies the boundary conditions*

$$\begin{aligned} y^{(i_1)}(a_1) &= 0 \qquad i_1 = n_{11}, n_{12}, \cdots, n_{1p_1} \\ y^{(i_2)}(a_2) &= 0 \qquad i_2 = n_{21}, n_{22}, \cdots, n_{2p_2} \\ &\vdots \\ y^{(i_k)}(a_k) &= 0 \qquad i_k = n_{k1}, n_{k2}, \cdots, n_{kp_k} \end{aligned}$$

where $0 \leqq n_{j1} < n_{j2} < \cdots < n_{jp_j} < n$ $(j = 1, \cdots, k)$ *and* $\sum_{j=1}^{k} p_j \leqq n - 1$.

Proof. Let $y_1(x), y_2(x), \cdots, y_n(x)$ be a fundamental set of solutions of (1). We wish to find a nontrivial set of constants $C_1, \cdots, C_n$ such that $y = C_1 y_1 + \cdots + C_n y_n$ satisfies the above boundary conditions.

Applying the boundary conditions to $y(x)$, we have

$$\begin{aligned} y^{(n_{11})}(a_1) &= C_1 y_1^{(n_{11})}(a_1) + \cdots + C_n y^{(n_{11})}(a_1) = 0 \\ &\vdots \\ y^{(n_{1p_1})}(a_1) &= C_1 y_1^{(n_{1p_1})}(a_1) + \cdots + C_n y_n^{(n_{1p_1})}(a_1) = 0 \\ &\vdots \\ y^{(n_{21})}(a_2) &= C_1 y_1^{(n_{21})}(a_2) + \cdots + C_n y_n^{(n_{21})}(a_2) = 0 \\ &\vdots \\ y^{(n_{2p_2})}(a_2) &= C_1 y_1^{(n_{2p_2})}(a_2) + \cdots + C_n y_n^{(n_{2p_2})}(a_2) = 0 \\ &\vdots \\ y^{(n_{k1})}(a_k) &= C_1 y_1^{(n_{kp_k})}(a_k) + \cdots + C_n y_n^{(n_{kp_k})}(a_k) = 0 \\ &\vdots \\ y^{(n_{kp_k})}(a_k) &= C_1 y_1^{(n_{k1})}(a_k) + \cdots + C_n y_n^{(n_{k1})}(a_k) = 0 . \end{aligned}$$

This is a system of $\sum_{j=1}^{k} p_j$ (i.e., less than n) homogeneous equations in n unknowns $C_1, \cdots, C_n$, and so there always exists a nontrivial solution.

We shall make repeated use of this theorem throughout the remainder of this paper.

2. Zero properties of solutions. In this section we shall be concerned with zero properties of solutions of (1). The principal stimulation for this investigation is the work of Leighton and Nehari [11]. Their concern was with the equation

$$(r(x)y'')'' - p(x)y = 0 , \tag{2}$$

where $r(x) \in C_2$, $p(x) \in C$, and $r(x) > 0$. They defined the first conjugate point of the point a, denoted by $\eta_1(a)$, to be the minimum point $b > a$ such that there is a nontrivial solution $y(x)$ of (2) vanishing at $x = a$, with four zeros, counted in their multiplicities, in $[a, b]$. They showed that if there is a nontrivial solution with four zeros on $[a, \infty)$, then $\eta_1(a)$ exists. It was then shown that, if $p(x) > 0$ in (2), a solution which vanishes four times in $[a, \eta_1(a)]$ has double zeros at $x = a$ and $x = \eta_1(a)$ and does not vanish in $(a, \eta_1(a))$. Barrett [3, 4, 5] then extended this work to equations of the form

$$[(r(x)y'')' + q(x)y']' + p(x)y = 0 ,$$

where $r(x)$, $q(x)$ and $p(x)$ are all continuous. Hunt [9] and Reid [13] in turn extended this work to self-adjoint differential equations of order $2n$, defining conjugate point as the minimum point $b > a$ such that there exists a nontrivial solution with a zero of order n at a and a zero of order n at b.

Hanan [7], using a definition of conjugate point similar to that of Leighton and Nehari, investigated the third order equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0 .$$

Investigations similar to Hanan's were undertaken by Azbelev and Chaluk [1].

We shall here be interested in the extension of this work to the equation (1), using the conjugate point concept of Leighton and Nehari. Some of the results obtained in this section have been reported independently by A. Ju. Levin [12] who uses the methods of Green's functions. It should be noted that self-adjointness and assumptions on the signs of the coefficients have been of fundamental importance in the work cited above (except for that of Levin); in general we make no such assumptions here. Following Barrett [2, 3], we make the following definition:

DEFINITION 1. Equation (1) is said to be disconjugate in $[a, \infty)$ if there exists no nontrivial solution of (1) which vanishes at a, and has n zeros, counting multiplicities, in $[a, \infty)$.

We now prove a basic theorem, special cases of which have been

used by Leighton and Nehari and Barrett, and a special case of which was proved by Hanan.

THEOREM 4. *Suppose that for each positive integer k there is an n^{th} order linear boundary operator U_k, operating at points $b_{k1}, \cdots, b_{kj_k}$, such that*

(i) (1) *has a nontrivial solution $y_k(x)$ satisfying $U_k(y_k(x)) = 0$;*

(ii) *there exist points $a_{k1}, \cdots, a_{kn}$ such that* $y_k^{(i-1)}(a_{ki}) = 0$ $(i = 1, \cdots, n)$;

(iii) $\min_i b_{ki} \leqq \min_i a_{ki} < \max_i a_{ki} \leqq \max_i b_{ki}$;

(iv) $\min_i b_{ki} \leqq \min_i b_{ji}$ *and* $\max_i b_{ji} \leqq \max_i b_{ki}$ *if* $k < j$.

Then $\lim_{k\to 0}(\max_i b_{ki} - \min_i b_{ki}) > 0$.

REMARK. If $y_k(x)$ is a nontrivial solution of (1) having n zeros on an interval, counted in accordance with their multiplicities, then repeated application of Rolle's Theorem shows the existence of the desired set of points $a_{k1}, a_{k2}, \cdots, a_{kn}$.

Proof of Theorem 4. Let $A_k = \min_i a_{ki}$, $B_k = \max_i a_{ki}$, $C_k = \min_i b_{ki}$ and $D_k = \max_i b_{ki}$. Conditions (iv) and (iii) assure us that $\{D_k\}$ forms a decreasing sequence bounded below; hence there is a D such that $D_k \to D$. Similarly $\{C_k\}$ forms an increasing sequence bounded above; hence there is a C such that $C_k \to C$ and $D \geqq C$. We need only show that equality does not hold.

Let $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ be the fundamental set of solutions of (1) satisfying $\phi_{i+1}^{(j)}(D) = \delta_{ij}$ $(i, j = 0, 1, \cdots, n-1)$. Then $y_k(x)$ can be written

$$y_k(x) = \sum_{i=1}^{n} c_{ik}\phi_i(x) .$$

We normalize $y_k(x)$ by letting

$$\bar{y}_k(x) = \frac{y_k(x)}{\left(\sum_{i=1}^{n} c_{ik}^2\right)^{1/2}} = \sum_{i=1}^{n} d_{ik}\phi_i(x) ,$$

so that $\sum_{i=1}^{n} d_{ik}^2 = 1$. Then $\bar{y}_k(x)$ is a solution of (1) possessing the same zero properties as $y_k(x)$; in particular, $\bar{y}_k(x)$ satisfies condition (ii). Further, on any closed interval about D, $\{\bar{y}_k(x)\}$ is a uniformly bounded and equicontinuous family. Hence there is a uniformly convergent subsequence $\{\bar{y}_{k_i}(x)\}$ whose limit function $\bar{y}(x)$ is a solution of (1).

Now suppose $D = C$; then by condition (iii)

$$A = \lim_{k\to 0} A_k = \lim_{k\to 0} B_k = B .$$

But now by condition (ii), for each $j(0 \leqq j < n)$, A is a limit point of zeros of $\{\bar{y}_{k_i}^{(j)}(x)\}$; hence $\bar{y}^{(j)}(A) = 0$ $(j = 0, 1, \cdots, n-1)$, and so $\bar{y}(x) \equiv 0$. This is impossible, since $\sum_{i=1}^{n} d_{ik}^2 = 1$, which completes the proof.

We are now in a position to make

DEFINITION 2. The first conjugate point $\eta_1(a)$ of the point a is the smallest number $b > a$ such that there exists a nontrivial solution of (1) which vanishes at a and has n zeros, counting multiplicities in $[a, b]$.

We now establish one of the main results of this section.

THEOREM 5. *If* (1) *is not disconjugate, then* $\eta_1(a)$ *exists, and there is a nontrivial solution of* (1) *which has a total of at least* n *zeros at* a *and* $\eta_1(a)$ *and does not vanish in* $(a, \eta_1(a))$.

Proof. The existence of $\eta_1(a)$ is guaranted by Theorem 4. Let $Y = \{y(x) \mid y(x)$ is a nontrivial solution of (1) which has n zeros in $[a, \eta_1(a)]\}$, $R = \{r \mid$ there is a $y(x) \in Y$ with a zero of order r at $x = a\}$, $n_0 = \max r \in R$, $M = \{m \mid$ there is a $y \in Y$ with a zero of order n_0 at $x = a$, and m zeros in $(a, \eta_1(a))\}$, $\bar{m} = \max m \in M$, and let $\phi(x) \in Y$ be a solution with $\bar{m}$ zeros in $(a, \eta_1(a))$ and a zero of order n_0 at $x = a$. Let $a = a_0 < a_1 < \cdots < a_p = \eta_1(a)$ be the zeros of $\phi(x)$ on $[a, \eta_1(a)]$, with respective multiplicities $n_0, n_1, \cdots, n_p$. Then $n_1 + n_2 + \cdots + n_{p-1} = \bar{m}$. Let $\bar{n} = n_0 + \cdots + n_{p-1}$. We claim that $\phi(x)$ exhibits the property asserted. For suppose the theorem false for $\phi(x)$, i.e., suppose $p > 1$. We shall show the existence of a $\psi(x) \in Y$ with a zero of order n_0 at $x = a$ and a zero of order $\bar{m} + 1$ in $(a, \eta_1(a))$, which will contradict the maximality of $\bar{m}$. To this end let us consider a nontrivial solution $z(x)$ of (1) with zeros at $a_0, a_1, \cdots, a_p$ of multiplicitis $n_0 + 1, n_1, \cdots, n_{p-1} - 1$, $n - \bar{n} - 1$ respectively; such a solution exists (note that $p > 1$ implies $n_0 < n - 1$) since only $n - 1$ zeros (counting multiplicities) are specified. Since the zero of $z(x)$ at a_0 is of order $n_0 + 1$, $z(x) \notin Y$, and so $z(x)$ does not vanish on (a_{p-1}, a_p), nor are any of the zeros of multiplicity greater than that specified. Hence we can apply Theorem 1 to $z(x)$ and $\phi(x)$ on (a_{p-1}, a_p) to obtain a linear combination $\psi(x)$ with a double zero at c, say, on (a_{p-1}, a_p). Now $\psi(x)$ has zeros at $a_0, a_1, \cdots, a_{p-2}$, a_{p-1}, c, a_p of multiplicities $n_0, n_1, \cdots, n_{p-2}, n_{p-1} - 1, 2, n - \bar{n} - 1$; i.e., $\psi(x) \in Y$ with a zero at a of order n_0, and with a total multiplicity of $\bar{m} + 1$ of zeros on $(a, \eta_1(a))$, contradicting the maximality of $\bar{m}$; this established the theorem.

COROLLARY. *Let* $\phi(x)$ *be a nontrivial solution of* (1) *with zeros*

of orders n_0 and at least $n - n_0$ at a and $\eta_1(a)$ respectively, where n_0 is maximal. Then $\phi(x)$ is essentially unique i.e., unique except for multiplication by a constant factor). (We note that a similar result holds for a $\psi(x) \in Y$, not vanishing on $(a, \eta_1(a))$, and with a zero of maximal order at $\eta_1(a)$).

Proof. Let $\psi(x)$ be any such solution. By the maximality of n_0 we have $\phi^{(n_0)}(a) \neq 0 \neq \psi^{(n_0)}(a)$. Consider the solution

$$\chi(x) = \psi^{(n_0)}(a)\phi(x) - \phi^{(n_0)}(a)\psi(x) .$$

$\chi(x)$ clearly has a zero of order $n + 1$ at $x = a$ and a zero of order $n - n_0$ at $x = \eta_1(a)$; hence, by the maximality of n_0, $\chi(x) \equiv 0$.

In order to examine the behavior of $\eta_1(a)$ as a function of a we prove the following theorems. The principal theorem, Theorem 7, in this sequence was obtained by Leighton and Nehari [11] for equation (2) and by Hunt [9] for the special definition of η_1 mentioned at the beginning of this section, for an equation more general than that in [11].

THEOREM 6. *Let a be any point for which $\eta_1(a)$ exists; then, for any $b < a$, $\eta_1(b)$ exists.*

Proof. Let a_0 be the largest number $c \geqq a$ such that $\eta_1(c) = \eta_1(a)$ (the existence of a_0 is guaranteed by Theorem 4). Let $\phi(x)$ be a solution such that $\phi(a_0) = 0$ and $\phi(x)$ has n zeros in $[a_0, \eta_1(a)]$, where the order of the zero of $\phi(x)$ at $\eta_1(a)$ is minimal over the set of all such solutions. Let $m_0, m_1, \cdots, m_r$ be the orders of the zeros of $\phi(x)$ at $a_0 < a_1 < \cdots < a_r = \eta_1(a)$ respectively. We are supposing m_r is minimal. Let $M = m_0 + \cdots + m_{r-1} \leqq n - 1$. Clearly $m_r \geqq n - M$.

Now suppose the theorem is not true. Let $\psi(x)$ be a nontrivial solution of (1) such that $\psi(b) = 0$ and $\psi(x)$ has zeros of multiplicities $m_0, m_1, \cdots, m_{r-2}, m_{r-1} - 1, n - M - 1$ at $a_0, a_1, \cdots, a_{r-2}, a_{r-1}, a_r$ respectively. We have specified (counting multiplicities) $n - 1$ zeros of $\psi(x)$ in $[b, \eta_1(a)]$; hence, since we are supposing $\eta_1(b)$ does not exist, $\psi(x)$ has zeros only at these points and only of the multiplicities specified. By Theorem 1, applied to $\phi(x)$ and $\psi(x)$, there must exist a linear combination $\chi(x)$ of $\phi(x)$ and $\psi(x)$ with a double zero in (a_{r-1}, a_r). Further, $\chi(x)$ has zeros of orders $m_0, m_1, \cdots, m_{r-2}, m_{r-1} - 1, n - M - 1$ at $a_0, a_1, \cdots, a_{r-1}, a_r$ respectively. Hence $\chi(x)$ has n zeros on $[a_0, \eta_1(a)]$. Now if $r = 1 = m_0 = m_r$ then $\chi(x)$ has a double zero and is also a solution of a second order equation; hence $\chi(x) \equiv 0$. If $r = 1 = m_0$ and $m_r > 1$ then $\chi(\eta_1(a)) = 0$ and $\chi(x)$ has n zeros in $(a_0, \eta_1(a)]$ which contradicts the maximality of a_0. If $r = 1 = m_r$ and $m_0 > 1$ then $\chi(x)$ has n zeros

in $[a_0, \eta_1(a))$ and vanishes at a_0, which contradicts the fact that $\eta_1(a^0) = \eta_1(a)$. In all other cases $\chi(x)$ has n zeros in $[a_0, \eta_1(a)]$ with a zero of order $n - M - 1 < m_r$ at $\eta_1(a)$, which contradicts the minimality of m_r.

COROLLARY 1. *Let a be any point for which $\eta_1(a)$ exists; then, for any $b < a$, there exists a solution which vanishes at b and which vanishes $n - 1$ times, counting multiplicity, in $[a, \eta_1(a)]$.*

Proof. The function $\psi(x)$ constructed in the proof of Theorem 5 is such a function.

COROLLARY 2. *Let a be any point for point for which $\eta_1(a)$ exists; then no solution has more than $n - 1$ zeros in $[a, \eta_1(a))$.*

Proof. Suppose this is not true. Then there exists a solution with n zeros in $[a, \eta_1(a))$. Hence there is a $b \in [a, \eta_1(a))$ such that $\eta_1(b) < \eta_1(a)$; clearly such a b cannot equal a. Then, by Corollary 1, there exists a solution $\phi(x)$ which vanishes at a and which vanishes $n - 1$ times, counting multiplicity, in $[b, \eta_1(b)]$. Hence $\phi(x)$ vanishes at a and has n zeros in $[a, \eta_1(b)]$, which contradicts the definition of $\eta_1(a)$.

It should be noted that, for $n = 2$, Corollaries 1 and 2 combine to give the classicial Sturm separation theorem.

COROLLARY 3. *$\eta_1(a)$ is a nondecreasing function of a.*

THEOREM 7. *$\eta_1(a)$ is an increasing function of a.*

Proof. Suppose the theorem is not true. We know by Corollary 3 that $\eta_1(a)$ is a nondecreasing function of a. Thus there exist $a, \eta_1(a), b$ and $\eta_1(b)$ such that $a < b$ and $\eta_1(a) = \eta_1(b)$. Hence, by Theorem 5 and Corollary 3, for every $x \in [a, b]$, $\eta_1(x)$ exists and equals $\eta_1(a)$. For each $\alpha \in [a, b]$ let $\phi_\alpha(x)$ be the essentially unique solution which vanishes at least n times at α and $\eta_1(a)$, where the order of the zero at $\eta_1(a)$ is maximal over all such solutions, and $\phi_\alpha(x) \neq 0$ for $x \in (\alpha, \eta_1(a))$, as in the corollary to Theorem 5.

Let $S_i = \{\alpha \mid \phi_\alpha(x)$ has a zero of exactly order i at $x = \eta_1(a)\}$, $(i = 1, \cdots, n - 1)$. Clearly $\bigcup_{i=1}^{n-1} S_i = [a, b]$. Hence there is a subinterval $[c, d] \subseteq [a, b]$ and an $i (1 \leqq i < n)$ such that S_i is dense in $[c, d]$ (see Simmons [14], page 74). Pick the maximum such i and denote it by m; denote the subinterval $[c, d]$ corresponding to this value of i by $[a_m, b_m]$.

Let $\{y_1, y_2, \cdots, y_n\}$ be the fundamental set of solution satisfying $y_{i+1}^{(j)}(\eta_1(a)) = \delta_{ij}$. Now the condition that S_m is dense in $[a_m, b_m]$ is equivalent to the condition that the determinant

$$W_{m+1} = \begin{vmatrix} y_{m+1} & y_{m+2} & \cdots & y_n \\ y'_{m+1} & y'_{m+2} & \cdots & y'_n \\ \vdots & & & \\ y_{m+1}^{(n-m-1)} & y_{m+2}^{(n-m-1)} & \cdots & y_n^{(n-m-1)} \end{vmatrix}$$

vanish at a dense set of points in $[a_m, b_m]$. This follows since $W_{m+1}(\alpha) = 0$ is the condition for a solution with a zero of order m at $\eta_1(a)$ to have a zero of order $n - m$ at a point $\alpha \neq \eta_1(a)$. Further, since W_{m+1} is continuous, we have $W_{m+1} \equiv 0$ in $[a_m, b_m]$. Moreover from the maximality of m we know that

$$W_{m+2} = \begin{vmatrix} y_{m+2} & y_{m+3} & \cdots & y_n \\ y'_{m+2} & y'_{m+3} & \cdots & y'_n \\ \vdots & & & \\ y_{m+2}^{(n-m-2)} & y_{m+3}^{(n-m-2)} & \cdots & y_n^{(n-m-2)} \end{vmatrix} \neq 0$$

on some subinterval $[a'_m, b'_m] \subseteq [a_m, b_m]$.

Now consider the differential equation

$$\begin{vmatrix} y & y_{m+2} & \cdots & y_n \\ y' & y'_{m+2} & \cdots & y'_n \\ \vdots & & & \\ y^{(n-m-1)} & y_{m+2}^{(n-m-1)} & \cdots & y_n^{(n-m-1)} \end{vmatrix} = 0$$

formed by substituting y for y_{m+1} in the equation $W_{m+1} \equiv 0$. This is an equation of order $n - m - 1$ with non-vanishing leading coefficient W_{m+2} on the interval $[a'_m, b'_m]$. The solutions are $y_{m+2}, y_{m+3}, \cdots, y_m$, a total of $n - m - 1$ solutions. These are linearly independent, since $W_{m+2} \neq 0$ on $[a'_m, b'_m]$; hence they form a solution basis. But y_{m+1} is also a solution, since $W_{m+1} \equiv 0$ on $[a'_m, b'_m]$; therefore $y_{m+1}, y_{m+2}, \cdots, y_n$ are linearly dependent on $[a'_m, b'_m]$, which contradicts the fact that $y_1, \cdots, y_n$ are solutions of (1) which are independent on $[a, b]$. This completes the proof.

3. Zero properties of derivatives of solutions. We now turn our attention to the behavior of the derivatives of solutions. The principal motivation for this type of investigation is the work of Barrett [2, 3, 4, 5], in which are references to earlier work in this field. Barrett [2] discussed the behavior of solutions of the equation

$$(p(x)y')' + f(x)y = 0$$

in relation to the boundary conditions $y(a) = y'(b) = 0$; the minimum such $b > a$ for which these conditions are satisfied was denoted by $\mu_1(a)$, and a was called a (left) focal point of the point $\mu_1(a)$. This discussion was extended to the equation (2), with a focal point being defined in terms of the first point $b > a$, for which conditions of the type $y(a) = y'(a) = y''(b) = y'''(b) = 0$ are satisfied. This was in turn extended by Hunt [9] and Reid [13] to more general even order self-adjoint differential equations of the form

$$[r_n(x)y^{(n)}]^{(n)} + \sum_{i=0}^{n-1} [(-1)^{n+i+1} r_i(x)y^{(i)}]^{(i)} = 0 .$$

Their definition of focal point involved the conditions

$$\begin{aligned} y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0 \\ = (r_n y^{(n)})(b) = (r_n y^{(n)})'(b) = \cdots = (r_n y^{(n)})^{(n-1)}(b) . \end{aligned}$$

It should be noted that the use of the terms "focal point" and "conjugate point" stems from considerations in the calculus of variations, in which field the majority of the investigations have taken place (see for example Reid [13] and the references in that paper).

In the present paper, we propose to extend the definition of focal point in a different direction, to which the techniques of differential equations seem more applicable. To this end we make the following definition.

DEFINITION 3. Suppose that there exists a point $c > a$, an integer $k(0 < k < n)$, and a nontrivial solution $\phi(x)$ of (1) which has a zero of order k at $x = a$ and such that $\phi^{(k)}(x)$ vanishes $n - k$ times, counting multiplicities, in $(a, c]$. Then the minimum such c, over all such solutions for all possible k, which exists by Theorem 4, will be denoted by $\tilde{\mu}_1(a)$.

DEFINITION 4. Suppose that there exists a point $c > a$, an integer $k(0 < k < n)$, and a nontrivial solution $\phi(x)$ of (1) such that $\phi(x)$ has a zero at $x = a$ of order k and such that $\phi^{(k)}(x)$ has a zero at $x = c$ of order $n - k$. By Theorem 4 there is a minimum such c, over all such solutions for all possible k, to be denoted by $\mu_1(a)$; a will then be called the first (left) focal point of $\mu_1(a)$.

THEOREM 8. *If $\tilde{\mu}_1(a)$ exists, then $\mu_1(a)$ exists and $\mu_1(a) = \tilde{\mu}_1(a)$.*

Proof. Let $U = \{u(x) \mid u(x)$ is a nontrivial solution of (1) and there exists a $k(0 < k < n)$ such that $u(x)$ has a zero of order k at $x = a$ and such that $u^{(k)}(x)$ has $n - k$ zeros (counting multiplicities) in $(a, \tilde{\mu}_1(a)]\}$,

$K = \{k \mid$ there is a $u(x) \in U$ with a zero of order k at $x = a\}$, $m_0 = \max k \in K$, $P = \{p \mid$ there is a $u(x) \in U$ with a zero of order m_0 at $x = a$, and such that $u^{(m_0)}(x)$ has p zeros in $(a, \tilde{\mu}_1(a))\}$, $p_0 = \max p \in P$, and let $\psi(x) \in U$ be a solution with a zero of order m_0 at $x = a$ and such that $\psi^{(m_0)}(x)$ has p_0 zeros in $(a, \tilde{\mu}_1(a))$. Let the zeros of $\psi^{(m_0)}(x)$ be of orders $m_1, m_2, \cdots, m_j$ at $a_1, a_2, \cdots, a_j$ respectively, where $a < a_1 < \cdots < a_j = \tilde{\mu}_1(a)$; by definition of p_0, $m_1 + m_2 + \cdots + m_{j-1} = p_0$.

Now suppose the conclusion does not hold; i.e., suppose $j > 1$. Let $z(x)$ be a nontrivial solution of (1) with a zero of order $m_0 + 1$ at $x = 0$ (note that $z(x) \notin U$) and with $z^{(m_0)}(x)$ having zeros of multiplicities $m_1, m_2, \cdots, m_{j-1}, m_{j-1} - 1, n - m_0 - 1 - p_0$ at $a_1, a_2, \cdots, a_{j-2}, a_{j-1}, a_j$ respectively, except that if $m_{j-1} = 1$ we make no specification at a_{j-1}. We have specified $n - 1$ zeros of $z(x)$ and $z^{(m_0)}$ on $[a, \tilde{\mu}_1(a)]$, $n - m_0 - 1$ zeros of $z^{(m_0)}(x)$ on $[a, \tilde{\mu}_1(a)]$, p_0 zeros of $z^{(m_0)}(x)$ on $[a, \tilde{\mu}_1(a))$, and $p_0 - 1$ zeros of $z^{(m_0)}(x)$ on $(a, \tilde{\mu}_1(a))$. Hence by Rolle's Theorem $z^{(m_0+1)}(x)$ has at least $n - m_0 - 2$ zeros on $[a, \tilde{\mu}_1(a)]$, and, in fact, since $z(x) \notin U$, $z^{(m_0)}(x)$ has exactly $n - m_0 - 2$ zeros on $[a, \tilde{\mu}_1(a)]$. Therefore, $z^{(m_0)}(x)$ does not vanish at any point on (a_{j-1}, a_j), nor are any of the specified zeros of multiplicities greater than that specified. We can now apply Theorem 1 to the functions $z^{(m_0)}(x)$ and $\psi^{(m_0)}(x)$ on the interval $[a_{j-1}, a_j]$, concluding that there is a linear combination $\chi(x)$ of $\psi(x)$ and $z(x)$ such that $\chi^{(m_0)}(x)$ has a double zero in (a_{j-1}, a_j). We now have a nontrivial solution $\chi(x)$ of (1) with a zero of order m_0 at $x = a$, such that $\chi^{(m_0)}(x)$ has at least $n - m_0$ zeros on $[a, \tilde{\mu}_1(a)]$ (hence $\chi(x) \in U$), and $\chi^{(m_0)}(x)$ has at least $p_0 + 1$ zeros on $(a, \tilde{\mu}_1(x))$ which contradicts the maximal property of p_0. Hence $j = 1$ and the conclusion holds.

THEOREM 9. *If $\eta_1(a)$ exists, then $\mu_1(a)$ exists and $\mu_1(a) < \eta_1(a)$.*

Proof. Let $\phi(x)$ be the essentially unique solution of Theorem 5, with its zero at a of order n_0 (where n_0 is maximal over the orders of the zero of all nontrivial solutions vanishing n times at a and $\eta_1(a)$); then its zero at $\eta_1(a)$ is of order at least $n - n_0$. By Rolle's Theorem, $\phi'(x)$ vanishes at least once on $(a, \eta_1(a))$ and at least $n - n_0$ times on $(a, \eta_1(a)]$. After repeated application of Rolle's Theorem we find that $\phi^{(n_0)}(x)$ vanishes at least n_0 times on $(a, \eta_1(a))$ and at least $n - n_0$ times on $(a, \eta_1(a)]$. Hence $\tilde{\mu}_1(a)$ exists, by Theorem 4; clearly $\tilde{\mu}_1(a) \leqq \eta_1(a)$, and by Theorem 8 $\mu_1(a) = \tilde{\mu}_1(a)$.

To show that $\mu_1(a) < \eta_1(a)$, suppose $\mu_1(a) \geqq \eta_1(a)$. Then, by the above, $\mu_1(a) = \tilde{\mu}_1(a) = \eta_1(a)$. Let $R = \{r \mid$ there exists a $u(x) \in U$ with a zero of order r at $x = a$ and such that $u^{(r)}(x)$ vanishes in $(a, \tilde{\mu}_1(a))\}$. By assumption, $\tilde{\mu}_1(a) = \eta_1(a)$; hence, by the first paragraph of this proof, R is nonvoid. Let $r_0 = \max r \in R$. Let $S = \{s \mid$ there exists a

$u(x) \in U$ with a zero of order r_0 at $x = a$ and such that $u^{(r_0)}(x)$ has s zeros (counting multiplicities) in $(a, \tilde{\mu}_1(a))\}$. Let $s_0 = \max s \in S$. We now apply the argument used in the proof of Theorem 8 to obtain a contradiction.

COROLLARY 1. *If $\phi(x)$ is a nontrivial solution of* (1) *with a zero of order k at $x = a$ and such that $\phi^{(k)}(x)$ has a zero of order $n - k$ at $x = \mu_1(a)$, where k is maximal over all such solutions, then $\phi^{(i)}(x) \neq 0$ in $(a, \mu_1(a)]$ $(i = 0, 1, \cdots, k - 1)$ and $\phi^{(k)}(x) \neq 0$ in $(a, \mu_1(a))$.*

Proof. Theorem 8 tells us that $\phi^{(k)}(x) \neq 0$ in $(a, \mu_1(a))$; and if $\phi^{(i)}(x) = 0$ at some point $x \in (a, \mu_1(a)]$, $(0 \leqq i \leqq k - 1)$, Rolle's Theorem allows us to conclude that there is a point $x_0 \in (a, \mu_1(a))$ where $\phi^{(k)}(x_0)$ is zero.

COROLLARY 2. *The conclusions of Theorems* 8 *and* 9 *are valid under the weaker conditions that there exists a nontrivial solution $\phi(x)$ of* (1) *satisfying the following: $\phi(x)$ has a zero of order q at $x = a$, and either*

(i) *$\phi(x)$ has $n - 1$ zeros (counting multiplicities) on $[a, \infty)$ and there exists a $p(1 \leqq p \leqq q)$ such that $\phi^{(p)}(x)$ vanishes at least once past the $(n - 1)^{st}$ zero; or*

(ii) *there exists a $p(0 \leqq p \leqq q)$ such that $\phi^{(p)}(x)$ has $n - q$ zeros (counting multiplicities) on (a, ∞).*

Proof. As in the proof of Theorem 9, Rolle's Theorem guarantees the existence of $\tilde{\mu}_1(a)$. The remainder of the proof is identical with that of Theorem 8.

4. An eigenfunction result. The basic relationships between conjugate points of equation (2) and eigenvalues of the associated boundary value problem were investigated in detail by Leighton and Nehari [11]. Subsequently Howard [8], using variational methods, discussed the relationships between focal points of (2) and eigenvalues of an eigenvalue problem, with focal point boundary conditions, associated with (2), making various assumptions on the coefficients. Barrett [3, 5] extended these results to fourth order equations of more general form. These results were extended by Reid [13] to more general even order self-adjoint differential equations.

After obtaining a general relationship between the solutions of (1) and those of

$$L_n^+ y = \sum_{i=0}^{n} (-1)^i (r_i y)^{(i)} = 0 \tag{3}$$

(the Lagrangian adjoint equation), we shall derive an eigenvalue relationship, from which we will develop a disconjugacy criterion. It should be noted that Theorem 10 is but a part of the more general theory of boundary operators ([10, Chapter 9], [6, Chapter 11]) included here to demonstrate the intimate relationship between boundary value problems associated with equation (1) and those associated with equation (3).

THEOREM 10. *If $u(x)$ is a solution of* (1) *with a zero of order exactly k at $x = a$ and a zero of order $n - k$ at $x = b$, then there exists a nontrivial solution $v(x)$ of* (3) *having a zero of order k at $x = b$ and a zero of order $n - k$ at $x = a$. Further, if k is maximal in the sense of Theorem* 5, *then $v(x)$ is essentially unique (i.e., unique except for multiplication by a constant factor).*

Proof. The operators L and L^+ in the equations (1) and (3) are connected by Green's identity:

$$(vL_n, u) - (u, L_n^+ v) = \int_a^b (vL_n u - uL_n^+ v)dx = \sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} (-1)^j (vr_i)^{(j)}(x) u^{(i-j-1)}(x) \right) \Bigg|_a^b . \tag{4}$$

Let u be the solution of $L_n y = 0$ whose existence we have assumed; let v be a nontrivial solution of $L_n^+ y = 0$ with a zero of order $n - k - 1$ at $x = a$ and a zero of order k at $x = b$. Then, upon applying Theorem 2, (4) becomes

$$0 = (-1)^{n-k} (vr_n)^{(n-k-1)}(a) u^{(k)}(a) .$$

Since $u^{(k)}(a) \neq 0$ by hypothesis, this implies that $(vr_n)^{n-k-1}(a) = 0$; hence, by the corollary to Theorem 2, $v^{(n-k-1)}(a) = 0$, and $v(x)$ has a zero of order $n - k$ at $x = a$.

To prove the second part of the theorem suppose k is maximal and $v(x)$ is not unique; i.e., suppose that there are two nontrivial solutions $v_1(x)$ and $v_2(x)$ of (3) with zeros of order $n - k$ at $x = a$, and zeros of order k at $x = b$. Either $v_1(x)$, $v_2(x)$ or a linear combination of $v_1(x)$ and $v_2(x)$ has a zero of order $k + 1$ at $x = b$ and a zero of order $n - k$ at a. Let $v(x)$ be the nontrivial solution of (3) with a zero of order at least $n - k$ at $x = a$ and a maximal order zero, at $x = b$. Interchanging the roles of u and v in the previous argument, we find that there is a solution $u(x)$ with a zero of order $k + 1$ at $x = a$ and a zero of order $n - k - p$ at $x = b$, where $p > 0$; this contradicts the maximality of k.

We now derive an eigenfunction relationship between equations (1) and (3).

THEOREM 11. *Consider the adjoint n^{th} order linear differential operators L_n and L_n^+ and the adjoint problems*

$$Ly = \lambda y\,, \qquad Uy = 0\,, \tag{5}$$

$$L^+y = \lambda y\,, \qquad U^+y = 0\,, \tag{6}$$

where U and U^+ are self-adjoint or regular adjoint linear homogeneous boundary operators and λ is a parameter. Suppose further that Green's function for (5) *has only simple poles. Let the eigenfunctions of* (5) *and* (6) *be $\phi_i(x)$ and $\phi_i^+(x)$ respectively, normalized so that $(\phi_i, \phi_i^+) = 1$. Let $f(x)$ and $g(x)$ be any two functions such that L_ng exists, $Ug = 0$, L_n^+f exists, and $U^+f = 0$. Further let $a_i = (f, \phi_i)$ and $b_i = (g, \phi_i^+)$ and let λ_i be the ith eigenvalue of* (5). *Then*

$$(f, L_ng) = \sum_{i=0}^{\infty} a_ib_i\lambda_i\,.$$

Proof. We freely use the well-known properties of problems (5), (6) (see for instance Ince [5] Chapter 11, and Coddington and Levenson [6] Chapter 12).

$$\begin{aligned}(f, L_ng) &= \Big(\sum_{i=0}^{\infty} a_i\phi_i^+, L_ng\Big) = \sum_{i=0}^{\infty} a_i(\phi_i^+, L_ng)\\ &= \sum_{i=0}^{\infty} a_i(L_n^+\phi_i^+, g) = \sum_{i=0}^{\infty} a_i\Big(\lambda_i\phi_i^+, \sum_{j=0}^{\infty} b_j\phi_j\Big)\\ &= \sum_{i=0}^{\infty} a_i\lambda_i \sum_{j=0}^{\infty} b_j(\phi_i^+, \phi_j)\\ &= \sum_{i=0}^{\infty} a_i\lambda_i \sum_{j=0}^{\infty} b_j\delta_{ij} = \sum_{i=0}^{\infty} a_ib_i\lambda_i\,.\end{aligned}$$

If we specialize this result to the self-adjoint case, letting $f = g$ in (7), we obtain

$$(f, L_nf) = \sum_{i=0}^{\infty} a_i^2\lambda_i\,. \tag{8}$$

Further suppose $L_ny = 0$ assumes the special form

$$L_ny = [r_n(x)y^{(n)}]^{(n)} + \sum_{i=0}^{n-1} [(-1)^{n+i+1}r_i(x)y^{(i)}]^{(i)} = 0 \tag{9}$$

where $r_i(x) > 0$ $(i = 0, 1, \cdots, n)$.

We now state a disconjugacy criterion due to Reid [13].

THEOREM (Reid [13]). *Equation* (9) *is disconjugate in the sense of Reid (i.e., no solution possesses two nth order zeros) on the interval $[a, \infty)$ if and only if for all values b on (a, ∞) and all functions*

$y(x)$ such that $y(x) \in C_{n-1}[a, b]$, $y(x)$ is absolutely continuous on $[a, b]$, $(y^{(n)})^2$ is integrable on $[a, b]$ and $y(x)$ has n^{th} order zeros at $x = a$ and $x = b$, $(y, L_n y) > 0$ (where $L_n y$ is as in equation (9)).

Using this result we prove the following theorem. (Our impression is that Theorem 12 is in the literature, but the author has been unable to find it).

THEOREM 12. *Equation (9) is disconjugate in the sense of Reid if and only if the minimum eigenvalue λ_0 is positive (all functions are assumed to be of the class described in the above theorem of Reid).*

Proof. Suppose (9) is disconjugate. Letting $f(x) = \phi_0(x)$ in (8) and applying the theorem of Reid, we have $0 < (\phi_0, L_n\phi_0) = \lambda_0$, since $(\phi_i, \phi_j) = \delta_{ij}$. The sufficiency is immediate from equation (8).

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