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# SOME RESULTS IN THE LOCATION OF ZEROS OF POLYNOMIALS

ZALMAN RUBINSTEIN

**Three out of the four theorems proved in this paper deal with the location of the zeros of a polynomial $P(z)$ whose zeros $z_i, i = 1, 2, \cdots, n$ satisfy the conditions $|z_i| \leqq 1$, and $\sum_{i=1}^{n} z_i^p = 0$ for $p = 1, 2, \cdots, l$. One of those estimates is**

$$\left|\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z}\right| < \frac{l+1}{|z|(|z|^{l+1} - 1)}$$

**for $|z| > 1$.**

**The fourth result is of a different nature. It refines, in particular, a theorem due to Eneström and Kakeya. It is shown that no zero of the polynomial $h(z) = \sum_{k=0}^{n} b_k z^k$ lies in the disk**

$$\left|z - \frac{\beta e^{-i\theta}}{(\beta + 1)}\right| < \frac{1}{\beta + 1},$$

**where $\beta = \max_{|z|=1} |h'(z)|/\max_{|z|=1} |h(z)|$, and $\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$.**

We generalize and strengthen certain well-known results due to Biernacki [1], Dieudonné [3, 5], and Kakeya [8].

We use repeatedly a recent result due to Walsh which is a generalized form of an earlier theorem of his [10]. It concerns the case in which all the zeros of a polynomial lie within a certain distance of their centroid.

THEOREM 1. *Let $h(z) = \sum_{k=0}^{n} b_k z^k (b_k$ complex$)$,*

$$\beta = \frac{\max\limits_{|z|=1} |h'(z)|}{\max\limits_{|z|=1} |h(z)|},$$

*$\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$, and let $C_\beta$ be the disc $|z - \beta e^{-i\theta}/(\beta + 1)| < 1/(\beta + 1)$, then no zero of $h$ lies in $C_\beta$.*

*Proof.* Consider the function $F(z) = e^{-i\varphi} h(ze^{i\theta})/m$, where $h(e^{i\theta}) = me^{i\varphi}$. Then $F$ satisfies the conditions, $|F(z)| < 1$ in $|z| < 1, F(1) = 1$. Let $x_n \to 1$ as $n \to \infty$, $0 < x_n < 1$, and let $\alpha = \lim_{n\to\infty} [(1 - |F(x_n)|)/(1 - x_n)]$. Then $\alpha \leqq |F'(1)|$. It follows readily (see [2] p. 57) that

$$\lim_{n\to\infty} [(1 - |F(x_n)|)/(1 - x_n)] = F'(1) = e^{i(\theta-\varphi)} h'(e^{i\theta})/m = |h'(e^{i\theta})|/m .$$

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We apply now the following result due to Julia [2]: If a function $f$ is regular in the unit disc and $|f(z)| < 1$ for $|z| < 1$, and there exists a sequence of number $z_1, \cdots, z_n, \cdots$ such that $\lim_{n\to\infty} z_n = 1$, $\lim_{n\to\infty} f(z_n) = 1$, $\lim_{n\to\infty} [(1 - |f(z_n)|)/(1 - |z_n|)] = \alpha$ then

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leqq \alpha \frac{|1 - z|^2}{1 - |z|^2} \qquad \text{for } |z| < 1 . \tag{1}$$

In (1), set $f(z) = F(z)$, $\alpha = |h'(e^{i\theta})|/m$. If $F(z_0) = 0$ and $|z_0| < 1$, then $(1 - |z_0|^2)/|1 - z_0|^2 \leqq \alpha$, which is equivalent to $e^{-i\theta} z_0 \notin C_\alpha$. Since $\alpha \leqq \beta$, it follows that $C_\beta \subset C_\alpha$; hence $e^{-i\theta} z_0 \notin C_\beta$, which concludes the proof.

COROLLARY 1. *Let* $h(z) = \sum_{k=0}^n b_k z^k, b_k > 0$. *Then* $\beta = \sum_{k=1}^n k b_k / \sum_{k=0}^n b_k$, *and no zero is in the disc*

$$\left| z - \frac{\sum_{k=0}^{n} k b_k}{\sum_{k=0}^{n} (k+1) b_k} \right| < \frac{\sum_{k=0}^{n} b_k}{\sum_{k=0}^{n} (k+1) b_k} .$$

*In particular, if* $b_k$ *is a strictly increasing sequence, then all the zeros of* $h(z)$ *lie in the complement of* $C_\beta$ *with respect to the unit disc. This makes more precise the theorem of Eneström and Kakeya* [8].

In a recent paper, Tchakaloff [9] (see also [7]) has proved that if all the zeros of the polynomials

$$P_k(z) = a_n^{(k)} z^n + \cdots + a_0^{(k)} (a_n^{(k)} > 0, k = 1, \cdots, m) \tag{2}$$

lie in the unit disc and if $A_k > 0 (k = 1, \cdots, m)$, then all the zeros of the polynomial $\sum_{k=1}^m A_k P_k(z)$ lie in the disc $|z| \leqq 1/\sin(\pi/2n)$, and that this is the best possible result. We prove a more precise result in the case where there is more information about the zeros of $P_k(z)$.

THEOREM 2. *Let the polynomials* $P_k(z) (k = 1, \cdots, m)$ *of the form* (2) *have all their zeros* $z_{ik} (i = 1, \cdots, n; k = 1, \cdots, m)$ *in the unit disc and let* $A_k > 0 (k = 1, \cdots, m)$. *Suppose that* $\sum_{i=1}^n z_{ik}^p = 0$ *for* $p = 1, \cdots, l (k = 1, \cdots, m)$. *Then all the zeros of the polynomial* $\sum_{k=1}^m A_k P_k(z)$ *lie in the disc* $|z| \leqq (\sin \pi/2n)^{-1/(l+1)}$. *For values of the form* $n = (l+1)r$, *the exact bound does not exceed* $(\sin(\pi(l+1))/2n))^{-1/(l+1)}$.

*Proof*. Without loss of generality we may assume that $a_n^{(k)} = 1$. By a recent result due to Walsh [11] the polynomials $P_k$ satisfy the equality $P_k(z) = (z - \varphi_k(z))^n$, where $|\varphi_k(z)| < |z|^{-l}$ for $|z| > 1$. Let $\zeta$ be a point outside the unit disc at which the circle $|z| = |\zeta|^{-l}$

subtends an angle $\Psi$. On the circle $|z| = |\zeta|^{-l}$ there exists a point $a$, such that $0 \leqq \arg((\zeta - \varphi_k)/(\zeta - a)) \leqq \Psi$, and

$$\sum_{k=1}^{m} A_k P_k(\zeta) = (\zeta - a)^n \sum_{k=1}^{m} A_k \left(\frac{\zeta - \varphi_k}{\zeta - a}\right)^n . \tag{3}$$

One deduces from equation (3) that

$$\sum_{k=1}^{m} A_k P_k(\zeta) \neq 0 \text{ if } \Psi < \frac{\pi}{n} .$$

For $\Psi = \pi/n$, $\sin(\pi/2n) = |\zeta|^{-(l+1)}$. This proves the first part of the theorem. The example $A_1 = A_2 = 1$, $m = 2$, $P_1(z) = (z^{l+1} + \mu)^r$, $P_2(z) = (z^{l+1} + \bar{\mu})^r$, where $\mu = i \exp(i\pi/2n)$, proves the second part of the theorem, since in this case the polynomial $P_1(z) + P_2(z)$ has the zero

$$z = \left[\sin \frac{\pi(l+1)}{2n}\right]^{-1/(l+1)} .$$

Dieudonné has proved [3], (for a different proof see [4]), that if the polynomial $P$ has all its zeros in the closed unit disc, then

$$\left|\frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)}\right| \leqq \frac{1}{|z| - 1} , \qquad \text{for } |z| > 1 . \tag{4}$$

We give a short proof of (4), which at the same time yields a stronger inequality in the case where the centroid of the zeros of $P$ is at the origin.

THEOREM 3. *If all the zeros* $z_i (i = 1, \cdots, n)$ *of the polynomial* $P(z)$ *lie in the closed unit disc and if* $\sum_{i=1}^{n} z_i^k = 0 (k = 1, \cdots, l)$, *then for* $|z| > 1$ *the following sharp estimate holds*

$$\left|\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z}\right| \leqq \frac{l+1}{|z|(|z|^{l+1} - 1)} . \tag{5}$$

*Inequality* (5) *holds also for* $l = 0$, *in which case the second condition imposed on the* $z_i$ *is to be omitted.*

*Proof.* By a recent result due to Walsh [12], there exists a function $\varphi(z)$, $|\varphi(z)| < |z|^{-l}$, such that for $|z| > 1$

$$\frac{P'(z)}{P(z)} = \frac{n}{z - \varphi(z)} . \tag{6}$$

An estimate due to Goluzin [6], applied to $\varphi$ yields the inequality

$$|\varphi'(z)| \leqq \frac{l|z|^{l-1}}{|z|^{2l} - 1}(1 - |\varphi(z)|^2) , \tag{7}$$

for $|z| > 1$. Since by (6)

$$\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} = \frac{\varphi(z) - z\varphi'(z)}{z(z - \varphi(z))} \tag{8}$$

is follows, using (7), that

$$\left|\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z}\right| \leqq \frac{1}{|z|}\left[\frac{|\varphi(z)|}{|z| - |\varphi(z)|} + \frac{l|z|^l}{|z|^{2l} - 1}\frac{1 - |\varphi(z)|^2}{|z| - |\varphi(z)|}\right]$$

It remains to prove the inequality

$$\frac{x}{a - x} + \frac{la^l}{a^{2l} - 1}\frac{1 - x^2}{a - x} \leqq \frac{l + 1}{a^{l+1} - 1} \tag{9}$$

for all $0 \leqq x \leqq a^{-l}$, and $a > 1$.

If we denote the left hand side of (9) by $f(x)$, then $f(a^{-l}) = (l + 1)/(a^{l+1} - 1)$, and $f'(x) \geqq 0$ provided the function $g(x) = a^{2l+1} - a + la^l(x^2 - 2ax + 1)$ is nonnegative. Since $g'(x) \leqq 0$ it is enough to show that $h(a) = g(a^{-l})$ is nonnegative. Indeed one verifies that $h(1) = 0$ and $h'(a) > 0$ for all $a > 1$.

The particular case $P(z) = z^n - 1$, $l = n - 1$, shows that the bound (5) cannot, in general, be improved.

The result due to Dieudonné follows from (7) and (8).

Finally, we discuss a problem raised by Biernacki [1], which was also treated by Dieudonné [5], namely that of determining a region containing all but, possibly, one zero of the polynomial $aP(z) + P'(z)$ for all complex $a$. Each of the above authors has proved that if all the zeros of $P$ lie in the unit disc, then the concentric disc of radius $2^{1/2}$ is the smallest concentric disc that has the above mentioned proporty. Assuming additional information about the zeros of $P$, we obtain a smaller disc for all but possibly $l + 1$ zeros of the polynomial $z^lP(z) + aP'(z)$.

THEOREM 4. *If all the zeros $z_i (i = 1, \cdots, n)$ of the polynomial $P(z)$ lie in the closed unit disc and if $\sum_{i=1}^n z_i^k = 0 (k = 1, \cdots, l)$, then for all complex $a$ at least $n - 1$ zeros of the polynomial $z^lP(z) + aP'(z)$ lie in the disc $|z| \leqq 2^{1/(2(l+1))}$.*

*Proof.* Proceeding as in the proof of Theorem 3, we have

$$\frac{P'(z)}{P(z)} = -\frac{z^l}{a} = \frac{n}{z - \varphi(z)},$$

satisfied by any zero of the polynomial $z^l P + aP'$ which exceeds 1 in modulus. Set $g(z) = z^{-l}\varphi(1/z)$, $w = z^{l+1}$ and $h(w) = g(z)$. Then $|g(z)| < 1$ if $|z| < 1$ and

$$g(z) = \frac{1}{z^{l+1}} + an \tag{10}$$

$$h(w) = \frac{1}{w} + an\ . \tag{11}$$

If for some $a$ the polynomial $z^l P + aP'$ has at most $n - 2$ zeros in the disc $|z| \leqq 2^{1/(2(l+1))}$, then equation (10) has at least $l + 2$ roots in the disc $|z| < 2^{-1/(2(l+1))}$, and hence equation (11) has at least two roots in the disc $|w| < 2^{-1/2}$. This was proved to be impossible in [5]

Theorem 4 is sharp for all $l$ and $n$ of the form $n = 2k(l + 1)$, $k = 1, 2, \cdots$. The upper limit is attained by the zeros of the polynomial

$$P(z) = (z^{2l+2} - 2^{1/2}z^{l+1} + 1)^{n/(2(l+1))}\ .$$

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HARVARD UNIVERSITY