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ON THE STABILITY OF THE SET OF EXPONENTS OF A CAUCHY EXPONENTIAL SERIES

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If $f \in L(-D, D)$ and $Q(z)$ is a meromorphic function whose poles, all simple, forms a sub-set of the set $\{\lambda_\nu\}$ ($\nu = 0, \pm 1, \pm 2, \cdots$), then the C.E.S. (Cauchy exponential series) of f with respect to $Q(z)$ is $\Sigma c_\nu e^{\lambda_\nu x}$, where

$$c_\nu e^{\lambda_\nu x} = \operatorname*{res}_{\lambda_\nu} Q(z)\int_{-D}^{D} f(t)e^{z(x-t)}dt\ .$$

Suppose we are given a class A of functions f each of which can be 'represented' in $(-D, D)$ by its C.E.S. with respect to $Q(z)$. We define a set of neighbourhoods U of $\{\lambda_\nu\}$. Then $\{\lambda_\nu\}$ is *stable* if there is a U such that to each $\{\kappa_\nu\} \in U$ there corresponds a meromorphic function $q(z)$ whose poles, all simple, form a sub-set of $\{\kappa_\nu\}$ and which is such that each $f \in A$ can be represented in $(-D, D)$ by its C.E.S. with respect to $q(z)$; and $\{\lambda_\nu\}$ is *unstable* if there is no such neighbourhood.

The case in which $\lambda_\nu = i\nu$, A is $BV[-D, D]$, 'representation of f in $(-D, D)$' means '$\sum_{|\nu| \leq n} c_\nu e^{\lambda_\nu x} \to 1/2\,(f(x+) + f(x-))$ boundedly within (D, D)' is considered. It is shown, in particular, that with reasonable conditions on the set of neighbourhoods U, $\{i\nu\}$ is *unstable* if $D > 1/2\,\pi$, and *stable* if $D = 1/2\,\pi$.

Let $D > 0$ and $f \in L(-D, D)$. Let $Q(z)$ be a meromorphic function whose poles, all simple, form a sub-set of the set $\{\lambda_\nu\}(\nu = 0, \pm 1, \cdots)$. Here, and in what follows, the use of the symbol $\{\lambda_\nu\}$ implies that $\lambda_\nu \neq \lambda_{\nu'}$ if $\nu \neq \nu'$. The C. E. S. (Cauchy exponential series) of f with respect to Q is $\sum c_\nu e^{\lambda_\nu x}$ where

$$c_\nu e^{\lambda_\nu x} = \operatorname*{res}_{\lambda_\nu} Q(z)\int_{-D}^{D} f(t)e^{z(x-t)}dt\ .$$

Suppose that the set $\{\lambda_\nu\}$ is such that, for a class A of functions f, the C.E.S. of f 'represents' f in $(-D, D)$. Then we may consider the question of the stability of the set $\{\lambda_\nu\}$. We define, in some way, a set of neighbourhoods U of $\{\lambda_\nu\}$. Then $\{\lambda_\nu\}$ is *stable* if there is a neighbourhood U such that to each $\{\kappa_\nu\} \in U$, there corresponds a meromorphic function $q(z)$ whose poles, all simple, form a sub-set of $\{\kappa_\nu\}$, and which is such that each $f \in A$ can be represented in $(-D, D)$ by its C.E.S. with respect to $q(z)$; and $\{\lambda_\nu\}$ is *unstable* if there is no such neighbourhood. The stability of $\{\lambda_\nu\}$ depends on the value of D, the class A, the, particular meaning we give to the 'representation' of f,

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and finally on the definition of the set of neighbourhoods U. In this note, we confine our attention to the simplest case: $\lambda_\nu = i\nu$, A is the class of functions f which are $BV[-D, D]$ and satisfy $2f(x) = f(x+) + f(x-)$ in $(-D, D)$, 'representation' of f means 'bounded convergence to $f(x)$ within $(-D, D)$', i.e., for each δ satisfying $0 < \delta < D$, $\sum_{|\nu| \leqq n} c_\nu e^{\lambda_\nu x} \to f(x)$ boundedly in the segment $|x| \leqq D - \delta$. We recall that if $D = \pi$, then each $f \in A$ can be represented by its C.E.S. with respect to $Q_0(z) = 1/2 \coth \pi z$, since, in this case, the C.E.S. is the Fourier series of f. Let us suppose that to each neighbourhood U there corresponds an $\varepsilon > 0$ such that $\{\mu_\nu\} \in U$ if $\sum |\mu_\nu - \lambda_\nu| < \varepsilon$; and to each $\delta > 0$ there corresponds a neighbourhood U_δ such that if $\{\mu_\nu\} \in U_\delta$ then $\sup |\mu_\nu - \lambda_\nu| < \delta$. What we prove, implies that $\{i\nu\}$ *is unstable if* $D > \pi/2$, *and stable if* $D = \pi/2$. We shall, however, prove more than this, viz.

THEOREM 1. *Let* $\{l_\nu\}$ *be a real set not containing every integer, such that* l_ν *is an integer for* $|\nu| \geqq N$. *If* $D > \pi/2$, *then there is no meromorphic function* $q(z)$ *whose poles, all simple, form a sub-set of* $\{il_\nu\}$ *and which is such that each* $f \in A$ *can be represented by its C.E.S. with respect to* q.

THEOREM 2. *Let* $l_\nu = \nu + \alpha_\nu + i\beta_\nu$ *where* α_ν, β_ν *are real numbers which satisfy*

$$\overline{\lim_{|\nu| \to \infty}} |\alpha_\nu| < \frac{1}{8}, \qquad \overline{\lim_{|\nu| \to \infty}} |\beta_\nu| < \infty .$$

If $D = \pi/2$, *there exists a meromorphic function* $q(z)$ *whose poles, all simple, form a sub-set of* $\{il_\nu\}$ *and which is such that each* $f \in A$ *can be represented by its C.E.S. with respect to* q.

THEOREM. 3. *The conclusion of Theorem* 2 *holds if the condition on* α_ν *is replaced by* $\sup |\alpha_\nu| < 1/4$.

The relation between Theorem 2 and the work of Korous [1] is explained in § 6. The relation between Theorem 3 and the work of Levinson [2] is explained in § 7.

2. Let $0 < D \leqq \pi$, and let A have the meaning specified in § 1.

LEMMA 1. *If* $H_n(t) \in L(-2D, 2D)$ *for* $n \geqq n_0$, *then, in order that for each* $f \in A$,

$$\int_{-D}^{D} f(t) H_n(t - x) dt \to f(x)$$

boundedly within $(-D, D)$, *it is necessary and sufficient that*

$$\int_0^t H_n(u)du \rightarrow \frac{1}{2} \operatorname{sgn} t$$

boundedly within $(-2D, 2D)$.

Proof. Let

$$J_n(u) = \frac{1}{2\pi} \frac{\sin\left(n + \frac{1}{2}\right)u}{\sin \frac{1}{2} u} .$$

Then for each $f \in A$,

$$\int_{-D}^{D} f(t)J_n(t - x)dt \rightarrow f(x)$$

boundedly within $(-D, D)$, and

$$\int_0^t J_n(u)du \rightarrow \frac{1}{2} \operatorname{sgn} t$$

boundedly within $(-2D, 2D)$. Let $K_n(u) = H_n(u) - J_n(u)$. It suffices to prove: in order that for each $f \in A$,

$$\int_{-D}^{D} f(t)K_n(t - x)dt \rightarrow 0$$

boundedly within $(-D, D)$, it is necessary and sufficient that

$$k_n(t) = \int_0^t K_n(u)du \rightarrow 0$$

boundedly within $(-2D, 2D)$.

Sufficiency. We have

$$(1) \quad \int_{-D}^{D} f(t)K_n(t - x)dt = f(D)k_n(D - x) - f(-D)k_n(-D - x) - \int_{-D}^{D} k_n(t - x)df(t)$$

and the second member tends to zero boundedly within $(-D, D)$.

Necessity. In the first place, it is necessary that for each $\tau \in (-2D, 2D)$, $k_n(\tau) \rightarrow 0$ as $n \rightarrow \infty$. For let $\alpha, \beta \in (-D, D)$ and let $x = \alpha$. Let $f(t) = 1$ in the open interval, and let $f(t) = 0$ outside the closed interval, whose end points are α, β. Then

$$k_n(\beta - \alpha) = \int_\alpha^\beta K_n(t-\alpha)dt \to 0 .$$

Since α, β can be chosen so that $\beta - \alpha$ has any assigned value in $(-2D, 2D)$, this proves our assertion.

By (1), for each $x \in (-D, D)$, the functions $k_n(t-x)$ of t, for $n \geqq n_0$, form a sequence of elements of $C[-D, D]$ such that

$$\int_{-D}^{D} k_n(t-x)df(t)$$

is convergent for each $f \in A$. By the principle of uniform boundedness, it follows that

$$\sup_{t \in [-D, D]} |k_n(t-x)| < \infty .$$

Choose $x = D - \delta$. Then $k_n(t)$ is uniformly bounded in $[-2D+\delta, \delta]$. Choose $x = -D + \delta$. Then $k_n(t)$ is uniformly bounded in $[-\delta, 2D-\delta]$. Hence $k_n(t)$ is uniformly bounded within $(-2D, 2D)$ as required.

3. *Proof of Theorem* 1. We may suppose that $D \leqq \pi$. Let ω be chosen to satisfy $\pi < \omega < 2D$. We choose the notation so that if $0 \in \{l_\nu\}$ then $0 = l_0$. If a meromorphic function $q(z)$, with the properties mentioned in the enunciation, exists, let C_n denote a contour which contains in its interior precisely those il_ν for which $|\nu| \leqq n$, and which does not pass through any of the il_ν. Let

$$H_n(u) = \frac{1}{2\pi i}\int_{C_n} q(z)e^{-zu}dz . \tag{2}$$

If $\sum c_\nu e^{il_\nu x}$ is the C.E.S. of f with respect to $q(z)$, then

$$\begin{aligned}\sum_{|\nu|\leqq n} c_\nu e^{il_\nu x} &= \sum_{|\nu|\leqq n} \operatorname*{res}_{il_\nu} q(z)\int_{-D}^{D} f(t)e^{z(x-t)}dt \\ &= \int_{-D}^{D} f(t)H_n(t-x)dt .\end{aligned} \tag{3}$$

We have

$$\begin{aligned}\int_0^x H_n(u)du &= \frac{1}{2\pi i}\int_{C_n} q(z)\frac{1-e^{-zx}}{z}dz \\ &= \sum_{|\nu|\leqq n} \frac{r_\nu}{il_\nu}(1-e^{-il_\nu x})\end{aligned} \tag{4}$$

where r_ν is the residue of $q(z)$ at il_ν and where, if $l_0 = 0$, we use the convention

$$\frac{1-e^{-il_0 t}}{il_0} = \lim_{l\to 0}\frac{1-e^{-ilt}}{il} = t . \tag{5}$$

By Lemma 1, it is necessary that

$$\sum_{|\nu|\leqq n} \frac{r_\nu}{il_\nu}(1 - e^{-il_\nu x}) \longrightarrow \frac{1}{2}\operatorname{sgn} x \tag{6}$$

boundedly within $(-2D, 2D)$, and hence in $[-\omega, \omega]$. Let $x \in (-\omega, \omega - 2\pi)$. Then for $|\nu| \geqq N$, the terms on the left are unaltered on replacing x by $x + 2\pi$. By subtraction, it follows that

$$\sum_{|\nu|<N} \frac{r_\nu}{il_\nu} e^{-il_\nu x}(e^{-il_\nu 2\pi} - 1) = -1 \tag{7}$$

for such x, and hence for all x. We note that if $l_0 = 0$, the term with $\nu = 0$ is $-r_0 2\pi$. At this point, we distinguish to cases, (a) $l_0 \neq 0$, (b) $l_0 = 0$.

In case (a), we integrate (7) over $(-X, X)$, divide by $2X$, and let $X \longrightarrow \infty$. We obtain a contradiction. In case (b), we take mean values as in case (a), and deduce that the term with $\nu = 0$ is -1. Then (7) implies that

$$\sum_{0<|\nu|<N} \frac{r_\nu}{il_\nu} e^{-il_\nu x}(e^{-il_\nu 2\pi} - 1) = 0$$

for all x. If we multiply this by its conjugate, and take mean values, we deduce that

$$\sum_{0<|\nu|<N} \frac{|r_\nu|^2}{l_\nu^2} \sin^2 \pi l_\nu = 0 \,. \tag{8}$$

By (6),

$$\sum_{0<|\nu|\leqq n} \frac{r_\nu}{il_\nu}(1 - e^{-il_\nu x}) \longrightarrow \frac{1}{2}\sin x - \frac{x}{2\pi}$$

boundedly within $(-2D, 2D)$. Considering odd parts, its follows that

$$\sum_{0<|\nu|\leqq n} \frac{r_\nu}{l_\nu} \sin l_\nu x \longrightarrow \frac{1}{2}\operatorname{sgn} x - \frac{x}{2\pi} \tag{9}$$

boundedly within $(-2D, 2D)$. By hypothesis, there is an integer μ say, which is not one of the l_ν; and $\mu \neq 0$ since $l_0 = 0$. By (8), $r_\nu = 0$ if l_ν is not an integer. Hence, on multiplying both sides of (9) by $\mu \sin \mu x$ and integrating over $(-\pi, \pi)$, we obtain $0 = 1$, a contradiction.

4. *Proof of Theorem* 2. For all sufficiently large n, the circle $\Gamma_n : |z| = n + 1/2$, contains in its interior the points il_ν for $|\nu| \leqq n$, and every point on Γ_n is at a distance greater than 3/8 from all the points il_ν. Let $q(z)$ be a meromorphic function whose poles, all simple,

form a sub-set of $\{il_\nu\}$, and define $H_n(u)$ by (2) with C_n replaced by Γ_n. Using the notation of §§ 1, 2, we have

$$J_n(u) = \frac{1}{2\pi i}\int_{\Gamma_n} Q_0(z)e^{-zu}dz ,$$

and therefore, as in § 2, it suffices to prove that we can choose $q(z)$ so that

$$\int_0^x K_n(u)du = \frac{1}{2\pi i}\int_{\Gamma_n}(q(z) - Q_0(z))\frac{1-e^{-zx}}{z}dz \to 0$$

boundedly within $(-\pi, \pi)$.

Write

$$P(z) = (z - il_0)\prod_1^\infty\left(1 - \frac{z}{il_\nu}\right)\left(1 - \frac{z}{il_{-\nu}}\right).$$

In § 5, we shall prove

LEMMA 2. *As* $|z| \to \infty$, $P(z) = o(|z|^{1/2}e^{\pi|rez|})$. *On* Γ_n, $|P(z)|^{-1} = o(n^{1/2}e^{-\pi|rez|})$ *as* $n \to \infty$.

The meromorphic function $Q_0(z)P(z)$ is regular, except possibly at the points $i\nu$, which are at most simple poles of residue $P(i\nu)/2\pi$. By Lemma 2, $P(i\nu) = o(|\nu|^{1/2})$. Hence we can define the meromorphic function

$$R(z) = \frac{1}{2\pi}\left[\frac{P(0)}{z} + \sum' P(i\nu)\left(\frac{1}{z - i\nu} + \frac{1}{i\nu}\right)\right]$$

which has the same principal parts as $Q_0(z)P(z)$. Thus

$$Q_0(z)P(z) = R(z) + S(z)$$

where $S(z)$ is an integral function. We can write $q(z)P(z) = F(z)$, where $F(z)$ is an integral function. Then

$$q(z) - Q_0(z) = \frac{F(z) - S(z) - R(z)}{P(z)} . \tag{10}$$

In § 5, we shall prove

LEMMA 3. *On* Γ_n, $R(z) = o(n^{1/2})$ *as* $n \to \infty$.

We choose $F(z)$ so that the numerator in (10) will not be of a greater order of magnitude than $R(z)$. This means, since F and S are integral functions, that $F = S + c$ where c is a constant. Theorem 2 will follow if we show that

$$I_n(x) = \int_{\Gamma_n} \frac{c - R(z)}{P(z)} \cdot \frac{1 - e^{-zx}}{z} dz$$

tends to zero boundedly within $(-\pi, \pi)$. Write $z = (n + 1/2)e^{i\theta}$. By Lemmas 2 and 3,

$$\frac{c - R(z)}{P(z)} = o(ne^{-n\pi|\cos\theta|}) .$$

If then $|x| \leqq \pi - \delta$, $\delta > 0$, we have

$$I_n(x) = o\left(n\int_0^{2\pi} e^{-n\delta|\cos\theta|} d\theta\right) = o(1) .$$

5. In order to prove Lemmas 2 and 3, it will be convenient to write

$$P(iz) = ip(z),$$

so that

$$p(z) = (z - l_0) \prod_1^\infty \left(1 - \frac{z}{l_\nu}\right)\left(1 - \frac{z}{l_{-\nu}}\right) ,$$

and

$$\begin{aligned} R(iz) &= r(z) \\ &= \frac{1}{2\pi}\left[\frac{p(0)}{z} + \sum{}' p(\nu)\left(\frac{1}{z-\nu} + \frac{1}{\nu}\right)\right] . \end{aligned} \tag{11}$$

We need the following result, which is a special case ($a = 0$) of [3] Theorem 1 (with a change of notation).

LEMMA 4. *Let L, M be positive numbers. Let $s_\nu = \nu + \sigma_\nu + i\tau_\nu$, where σ_ν, τ_ν are real numbers which satisfy $|\sigma_\nu| \leqq L$, $|\tau_\nu| \leqq M$ for all ν. Suppose that there is a $\delta > 0$ such that $|s_\nu| \geqq \delta$ for all ν. Let*

$$\psi(z) = \left(1 - \frac{z}{s_0}\right) \prod_1^\infty \left(1 - \frac{z}{s_\nu}\right)\left(1 - \frac{z}{s_{-\nu}}\right) .$$

Then there is a positive constant C (depending only on L, M, δ) such that,

(i) *for all z, $|\psi(z)| < C(1 + |z|)^{4L} e^{\pi|imz|}$;*
(ii) *if $|z - s_\nu| \geqq \delta$ for all ν, then $|\psi(z)|^{-1} < C(1 + |z|)^{4L} e^{-\pi|imz|}$.*

Proof of Lemma 2. We can find a positive number $L < 1/8$ such that $|\alpha_\nu| \leqq L$ for $|\nu| > N$ say; and a positive number M such that $|\beta_\nu| \leqq M$ for all ν. In Lemma 4, choose $s_\nu = l_\nu$ for $|\nu| > N$; $= \nu$ for

$0 < |\nu| \leqq N$; $= 3/8$ for $\nu = 0$. Then $p(z)/\psi(z)$ tends to a nonzero constant as $|z| \to \infty$. By Lemma 4 (with $\delta = 3/8$), there is a positive constant D such that

(i) $|p(z)| < D|z|^{4L}e^{\pi|imz|}$ if $|z|$ is sufficiently large;

(ii) if z is on Γ_n and n is sufficiently large then $|p(z)|^{-1} < Dn^{4L}e^{-\pi|imz|}$ (the condition $|z - s_\nu| \geqq 3/8$ for all ν being satisfied). Since $P(z) = ip(-iz)$, and $4L < 1/2$, the lemma follows.

Proof of Lemma 3. By (i) above, $p(\nu) = O(|\nu|^{4L})$. By (11), it will suffice to prove that if z is on Γ_n, then

$$\sum{}' \frac{zp(\nu)}{\nu(z-\nu)} = o(n^{1/2}) \, .$$

The left hand side is

$$O\left[\sum_{0<\nu\leqq n} \frac{n\nu^{4L}}{\nu\left(n + \dfrac{1}{2} - \nu\right)} + \sum_{n<\nu\leqq 2n} \frac{n\nu^{4L}}{\nu\left(\nu - n - \dfrac{1}{2}\right)} + \sum_{\nu>2n} n\nu^{4L-2}\right] .$$

The first and second sums are $O(n^{4L}\log n)$. The third sum is $O(n^{4L})$. This proves the lemma.

In Lemma 4, we could replace $4L$ by $2L$, if the σ_ν satisfy the further condition

$$\sum_{|\nu|\leqq n} \frac{\sigma_\nu}{\nu + \dfrac{1}{2}} = O(1) \, .$$

This follows from [3] Theorem 2. Hence, as the preceding proof shows, we can replace 1/8 by 1/4 in Theorem 2 if we add the condition

$$\sum_{|\nu|\leqq n} \frac{\alpha_\nu}{\nu + \dfrac{1}{2}} = O\,(1) \, .$$

6. The function $q(z)$ of § 4 is given by

$$q(z) = \frac{1}{2}\coth \pi z + \frac{c - R(z)}{P(z)} \, .$$

Let

$$\begin{aligned} q_0(z) &= iq(iz) \\ &= \frac{1}{2}\cot \pi z + \frac{c - r(z)}{p(z)} \, . \end{aligned}$$

If $\sum c_\nu e^{il_\nu x}$ is the C.E.S. of f with respect to $q(z)$, then, for all sufficiently large n,

$$\sum_{|\nu|\leqq n} c_\nu e^{il_\nu x} = \frac{1}{2\pi i}\int_{\Gamma_n} q(z)dz\int_{-\pi/2}^{\pi/2} f(t)e^{z(x-t)}dt \tag{12}$$
$$= \frac{1}{2\pi i}\int_{\Gamma_n} q_0(z)dz\int_{-\pi/2}^{\pi/2} f(t)e^{iz(x-t)}dt\ .$$

Suppose now that $\beta_\nu = 0$ for all ν, and that c is real. Then $q_0(z)$ is real for real z, so that $q_0(\bar{z}) = \overline{q_0(z)}$. If

$$r_\nu = \operatorname*{res}_{il_\nu} q(z) = \operatorname*{res}_{l_\nu} q_0(z)\ ,$$

then r_ν is real. Let f be real. Write

$$a_\nu - ib_\nu = c_\nu = r_\nu\int_{-\pi/2}^{\pi/2} f(t)e^{-il_\nu t}dt\ .$$

Equating real parts in (12), we get

$$\sum_{|\nu|\leqq n} a_\nu \cos l_\nu x + b_\nu \sin l_\nu x = \frac{1}{2\pi i}\int_{\Gamma_n} q_0(z)dz\int_{-\pi/2}^{\pi/2} f(t)\cos z(x-t)dt \tag{13}$$

We thus obtain the class of trigonometric series investigated by Korous [1]. Theorem 2 shows, in this special case, not only that (13) converges boundedly to $f(x)$ within $(-\pi/2, \pi/2)$, but also that

$$\sum_{|\nu|\leqq n} a_\nu \sin l_\nu x - b_\nu \cos l_\nu x$$

converges boundedly to zero.

7. We now turn to the proof of Theorem 3. We again suppose that the notation has been chosen so that if $0 \in \{l_\nu\}$, then $0 = l_0$. It will suffice to prove

LEMMA 5. *Under the conditions of Theorem* 3, *there are complex numbers* w_ν *such that*

$$\sum_{|\nu|\leqq n} w_\nu e^{i_\nu x} \longrightarrow \frac{1}{2}\operatorname{sgn} x$$

boundedly within $(-\pi, \pi)$.

For then, by the classical theorem of Mittag-Leffler, there is a meromorphic function $q(z)$ whose poles form a sub-set of $\{il_\nu\}$, the principal part at il_ν being $il_\nu w_\nu/(z - il_\nu)$ if $l_\nu \neq 0$. If $l_0 = 0$, we allow the origin to be a regular point. Defining $H_n(u)$ by (2), we have

$$\int_0^x H_n(u)du = \frac{1}{2\pi i}\int_{\sigma_n} q(z)\frac{1-e^{-zx}}{z}dz$$

$$= \sum_{|\nu|\leqq n} w_\nu(1 - e^{-il_\nu x}) .$$

By Lemma 5,

$$\sum_{|\nu|\leqq n} w_\nu \to 0 , \qquad \sum_{|\nu|\leqq n} w_\nu e^{-il_\nu x} \to -\frac{1}{2}\operatorname{sgn} x$$

boundedly within $(-\pi, \pi)$. Thus, Theorem 3 will follow from Lemma 1.

One way of proving Lemma 5 is to generalize the following theorem of Levinson [2, 48]: *if the real numbers* λ_ν *satisfy* $|\lambda_\nu| \leqq P < 1/4$, *then there are numbers* w_ν *such that*

$$\sum_{|\nu|\leqq n}\left[w_\nu e^{i\lambda_\nu x} - \frac{e^{-i\nu x}}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-i\nu t}dt\right]$$

converges uniformly to zero within $(-\pi, \pi)$ *if* $f \in L^2(-\pi, \pi)$. The generalization consists in showing that we can replace the real λ_ν by $\nu + \alpha_\nu + i\beta_\nu$, where $|\alpha_\nu| \leqq P$ and $\overline{\lim}_{|\nu|\to\infty} |\beta_\nu| < \infty$. However, we only need the result for the function $f(t) = 1/2 \operatorname{sgn} t$. It seems worthwhile to prove this special case, for which the argument of Levinson can be given a rather simple form. This is done in § 9.

8. We need the following deduction from Lemma 4.

LEMMA 6. *Let* $S_\nu = \nu + \sigma_\nu + i\tau_\nu$, *where* σ_ν, τ_ν *are real numbers which satisfy* $|\sigma_\nu| \leqq P$, $|\tau_\nu| \leqq Q$ *for all* ν, *where* $0 < P < 1/4$ *and* $Q > 0$. *Let*

$$\Psi(z) = (z - S_0)\prod_1^\infty \left(1 - \frac{z}{S_\nu}\right)\left(1 - \frac{z}{S_{-\nu}}\right) .$$

Then there is a constant K (*depending only on* P *and* Q) *such that*

$$|\Psi(z)| < K(1 + |z|)^{4P}e^{\pi|imz|} . \tag{14}$$

and there is a constant K_ε (*depending only on* P, Q *and* ε) *such that*

$$|\Psi(g)|^{-1} < K_\varepsilon(1 + |z|^{4P}e^{-\pi|imz|} \tag{15}$$

if $|z - S_\nu| \geqq \varepsilon$ *for all* ν.

Proof. In the following proof, and in § 9, the symbols K, K_ε do not necessarily denote the same constants at each occurrence. In Lemma 4, choose $s_0 = \frac{1}{2}P$, $s_\nu = S_\nu$ for $\nu \neq 0$. For $|\nu| \geqq 1$, we have $|s_\nu| > \frac{3}{4}$. By Lemma 4 (with $\delta = \min(1/2P, 3/4)$),

$$|\psi(z)| < K(1 + |z|)^{4P}e^{\pi|imz|} . \tag{16}$$

Now

$$\Psi(z) = -\frac{P}{2}\left(\frac{z - S_0}{z - s_0}\right)\psi(z) \tag{17}$$

and $|(z - S_0)/(z - s_0)| < K$ for $|z - s_0| \geqq 1/4$. For such z, (14) follows from (16). Finally, $|\Psi(z)| \leqq K$ inside $|z - s_0| \leqq 1/4$ since this is true on the boundary. This proves (14).

Let $|z - S_\nu| \geqq \varepsilon$ for all ν. If $|z - s_0| \geqq \varepsilon$ then

$$|\psi(z)|^{-1} < K_\varepsilon(1 + |z|)^{4P} e^{-\pi|imz|} \tag{18}$$

by Lemma 4, and $|(z - s_0)/(z - S_0)| < K_\varepsilon$ so that (15) follows from (17) and (18). If, however, $|z - s_0| < \varepsilon$, then for small ε the disc $\Delta : |z - s_0| < \varepsilon$ is outside each disc $|z - S_\nu| < \varepsilon$ $(\nu = \pm 1, \pm 2, \cdots)$. If it is outside the disc $\Delta' : |z - S_0| < \varepsilon$, then $(\Psi(z))^{-1}$ is regular in Δ and so $|\Psi(z)|^{-1} \leqq K_\varepsilon$ in Δ since this is true on the boundary. If Δ meets Δ' we apply this argument to the portion of Δ which is outside Δ'.

9. *Proof of Lemma* 5. By the hypothesis (of Theorem 3), there are positive numbers P, Q such that $|\alpha_\nu| \leqq P < 1/4$, $|\beta_\nu| \leqq Q$, for all ν. Let C_n denote the rectangular contour whose vertices are $\pm(n + 1/2) \pm ni$. Let

$$G(z) = (z - l_0)\prod_1^\infty \left(1 - \frac{z}{l_\nu}\right)\left(1 - \frac{z}{l_{-\nu}}\right).$$

We define

$$w_\nu = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G(u)\varphi(u)}{G'(l_\nu)(u - l_\nu)} du$$

where

$$\varphi(u) = \frac{1 - \cos \pi u}{u}.$$

Then

$$\sum_{|\nu| \leqq n} w_\nu e^{il_\nu x} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} G(u)\varphi(u)du \int_{C_n} \frac{e^{i\zeta x}}{G(\zeta)(u - \zeta)} d\zeta - \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \varphi(u)e^{iux}du \int_{C_n} \frac{d_\zeta}{u - \zeta}.$$

The last term is

$$\frac{1}{2\pi i}\int_{-(n+1/2)}^{n+1/2} \varphi(u)e^{iux}du = \frac{1}{2\pi}\int_{-(n+1/2)}^{n+1/2} \frac{1 - \cos \pi u}{u} \sin uxdu \longrightarrow \frac{1}{2} \operatorname{sgn} x$$

boundedly within $(-\pi, \pi)$. Hence it suffices to prove that $I_n(x) \to 0$ boundedly within $(-\pi, \pi)$, where

$$I_n(x) = \int_{-\infty}^{\infty} G(u)\varphi(u)du \int_{\sigma_n} \frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)} d\zeta .$$

Since $G(z)$ is a function $\Psi(z)$, we have by (15), $|G(\zeta)|^{-1} < Kn^{4P}e^{-\pi n}$ on the horizontal sides of C_n. Further,

$$|e^{i\zeta x}| \leqq e^{n|x|}, \quad |u-\zeta|^{-1} < K(1+|u|)^{-1}, \quad |\varphi(u)| < K(1+|u|)^{-1} .$$

Since $|G(u)| < K(1+|u|)^{4P}$ by (14), the contribution to I_n of a horizontal side of C_n does not exceed in absolute value

$$Kn^{1+4P}e^{-n(\pi-|x|)} \int_{-\infty}^{\infty} \frac{du}{(1+|u|)^{2-4P}} ,$$

and tends to zero uniformly within $(-\pi, \pi)$. It remains to consider the contribution to I_n of a vertical side of C_n, say the right side. This contribution is

$$J_n(x) = \int_{-\infty}^{\infty} G(u)\varphi(u)du \int_{-in}^{in} \frac{e^{ix(n+1/2+\zeta)}}{G\left(n+\frac{1}{2}+\zeta\right)\left(u-n-\frac{1}{2}-\zeta\right)} d\zeta$$

$$(19) \qquad = e^{ix(n+1/2)} \int_{-\infty}^{\infty} G\left(u+n+\frac{1}{2}\right)\varphi\left(u+n+\frac{1}{2}\right)du \times \int_{-in}^{in} \frac{e^{ix\zeta}}{G\left(n+\frac{1}{2}+\zeta\right)(u-\zeta)} d\zeta .$$

For all ν, we define $l'_\nu = -n + l_{\nu+n}$. Then

$$\begin{aligned} \frac{G(z)}{G(w)} &= \frac{(z-l_0)}{(w-l_0)} \prod_1^\infty \frac{(z-l_\nu)(z-l_{-\nu})}{(w-l_\nu)(w-l_{-\nu})} \\ &= \frac{z-n-l'_0}{w-n-l'_0} \prod_1^\infty \frac{(z-n-l'_{\nu-n})(z-n-l'_{-\nu-n})}{(w-n-l'_{\nu-n})(w-n-l'_{-\nu-n})} \\ &= \frac{G_n(z-n)}{G_n(w-n)} \end{aligned}$$

where

$$G_n(z) = (z-l'_0) \prod_1^\infty \left(1-\frac{z}{l'_\nu}\right)\left(1-\frac{z}{l'_{-\nu}}\right)$$

and $l'_\nu = \nu + \alpha'_\nu + i\beta'_\nu$, $\alpha'_\nu = \alpha_{\nu+n}$, $\beta'_\nu = \beta_{\nu+n}$. Then $|\alpha'_\nu| \leqq P$, $|\beta'_\nu| \leqq Q$. Hence $G_n(z)$ is a function $\Psi(z)$ (of Lemma 6) and satisfies the inequalities (14), (15) with constants K, K_ε independent of n. In (19), we use the equation

$$\frac{G\left(u+n+\frac{1}{2}\right)}{G\left(\zeta+n+\frac{1}{2}\right)}=\frac{G_n\left(u+\frac{1}{2}\right)}{G_n\left(\zeta+\frac{1}{2}\right)}.$$

It follows that

$$|J_n(x)| \leqq \int_{-\infty}^{\infty}\left|G_n\left(u+\frac{1}{2}\right)\right| \varphi\left(u+n+\frac{1}{2}\right)|J|\,du$$

where

$$J=\int_{\gamma}\frac{e^{ix\zeta}}{G_n\left(\zeta+\frac{1}{2}\right)(u-\zeta)}\,d\zeta$$

and γ denotes the path from $-in$ to in modified by replacing the segment $(-i/8, i/8)$ by the right half or the left half of the circle $|\zeta|=1/8$, according as $u<0$ or $u>0$. On γ, $re(\zeta+1/2)$ is between 3/8 and 5/8, and therefore $\zeta+1/2$ is at a distance greater than 1/8 from all the zeros of $G_n(z)$. By Lemma 6, $|G_n(\zeta+1/2)|^{-1}<Ke^{-\pi|\eta|}(1+|\eta|)$, where $\eta=im\,\zeta$. Further $|u-\zeta|^{-1}<K(1+|u|)^{-1}$, and so

$$\begin{aligned}|J| &< \frac{K}{1+|u|}\int_{-\infty}^{\infty}e^{-|\eta|(\pi-|x|)}(1+|\eta|)d\eta\\ &< \frac{K}{(1+|u|)(\pi-|x|)^2}.\end{aligned}$$

Since $|G_n(u+1/2)|<K(1+|u|)^{4P}$, it remains to prove that $H_n \to 0$ where

$$H_n=\int_{-\infty}^{\infty}\frac{du}{(1+|u|)^d\left(1+\left|u+n+\frac{1}{2}\right|\right)}$$

and $d=1-4P>0$.

If m is a positive integer, then

$$H_n=\int_{|u|\leqq m}+\int_{|u|>m}$$

and the first integral tends to zero as $n\to\infty$. Choose p so that $pd>1$ and let $q^{-1}+p^{-1}=1$. Then

$$\begin{aligned}\int_{|u|>m} &\leqq \left(\int_{|u|>m}\frac{du}{(1+|u|)^{pd}}\right)^{1/p}\left(\frac{du}{\int_{-\infty}^{\infty}\left(1+\left|u+n+\frac{1}{2}\right)^q\right.}\right)^{1/q}\\ &< Km^{1/p-d},\end{aligned}$$

so that $\overline{\lim} H_n = 0$, as required.

Added in proof. A result similar to Theorem 2 was proved in a Ph. D thesis by J. A. Anderson.

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