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A DESCRIPTION OF $\mathrm{MULT}_i(A^1, \cdots, A^n)$ BY GENERATORS AND RELATIONS

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If R is a ring (with unit) and $A^1_R, {}_RA^2_R, \cdots, {}_RA^{n-1}_{R,R}A^n$ are R-(bi)modules, then $\mathrm{Mult}_i^{R,n}(A^1, \cdots, A^n)$ is defined to be the ith left derived functor of the multiple tensor product $A^1 \otimes \cdots \otimes A^n$ $(\otimes = \otimes_R)$; i.e., $H_i(K^1 \otimes \cdots \otimes K^n)$, where each K^r is a projective resolution of A^r.

The purpose of this paper is to give a description of $\mathrm{Mult}_i^{R,n}(A^1, \cdots, A^n)$ in terms of generators and relations, analogous to that given by MacLane in the case $n=2$ [and $\mathrm{Mult}_i = \mathrm{Tor}_i^R(A^1, A^2)$].

Throughout this paper R is a ring with unit, all modules are unitary, and $\otimes$ means $\otimes_R$. If $A^1_R, {}_RA^2_R, \cdots, {}_RA^{n-1}_R, {}_RA^n$ are R-modules (or bimodules, as indicated), then

$$\mathrm{Mult}_i^{R,n}(A^1, \cdots, A^n)$$

is defined to be the ith left derived functor of the multiple tensor product $A^1 \otimes \cdots \otimes A^n$; i.e.

$$H_i(K^1 \otimes \cdots \otimes K^n) ,$$

where each K^r is a projective resolution of A^r. When no confusion can arise we shall often write Mult_i or Mult_i^n in place of $\mathrm{Mult}_i^{R,n}$. Note that for $n = 2$, Mult_i is simply the functor $\mathrm{Tor}_i^R(A^1, A^2)$.

A description of $\mathrm{Mult}_i^{Z,n}$ $(A^1, \cdots, A^n)$ is given in [1]. MacLane [2] has described $\mathrm{Tor}_i^R(A^1, A^2)$ in terms of generators and relations. The purpose of this paper is to extend this description to the functors $\mathrm{Mult}_i^{R,n}(A^1, \cdots, A^n)$. The first difficulty in doing this is to formulate the proper definition of the generators and defining relations. Once this is done, however, most of the proofs are analogous to (though usually considerably more complicated than) the proofs given for $\mathrm{Tor}_i^R(A^1, A^2)$.

A notable exception to this is Theorem 3.1, in which the results for $n = 2$ are used as the first step in an inductive procedure, which is much simpler than a direct proof. Unfortunately, this technique apparently cannot be applied in the proof of the crucial Theorem 3.6, where we must resort to a long and somewhat involved procedure.

Throughout this paper we shall often use the term R-module for left-R-modules, right R-modules, or R-bimodules, the specific meaning being indicated by the context.

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2. Definition and basic properties. For a fixed $i \geqq 0$, we consider chain complexes E of length i

$$E\colon E_0 \xleftarrow{\partial} E_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} E_i\ ,$$

with each E_r a finitely generated free R-(bi)module. The dual $E^* = \operatorname{Hom}_R(E, R)$ can also be regarded as a chain complex of length i.

$$E^*\colon E_i^* \xleftarrow{\partial^*} E^*_{i-1} \xleftarrow{\partial^*} \cdots \xleftarrow{\partial^*} E_0^*\ ,$$

where $\partial^* = \operatorname{Hom}(\partial, 1)$; each E_r^* is also a finitely generated free R-(bi)module. (Note: our definition of the boundary operator in E^* differs by a sign from that given in [2].)

If A is an R-module, it can be considered as a complex (in dimension zero) with trivial boundary operator. If E is a complex as in the previous paragraph, then by a *map* $\mu\colon E \to A$ we mean a chain transformation of complexes, i.e. an R-module homomorphism $\mu\colon E_0 \to A$ such that the composition

$$E_1 \xrightarrow{\partial} E_0 \xrightarrow{\mu} A$$

is zero. If E and F are two complexes as above then $E \otimes F$ and $E^* \otimes F^*$ are chain complexes of length $2i$ of finitely generated free R-bimodules (denote the boundary in these complexes by $\bar{\partial}$ and $\bar{\delta}$ respectively). If A is an R-bimodule, then by a *map* $\mu\colon (E \otimes F)_i \to A$ [or $\mu\colon (E^* \otimes F^*)_i \to A$] is meant a bimodule homomorphism such that the composition

$$(E \otimes F)_{i+1} \xrightarrow{\bar{\partial}} (E \otimes F)_i \xrightarrow{\mu} A$$

[or

$$(E^* \otimes F^*)_{i-1} \xrightarrow{\bar{\delta}} (E^* \otimes F^*)_i \xrightarrow{\mu} A]$$

is zero.

If $A^1_R, {}_RA^2_R, \cdots, {}_RA^{n-1}_R, {}_RA^n$ are R-modules, we shall define a certain group in terms of generators and relations, which (to avoid confusion in the long run) we call $\operatorname{Mult}_i^{R,n}(A^1, \cdots, A^n)$. We shall eventually show that this is precisely the group defined in § 1. But until that time we shall use Mult_i to refer to the group defined below and not to the group defined in § 1.

We take as generators of $\operatorname{Mult}_i^{R,n}(A^1, \cdots, A^n)$ all elements: $\langle \mu(1), E^1, \mu(1, 2), E^2, \mu(2, 3), E^3, \cdots, \mu(n-2, n-1), E^{n-1}, \mu(n)\rangle$, where (for $r = 1, 2, \cdots, n-1$) E^r is a chain complex of length i, with each E_k^r finitely generated free R-module; the μ's are maps,

$u(1)\colon E^1 \to A^1$;

$\mu(r, r+1)\colon (E^r \otimes E^{r+1})_i \to A^{r+1} \qquad (2 \leqq r \leqq n-1,\ r \text{ even})$;

$\mu(r, r+1)\colon (E^{r^*} \otimes E^{r+1^*})_i \to A^{r+1} \quad (1 \leqq r \leqq n-1,\ r \text{ odd})$;
$\mu(n)\colon E^{n-1} \to A^n \quad (n \text{ odd})$;
$\mu(n)\colon E^{n-1^*} \to A^n \quad (n \text{ even})$.

These generators are subject to the following relations. Suppose (for $r = 1, \cdots, n-1$) E^r and $\bar{E}^r$ are chain complexes of length i as above, $\lambda_r\colon E^r \to \bar{E}^r$ is a chain transformation, and there are maps

$$\begin{aligned}
&\bar{\mu}(1)\colon \bar{E}^1 \to A^1 ;\\
&\bar{\mu}(r, r+1)\colon (\bar{E}^r \otimes \bar{E}^{r+1})_i \to A^{r+1} \quad (r \text{ even}) ;\\
&\mu(r, r+1)\colon (E^{r^*} \otimes E^{r+1^*})_i \to A^{r+1} \quad (r \text{ odd}) ;\\
&\bar{\mu}(n)\colon \bar{E}^{n-1} \to A^n \quad (n \text{ odd}) ;\\
&\mu(n)\colon E^{n-1^*} \to A^n \quad (n \text{ even}) .
\end{aligned}$$

Then we require that the following relation hold.

$$\begin{aligned}
(1)\quad &\langle \bar{\mu}(1)\lambda_1, E^1, \mu(1, 2)(1^* \otimes \lambda_2^*), \bar{E}^2, \bar{\mu}(2, 3)(1 \otimes \lambda_3), E^3, \mu(3, 4)\\
&(1^* \otimes \lambda_4^*), \cdots, \mu(n-2, n-1)(1^* \otimes \lambda_{n-1}^*), \bar{E}^{n-1}, \bar{\mu}(n)\rangle\\
&= \langle \bar{\mu}(1), \bar{E}^1, \mu(1, 2)(\lambda_1^* \otimes 1^*), E^2, \bar{\mu}(2, 3)(\lambda_2 \otimes 1), \bar{E}^3, \mu(3, 4)\\
&(\lambda_3^* \otimes 1^*), \cdots, \mu(n-2, n-1)(\lambda_{n-2}^* \otimes 1^*), E^{n-1}, \bar{\mu}(n)\lambda_{n-1}\rangle ;
\end{aligned}$$

(n is assumed odd here; the same relation, with the obvious changes in the last entry holds for even n). Thus two generators of Mult_i are equal, provided one can be obtained from the other by a finite number of applications of the above relation. When no confusion can arise we shall often write generators of Mult_i as $\langle \mu, E^1, \mu, E^2, \cdots\rangle$.

$\mathrm{Mult}_i(A^1, \cdots, A^n)$ is made into an abelian group by definining addition as follows. If $\alpha\colon X \otimes Y \to D$ and $\beta\colon \bar{X} \otimes \bar{Y} \to D$ are R-module homomorphisms, we denote by $\alpha^*\beta$ the map

$$\alpha^*\beta\colon (X \oplus \bar{X}) \otimes (Y \otimes \bar{Y}) \to D ,$$

which is the composition

$$\begin{aligned}
&(X \oplus \bar{X}) \otimes (Y \oplus \bar{Y}) \cong (X \otimes Y) \oplus (\bar{X} \otimes Y) \oplus (X \otimes \bar{Y}) \oplus (\bar{X} \otimes \bar{Y})\\
&\xrightarrow{\pi} (X \otimes Y) \oplus (\bar{X} \otimes \bar{Y}) \xrightarrow{\alpha \oplus \beta} D \oplus D \xrightarrow{\nabla_D} D ,
\end{aligned}$$

where π is the projection onto the two end summands and ∇_D is the usual codiagonal map. This definition is extended in the obvious way to the situation where $X, \bar{X}, Y, \bar{Y}$ are chain complexes of finite length and $\alpha\colon (X \otimes Y)_i \to D$, $\beta\colon (\bar{X} \otimes \bar{Y})_i \to D$. Now define

$$\langle \mu, E^1, \mu, E^2, \cdots, E^{n-1}, \mu\rangle + \langle \bar{\mu}, \bar{E}^1, \bar{\mu}, \bar{E}^2, \cdots, \bar{E}^{n-1}, \bar{\mu}\rangle$$

to be the element

$$\langle \nabla_{A^1}(\bar{\mu} \oplus \mu), E^1 \oplus \bar{E}^1, \mu^*\bar{\mu}, E^2 \oplus \bar{E}^2, \mu^*\bar{\mu},$$
$$\cdots, \mu^*\bar{\mu}, E^{n-1} \oplus \bar{E}^{n-1}, \nabla_{A^n}(\mu \oplus \bar{\mu})\rangle .$$

It is easily verified that this addition respects the defining relation (1).

For R-modules X, Y let $\omega = \omega(X, Y): X \oplus Y \to Y \oplus X$ be the map given by $\omega(x, y) = (y, x)$. Let $\Delta_X: X \to X \oplus X$ and $\nabla_X: X \oplus X \to X$ be the usual diagonal and codiagonal maps. Then the following identities hold.

(2) $\nabla_X = \nabla_X \omega$;

(3) if $\alpha: X \to \bar{X}, \beta: Y \to \bar{Y}$, then
$$\omega(\alpha \oplus \beta) = (\beta \oplus \alpha)\omega: X \oplus Y \to \bar{Y} \oplus \bar{X} ;$$

(4) if $\alpha: X \otimes Y \to D, \beta: \bar{X} \otimes \bar{Y} \to D$, then
$$(\alpha^*\beta)(\omega \otimes 1) = (\beta^*\alpha)(1 \otimes \omega): (\bar{X} \oplus X) \otimes (Y \oplus \bar{Y}) \to D$$

(5) if α, β are as in (4), then
$$\alpha^*\beta(\omega \otimes \omega) = \beta^*\alpha: (\bar{X} \oplus X) \otimes (\bar{Y} \oplus Y) \to D$$

(6) $\omega(X, Y)^* = \omega(X^*, Y^*)$;

(7) if α, β are as in (4) and $\gamma: \tilde{X} \otimes \tilde{Y} \to D$, then
$$(\alpha^*\beta)^*\gamma = \alpha^*(\beta^*\gamma):$$

(8) if $\alpha: X \otimes Y \to D$, then
$$\alpha(\nabla_X \otimes 1) = \alpha^*\alpha(1 \otimes \Delta_Y): (X \oplus X) \otimes Y \to D;$$

(9) $(\Delta_X)^* = \nabla_{X^*}$ and $(\nabla_X)^* = \Delta_{X^*}$;

(10) if $\beta: X \to D$, then
$$\beta\nabla_X = \nabla_D(\beta \oplus \beta): X \oplus X \to D .$$

Using (1)–(6) in a manner analogous to that in [2] one verifies that addition in $\mathrm{Mult}_i(A^1, \cdots, A^n)$ is commutative. Associativity follows from (7) and the associativity of the diagonal and codiagonal maps. The zero element is $\langle 0, 0, \cdots, 0\rangle$, (where the zeros are either zero maps or zero complexes of length i). The inverse of $\langle \mu, E^1, \cdots\rangle$ is $\langle -\mu, E^1, \cdots\rangle$ since $\nabla_{A^1}(\mu \oplus (-\mu)) = 0$.

Using (1) and (8)–(10) one verifies that the generators

$$\langle \mu, E^1, \mu, \cdots, E^{n-1}, \mu\rangle$$

are additive in the μ's; i.e.

$$\langle \mu, E^1, \cdots, \mu(r, r+1), \cdots, E^{n-1}, \mu\rangle$$
$$+ \langle \mu, E^1, \cdots, \bar{\mu}(r, r+1), \cdots, E^{n-1}, \mu\rangle$$
$$= \langle \mu, E^1, \cdots, \mu(r, r+1) + \bar{\mu}(r, r+1), \cdots, E^{n-1}, \mu\rangle$$

Finally if (for $r = 1, 2, \cdots, n$) $\alpha(r): A^r \to \bar{A}^r$ are R-module homomorphisms, $\mathrm{Mult}_i(A^1, \cdots, A^n)$ becomes a covariant functor of n variables to the category of abelian groups by defining

$$\alpha(r)_* \langle \mu, E^1, \cdots, \mu(r-1, r), \cdots, E^{n-1}, \mu \rangle$$
$$= \langle \mu, E^1, \cdots, \alpha(r)\mu(r-1, r), E^{n-1}, \mu \rangle .$$

3. The main theorems.

THEOREM 3.1. *If* $A^1, \cdots, A^n$ *are R-modules, then there is a natural isomorphism*:

$$A^1 \otimes A^2 \otimes \cdots \otimes A^n \cong \mathrm{Mult}_0^{R,n}(A^1, \cdots, A^n) .$$

Proof. Define a map

$$f: A^1 \otimes \cdots \otimes A^n \to \mathrm{Mult}_0(A^1, \cdots, A^n)$$

by $f(a_1 \otimes \cdots \otimes a_n) = \langle \mu(a_1), R, \mu(a_2), \cdots, R, \mu(a_n) \rangle$, where $\mu(a_r): R = R \otimes R [= R^* \otimes R^*] \to A^r$ is given by $\mu(a_r)(1) = a_r$. f respects the defining relations on the generators of the tensor product and hence induces a well defined homomorphism. If $\alpha: A \to \bar{A}$ and $a \in A$, then

$$\alpha \circ \mu(a) = \mu(\alpha a): R \to \bar{A} ;$$

it follows that f is natural in A^r ($r = 1, \cdots, n$).

Next define a map

$$g: \mathrm{Mult}_0(A^1, \cdots, A^n) \to A^1 \otimes \cdots \otimes A^n$$

as follows. If $\langle \mu, E^1, \cdots, E^{n-1}, \mu \rangle$ is a generator of Mult_0, with each E^r finitely generated free, choose a basis $\{{}^r e(i_r) \mid i_r \in I_r\}$ for each E^r. Let ${}^r e^*(i_r)$ be the dual basis for E^{r*}. Then define $g\langle \mu, E^1, \cdots, E^{n-1}, \mu \rangle$ to be the element

$$\sum \mu[{}^1e(i_1)] \otimes \mu[{}^1e^*(i_1) \otimes {}^2e^*(i_2)] \otimes \mu[{}^2e(i_2) \otimes {}^3e(i_3)] \otimes$$
$$\cdots \otimes \mu[{}^{n-2}e^*(i_{n-2}) \otimes {}^{n-2}e^*(i_{n-1})] \otimes \mu[{}^{n-1}e(i_{n-1})] ,$$

where $i_r \in I_r$ and the sum is taken over $I_1 \times \cdots \times I_{n-1}$; ($n$ is assumed odd here; for n even the final terms should be changed in the obvious way). The proof that g is well defined is straight-forward (and analogous to the proof Theorem V. 7.3 of [2]).

It is immediately verified that $gf = 1$ and hence f is an epimorphism. In order to show that f is in fact an isomorphism we need the following two lemmas.

LEMMA 3.2. *If* A^n *is free, then* $f: A^1 \otimes \cdots \otimes A^n \to \mathrm{Mult}_0$ $(A^1, \cdots, A^n)$ *is an isomorphism.*

LEMMA 3.3. *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is a short exact sequence of R-modules, then there is an exact sequence*:

$$\begin{aligned}\mathrm{Mult}_0(A^1, \cdots, A^{n-1}, A) &\xrightarrow{\alpha_*} \mathrm{Mult}_0(A^1, \cdots, A^{n-1}, B)\\ &\xrightarrow{\beta_*} \mathrm{Mult}_0(A^1, \cdots, A^{n-1}, C) \longrightarrow 0\end{aligned}$$

The proofs of these lemmas will be given below. Let F be a free R-module such that

$$0 \longrightarrow K \xrightarrow{\alpha} F \xrightarrow{\beta} A^n \longrightarrow 0 \tag{1}$$

is exact ($K = \ker \beta$). Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} A^1\otimes\cdots\otimes A^{n-1}\otimes K & \to & A^1\otimes\cdots\otimes A^{n-1}\otimes F & \to & A^1\otimes\cdots\otimes A^{n-1}\otimes A^n & \to & 0\\ \downarrow f & & \downarrow f & & \downarrow f & & \\ \mathrm{Mult}_0(A^1, \cdots, A^{n-1}, K) & \to & \mathrm{Mult}_0(A^1, \cdots, F) & \to & \mathrm{Mult}_0(A^1, \cdots, A^n) & \to & 0 ,\end{array}$$

with horizontal maps induced by the sequence (1). Since F is free the middle map f is an isomorphism; since the other maps f are epimorphisms, it follows from the five-lemma that

$$f\colon A^1 \otimes \cdots \otimes A^n \to \mathrm{Mult}_0(A^1, \cdots, A^n)$$

is an isomorphism. Except for the proofs of the lemmas this completes the proof of Theorem 3.1.

Proof of Lemma 3.2. It suffices to assume that A^n is finitely generated and hence that $A^n = R$. Consider the diagram:

$$\begin{array}{ccc} A^1\otimes\cdots\otimes A^{n-1}\otimes R & \xrightarrow{f_n} & \mathrm{Mult}_0^n(A^1, \cdots, A^{n-1}, R)\\ \downarrow \lambda & & \uparrow G\\ A^1\otimes\cdots\otimes A^{n-1} & \xrightarrow{f_{n-1}} & \mathrm{Mult}_0^{n-1}(A^1, \cdots, A^{n-1}) ,\end{array}$$

where λ is the usual isomorphism and G is defined by

$$G\langle \mu, E^1, \cdots, E^{n-2}, \mu\rangle = \langle \mu, E^1, \cdots, E^{n-2}, \mu, R, 1\rangle ,$$

(this makes sense since $E^{n-2} \otimes R$ [or $E^{n-2*} \otimes R^*$] can be identified with E^{n-2} [or E^{n-2*}]). It can easily be verified that G respects the defining relations in Mult_0^{n-1} and hence induces a well defined homomorphism. Define a map

$$H\colon \mathrm{Mult}_0^n(A^1, \cdots, A^{n-1}, R) \to \mathrm{Mult}_0^{n-1}(A^1, \cdots, A^{n-1})$$

by

$$\begin{aligned} &H\langle \mu, E^1, \cdots, E^{n-2}, \mu, E^{n-1}, \nu\rangle\\ &\quad = \langle \mu, E^1, \cdots, E^{n-2}, \mu(1^* \otimes \nu^*)\rangle ;\end{aligned}$$

(this is for n odd; for n even, last entry is $\mu(1 \otimes \nu)$). This makes sense if we consider $\mu(1^* \otimes \nu^*)$ as a map on $E^{n-2} \otimes R^* = E^{n-2*}$ (similarly for n even). It can be verified that H induces a well defined homomorphism and that $HG = 1$ and $GH = 1$; hence G is an isomorphism. Finally one verified that the above diagram is commutative, i.e. $f_n = Gf_{n-1}\lambda$. Since f_{n-1} is known to be an isomorphism for $n = 3$ (cf. [2]) the conclusion of the lemma now follows by induction on n.

Proof of Lemma 3.3. If $\langle \mu, E^1, \cdots, E^{n-1}, \nu \rangle$ is a generator of $\mathrm{Mult}_0(A^1, \cdots, A^{n-1}, C)$, then the fact that E^{n-1}, is free implies that there is a map $\gamma: E^{n-1} \to B$ such that $\beta\gamma = \nu$. Hence

$$\beta_* \langle \mu, E^1, \cdots, E^{n-1}, \gamma \rangle = \langle \mu, E^1, \cdots, E^{n-1}, \nu \rangle$$

and β_* is an epimorphism. The rest of the proof is analogous to the proof of Theorem V. 5.1 of [2] and is omitted here.

PROPOSITION 3.4. If $F^1, \cdots, F^{n-1}, A$ are R-modules and each F^r is finitely generated free, with basis $\{{}^r e(i_r) \mid i_r \in I_r\}$, then every element of $F^1 \otimes F^2 \otimes \cdots \otimes F^{n-1} \otimes A$ can be written uniquely in the form:

$$\sum {}^1e(i_1) \otimes {}^2e(i_2) \otimes \cdots \otimes {}^{n-1}e(i_{n-1}) \otimes a(i_1, i_2, \cdots, i_{n-1}) ,$$

where $a(i_1, \cdots, i_{n-1}) \in A$ and the sum is taken over $I_1 \times \cdots \times I_{n-1}$.

The proof follows from the fact that $R \otimes \cdots \otimes R \otimes A$ is naturally isomorphic to A under the map given by

$$r_1 \otimes \cdots \otimes r_{n-1} \otimes a \to (r_1 \cdots r_{n-1})a .$$

Suppose that $F^1, \cdots, F^{n-1}$ are finitely generated free R-modules, the basis of F^r being $\{{}^r e(i_r) \mid i_r \in I_r\}$. Denote the dual basis of F^*_r by $\{{}^r e^*(i_r)\}$. For r odd let $\bar{F}^r$ be the finitely generated free R-module $F^1 \otimes \cdots \otimes F^r$; it has a basis $\{{}^1e(i_1) \otimes \cdots \otimes {}^r e(i_r)\}$ which we shall denote by $\{{}^r e(i_1, \cdots, i_r)\}$. For r even, let $\bar{F}^r$ be the finitely generated free R-module $F^{1*} \otimes \cdots \otimes F^{r*}$; denote its basis by $\{{}^r e^*(i_1, \cdots, i_r)\}$.

Define maps:

$$\pi(r): \bar{F}^{r-1} \otimes \bar{F}^r \to F^r \qquad (r \text{ odd}, \ r \geqq 3) ;$$
$$\pi(r): \bar{F}^{r-1*} \otimes \bar{F}^{r*} \to F^r \qquad (r \text{ even}, \ r \geqq 0) ;$$

as follows.

$$\pi(r)[{}^{r-1}e(i_1, \cdots, i_{r-1}) \otimes {}^r e(j_1, \cdots, j_r)] = \prod_{k \leqq r-1} \delta(i_k, j_k) {}^r e(j_r) ;$$
$$\pi(r)[{}^{r-1}e^*(i_1, \cdots, i_{r-1}) \otimes {}^r e^*(j_1, \cdots, j_r)] = \prod_{k \leqq r-1} \delta(i_k, j_k)] {}^r e(j_r) ;$$

where $\delta(i, j)$ is the Kronecker delta, and $i_k, j_k \in I_k$.

PROPOSITION 3.5. If $F^1, \cdots, F^{n-1}, A$ are R-modules, with each F^r finitely generated free, then every element of $\mathrm{Mult}_0(F^1, \cdots, F^{n-1}, A)$ can be written uniquely in the form:

$$\langle 1, \bar{F}^1, \pi(2), \bar{F}^2, \cdots, \pi(n-1), \bar{F}^{n-1}, \nu\rangle ,$$

where $\nu\colon \bar{F}^{n-1(*)} \to A$.

Proof. Under the natural isomorphism of Theorem 3.1, the element $\langle 1, F^1, \pi(2), \bar{F}^2, \cdots, \bar{F}^{n-1}, \nu\rangle$ is mapped onto

$$\sum {}^1e(i_1) \otimes {}^2e(i_2) \otimes \cdots \otimes {}^{n-1}e(i_{n-1}) \otimes \nu[{}^{n-1}e(i_1, \cdots, i_{n-1})] ,$$

where the sum is taken over $I_1 \times \cdots \times I_{n-1}$. Hence by Proposition 3.4 the values $\nu[{}^{n-1}e(i_1, \cdots, i_{n-1})]$ are uniquely determined and therefore so is ν. It is also clear from Proposition 3.4 and Theorem 3.1 that every element of Mult_0 can be written in the required form.

We are now in a position to prove the main result, that $\mathrm{Mult}_i^{R,n}(A^1, \cdots, A^n)$ as defined by generators and relations is isomorphic to the ith left derived functor of the functor $A^1 \otimes \cdots, \otimes A^n$. Recall that to define this functor it suffices to take free resolutions of only $n-1$ of the n modules.

THEOREM 3.6. *Let $A^1, \cdots, A^n$ be R-modules and $K^1, \cdots, K^{n-1}$ free resolutions of $A^1, \cdots, A^{n-1}$. Then there is a natural isomorphism (for each i)*

$$F\colon \mathrm{Mult}_i^{R,n}(A^1, \cdots, A^n) \cong H_i(K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n) ,$$

Proof. Let $\langle \mu, E^1, \cdots, E^{n-1}, \mu\rangle$ be a generator of $\mathrm{Mult}_i(A^1, \cdots, A^n)$. By the lifting theorem for chain complexes there exist chain transformations h over the respective identity maps as follows.

$$\begin{array}{ccccccccc}
E_i^1 & \longrightarrow \cdots \longrightarrow & E_1^1 & \longrightarrow & E_0^1 & \xrightarrow{\mu} & A^1 \\
\downarrow{\scriptstyle h(1,i)} & & \downarrow{\scriptstyle h(1,1)} & & \downarrow{\scriptstyle h(1,0)} & & \| \\
K_i^1 & \longrightarrow \cdots \longrightarrow & K_1^1 & \longrightarrow & K_0^1 & \xrightarrow{\varepsilon} & A^1 ;
\end{array}$$

$$\begin{array}{ccccc}
(E^{r*} \otimes E^{r+1*})_0 & \longrightarrow \cdots \text{---} & (E^{r*} \otimes E^{r+1*})_i & \xrightarrow{\mu} & A^{r+1} \\
\downarrow{\scriptstyle h(r+1,i)} & & \downarrow{\scriptstyle h(r+1,0)} & & \| \\
K_i^{r+1} & \longrightarrow \cdots \longrightarrow & K_0^{r+1} & \xrightarrow{\varepsilon} & A^{r+1}
\end{array}$$

for r odd, $r \geqq 1$;

$$\begin{array}{ccccc}
(E^{r} \otimes E^{r+1})_{2i} & \longrightarrow \cdots \longrightarrow & (E^{r} \otimes E^{r+1})_i & \xrightarrow{\mu} & A^{r+1} \\
\downarrow{\scriptstyle h(r+1,i)} & & \downarrow{\scriptstyle h(r+1,0)} & & \| \\
K_i^{r+1} & \longrightarrow \cdots \longrightarrow & K_0^{r+1} & \xrightarrow{\varepsilon} & A^{r+1}
\end{array}$$

for r even, $r \geqq 2$. Note that

$$h(1, p)\colon E^1_p \to K^1_p\ ;$$
$$h(r+1, p)\colon (E^{r*} \otimes E^{r+1*})_{i-p} \to K^{r+1}_p \qquad (r \text{ odd})\ ;$$
$$h(r+1, p)\colon (E^{r} \otimes E^{r+1})_{i+p} \to K^{r+1}_p \qquad (r \text{ even})\ .$$

We define $F\langle \mu, E^1, \cdots, E^{n-1}, \mu\rangle$ to be the homology class [in $(K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n)_i$] of the element

$$(2) \qquad \sum(-1)^*\langle h(1, p_1), E^1_{\bar p_2}, h(2, p_2), E^2_{\bar p_2}, \cdots, h(n-1, p_{n-1}), E^{n-1}_{\bar p_{n-1}}, \mu\rangle$$

where the sum is taken over all $(p_1, \cdots, p_{n-1})$ such that $\sum_{r=1}^{n-1} p_r = i$, and

$$\bar p_r = p_1 + p_2 + \cdots + p_r \qquad (r \text{ odd});$$
$$\bar p_r = i - p_1 - p_2 - \cdots - p_r \qquad (r \text{ even})\ ;$$

the sign $(-1)^*$ is determined as follows. For any positive integer k, let $\varepsilon(k) = \sum_{j=1}^{k} j$. Given $(p_1, \cdots, p_{n-1})$ such that $\sum_r p_r = i$, let

$$\zeta(p_1) = \varepsilon(p_1);\ \zeta(p_2) = \varepsilon(p_1 + p_2)\ ;$$
$$\zeta(p_r) = \varepsilon(i - p_1 - p_2 - \cdots - p_r) + p_{r+1} \qquad (r \text{ odd}, r \geqq 3)\ ;$$
$$\zeta(p_r) = \varepsilon(p_1 + \cdots + p_r) + p_{r+1} \qquad (r \text{ even}, r \geqq 4)\ .$$

Then set $(-1)^* = (-1)^{\zeta(p_1)+\zeta(p_2)+\cdots+\zeta(p_{n-2})}$.

Strictly speaking the maps $h(r, p_r)$ in (2) are actually the restrictions of these maps to suitable sub-modules; for example, if r is odd $h(r, p_r)$ is defined on $(E^{r-1} \otimes E^r)_{i+p_r}$ and the map $h(r, p_r)$ in (2) is the restriction to $E^{r-1}_{\bar p_{r-1}} \otimes E^r_{\bar p_r} \subseteqq (E^{r-1} \otimes E^r)_{i+p_r}$. Note that for each r, $h(r, p_r)$ is a map into $K^r_{p_r}$; if n is even $\bar p_{n-1} = i$ and $\mu\colon E^{n-1*}_i \to A^n$; if n is odd $\bar p_{n-1} = 0$ and $\mu\colon E^{n-1}_0 \to A^n$. Thus in every case $F\langle \mu, E^1, \cdots, E^{n-1}, \mu\rangle$ is an element of degree i of the group

$$\mathrm{Mult}_0(K^1, \cdots, K^{n-1}, A^n) = K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n\ .$$

In order to show that F is well defined we must verify that F is independent of the choice of the maps $h(r, -)$ and that the image of F is in fact contained in the group of cycles of $(K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n)_i$. Let $x = \langle \mu, E^1, \cdots, E^{n-1}, \mu \rangle \in \mathrm{Mult}_i(A^1, \cdots, A^n)$. As an element of $K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n$, Fx has boundary,

$$(3) \quad \sum(-1)^*\left[\sum_{r=1}^{n-1}(-1)^{u(r)} \langle h(1, p_1), E^1_{\bar p_1}, \cdots, \partial h(r, p_r), \cdots, E^{n-1}_{\bar p_{n-1}}, \mu \rangle\right],$$

where $p_0 = 0$, $u(r) = \sum_{k=1}^{r-1} p_k$ and $\sum(-1)^*$ is as in (2).

Using the facts that the maps $h(r, -)$ are chain maps (and thus commute with the various boundary operators), the additivity (in μ) of the $\langle \cdots, \mu, \cdots \rangle$, the defining relations in Mult_0, and the fact that $\mu \circ$(boundary) is zero in each case, it follows that (3) becomes (for n odd):

$$\begin{aligned}
&\sum (-1)^* \langle h(1, p_1 - 1)\partial, E^1_{\bar{p}_1} h(2, p_2), \cdots, E^{n-1}_{\bar{p}_{n-1}}, \mu \rangle \\
&+ \sum (-1)^* [\sum_{\substack{2 \leqq r \leqq n-2 \\ r \text{ even}}} (-1)^{u(r)} \langle \cdots, E^{r-2}_{\bar{p}_{r-2}}, h(r-1, p_{r-1})\ (1 \otimes \partial), \\
&\quad E^{r-1}_{\bar{p}_{r-1}+1}, h(r, p_r - 1), E^r_{\bar{p}_r}, \cdots \rangle + (-1)^{u(r)+\bar{p}_r} \langle \cdots, E^{r-1}_{\bar{p}_{r-1}}, \\
&\quad h(r, p_{r-1}), E^r_{\bar{p}_r+1}, h(r+1, p_{r+1})(\partial \otimes 1), E^{r+1}_{\bar{p}_{r+1}}, \cdots \rangle] \\
&+ \sum (-1)^* [\sum_{\substack{3 \leqq r \leqq n-2 \\ r \text{ odd}}} (-1)^{u(r)} \langle \cdots, E^{r-1}_{\bar{p}_{r-1}}, h(r, p_r - 1)(\partial \otimes 1), E^r_{\bar{p}_r}, \cdots \rangle \\
&+ (-1)^{u(r)+\bar{p}_r} \langle \cdots, E^{r-1}_{\bar{p}_{r-1}}, h(r, p_{r-1})(1 \otimes \partial), E^r_{\bar{p}_r}, \cdots \rangle] \\
&+ \sum (-1)^* \langle \cdots, E^{n-3}_{\bar{p}_{n-3}}, h(n-2, p_{n-2})(1 \otimes \partial), E^{n-2}_{\bar{p}_{n-2}+1}, \\
&\quad h(n-1, p_{n-1} - 1), E^{n-1}_{\bar{p}_{n-1}}, \mu \rangle .
\end{aligned}$$

(A similar statement holds for n even.) After a suitable change of indices (in the terms with r even) and careful attention to signs, it follows that all the terms cancel and hence the boundary of Fx is zero.

To show that F is independent of the choice of the maps $h(r, -)$, it suffices to assume that for some t, $g(t, -)$ is another such choice. (For convenience, assume t is odd; similar statements hold for even t.) Then there is a chain homotopy

$$s\colon (E^{t-1} \otimes E^t) \to K^t \ ;$$

specifically,

$$s(p+1); (E^{t-1} \otimes E^t)_{i+p} \to K^t_{p+1} \ ;$$

and

$$g(t, p) = h(t, p) + \bar{\partial} s(p+1) + s(p)\partial .$$

(where $\bar{\partial}$ is the boundary in $E^{t-1} \otimes E^t$). Thus it suffices to show that the element

$$\begin{aligned}
\sum (-1)^* \langle h(1, p_1), E^1_{\bar{p}_1}, \cdots, E^{t-1}_{\bar{p}_{t-1}}, \bar{\partial} s(p+1) \\
+ s(p)\partial, E^t_{\bar{p}_t}, \cdots, \mu \rangle
\end{aligned}$$

is a boundary in $K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n$. This fact follows from the repeated use of the defining relations for Mult_0 and the fact that maps $h(r-)$ are chain maps.

For convenience we shall now assume that $K^1, \cdots, K^{n-1}$ are finitely generated; (more precisely, we use suitably chosen finitely generated subcomplexes, cf. the argument in Theorem V.8.1 of [2]). Denote by $\hat{K}^r$ the complex K^r "cut off" beyond dimension i and let $\tilde{K}^r$ be the complex $\hat{K}^1 \otimes \cdots \otimes \hat{K}^r$ (from dimension i through 0) for r odd and $\hat{K}^{1*} \otimes \cdots \otimes \hat{K}^{r*}$ (from dimension 0 through i) for r even. Denote a free basis of $\hat{K}^r_p$ by $\{{}^rk_p(u_r)\}$ where u_r runs over a finite index set; the dual basis of $\hat{K}^{r*}_p$ is denoted by $\{{}^rk^*_p(u_r)\}$. If $(r_1, \cdots, r_t)$ is a t-tuple of nonnegative integers such that $\sum r_j = r$, we denote by $\{{}^tk_{(r)}(u, \cdots, u_t)\}$ the free basis

$$\{{}^1k_{r_1}(u_1) \otimes {}^2k_{r_2}(u_2) \otimes \cdots \otimes {}^tk_{r_t}(u_t)\}$$

of

$$\hat{K}^1_{r_1} \otimes \cdots \otimes \hat{K}^t_{r_t} \subseteqq \tilde{K}^t_r \qquad (t \text{ odd}).$$

Similarly $\{{}^tk^*_{(r)}(u_1, \cdots, u_t)\}$ denotes the free basis of

$$\hat{K}^{1*}_{r_1} \otimes \cdots \otimes \hat{K}^{t*}_{r_t} \subseteqq \tilde{K}^t_{i-r} \qquad (t \text{ even}) .$$

Strictly speaking this notation is somewhat ambiguous; but in context it will be clear.

Define as follows chain transformations

$$\pi\colon (\tilde{K}^{t*} \otimes \tilde{K}^{t+1*}) \to \hat{K}^{t+1} \qquad (t \text{ odd}) ;$$
$$\pi\colon (\tilde{K}^{t} \otimes \tilde{K}^{t+1}) \to \hat{K}^{t+1} \qquad (t \text{ even}) ,$$

where $(\tilde{K}^{t*} \otimes \tilde{K}^{t+1*})$ runs from dimension 0 to i and $(\tilde{K}^{t} \otimes \tilde{K}^{t+1})$ from dimension $2i$ to i. For t odd, let

$$y = {}^tk^*_{(r)}(u_1, \cdots, u_t) \otimes {}^{t+1}k_{(i-s)}(v_1, \cdots, v_{t+1}) ,$$

(where $(r_1, \cdots, r_t) = (r)$; $(s_1, \cdots, s_t) = (i - s)$; $r + s = n$), be a generator of $(\tilde{K}^{t*} \otimes \tilde{K}^{t+1*})_n$. Define

$$\pi y = \left[\prod_{j=1}^{t} \delta(r_j, s_j) \cdot \delta(u_j, v_j)\right](-1)^{\varepsilon(r)}\, {}^{t+1}k_{s_{t+1}}(v_{t+1}) ,$$

where $\varepsilon(r)$ is as above and δ is the Kronecker delta.

If $\pi y \neq 0$, then $r_j = s_j$ $(j \leqq t)$ and

$$i - s = \sum_{j=1}^{t+1} s_j = \sum_{j=1}^{t} r_j + s_{t+1} = r + s_{t+1} ;$$

hence $s_{t+1} = i - r - s = i - n$ and therefore

$$\pi\colon (\tilde{K}^{t*} \otimes \tilde{K}^{t+1*})_n \to \hat{K}^{t+1}_{i+n}$$

as desired (if $\pi y = 0$ there is no difficulty).

For t even, let

$$y = {}^{t}k^{*}_{(i-r)}(u_1, \cdots, u_t) \otimes {}^{t+1}k_{(s)}(v_1, \cdots, v_t) ,$$

(where $(r_1, \cdots, r_t) = (i-r)$, $(s_1, \cdots, s_t) = s$, $r + s = i + n$), be a generator of $(\tilde{K}^t \otimes \tilde{K}^{t+1})_{i+n}$. Define

$$\pi y = \left[\prod_{j=1}^{t} \delta(r_j, s_j)\cdot\delta(u_j, v_j)\right](-1)^{\varepsilon(r)}\, {}^{t+1}k_{s_{t+1}}(v_{t+1}) .$$

Note that if $\pi y \neq 0$, $s_j = r_j$ $(j \leqq t)$ and

$$s = \sum_{j=1}^{t+1} s_j = \sum_{j=1}^{t} r_j + s_{t+1} = i - r + s_{t+1} ;$$

hence $s_{t+1} = r + s - i = i + n - i = n$ and therefore

$$\pi\colon (\tilde{K}^t \otimes \tilde{K}^{t+1})_{i+n} \longrightarrow \hat{K}^{t+1}_n$$

as desired.

A laborious calculation shows that the maps π commute with the various boundary operators and thus are chain transformations. This calculation depends in part on the following facts (which will also be used below). Suppose E is a finitely generated free chain complex of finite length; denote the free basis of E_r by $\{e_r(u)\}$ and the dual basis of E^*_r by $\{e^*_r(u)\}$. Let G be a finitely generated free R-module with basis $\{f(w)\}$; define a map

$$\pi\colon E^* \otimes E \otimes G \longrightarrow G$$

by

$$\pi(e^*_r(u) \otimes e_s(v) \otimes f(w)) = \delta(r, s)\cdot\delta(u, v)\cdot f(w) ,$$

where δ is the Kronecker delta. Then

$$\pi[\partial^* e^*_r(u) \otimes e_{r+1}(v) \otimes f(w)] = \pi[e_r(u) \otimes \partial e_{r+1}(v) \otimes f(w)] .$$

This is true since the map $\partial\colon E_{r+1} \longrightarrow E_r$ can be described by matrix (r_{uv}) such that $\partial(e_{r+1}(v)) = \sum_u r_{uv} e_r(u)$; hence $\partial^*\colon E_r \longrightarrow E_{r+1}$ is given by $\partial^*(e^*_r(u)) = \sum_v r_{uv} e^*_{r+1}(v)$.

To show that the map F is an epimorphism, let z be a cycle in $(K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n)_i$. Then z can be written uniquely in the form $\sum z(p_1, \cdots, p_{n-1})$, where the sum is over all $(p_1, \cdots, p_{n-1})$ such that $\sum_j p_j = i$ and $z(p_1, \cdots, p_{n-1}) \in K^1_{p_1} \otimes \cdots \otimes K^{n-1}_{p_{n-1}} \otimes A^n$. Each $z(p_1, \cdots, p_{n-1})$ can be written uniquely in the form:

$$\langle 1, \bar{K}^1_{p_1}, \pi(2), \bar{K}^2_{p_1,p_2}, \pi(3), \bar{K}^3_{p_1,p_2,p_3}, \pi(4), \cdots, \bar{K}^{n-1}_{p_1,\cdots p_{n-1}}, \nu(p_1, \cdots, p_{n-1})\rangle ;$$

[cf. Proposition 3.5; the subscripts on the $\bar{K}^r$ are necessary to distinguish the $z(p_1, \cdots, p_{n-1})$]. Hence

$$z = \sum \langle 1, \bar{K}^1_{p_1}, \pi(2), \bar{K}^2_{p_1,p_2,\ldots,p_{n-1}}, \nu(p_1, \cdots, p_{n-1})\rangle .$$

where the sum is as above. Let $\tilde{K}^r$ and π be as in the previous paragraphs; we can consider the various $\bar{K}^r_{p_1,\ldots,p_r}$ as submodules (in various dimensions) of $\tilde{K}^r$; then the maps $\pi(r)$ are just the restrictions of the maps π to these submodules (except perhaps for a sign).

Consider the element

$$x = \langle \varepsilon, \tilde{K}^1, \varepsilon\pi, \tilde{K}^2, \varepsilon\pi, \tilde{K}^3, \varepsilon\pi, \cdots, \varepsilon\pi, \tilde{K}^{n-1}, \nu\rangle ,$$

where ν is defined as follows. If n is even, then

$$\tilde{K}^{n-1*}_i = \sum \bar{K}^{n-1*}_{p_1,\cdots,p_{n-1}}$$

(where the sum is over all $(p_1, \cdots, p_{n-1})$ such that $\sum_j p_j = i$), and $\nu\colon \tilde{K}^{n-1*}_i \to A$ is given on $\bar{K}^*_{p_1,\ldots,p_{n-1}}$ by $(-1)^{**}\nu(p_1, \cdots, p_{n-1})$, where

$$(-1)^{**} = (-1)^{\zeta(p_1)+\cdots+\zeta(p_{n-2})+\varepsilon(\bar{p}_2)+\varepsilon(\bar{p}_3)+\cdots+\varepsilon(\bar{p}_{n-1})} ,$$

($\varepsilon(\bar{p}_j)$ and $\zeta(p_j)$ as above). Similarly, if n is odd,

$$\tilde{K}^{n-1}_0 = \sum \bar{K}^{n-1}_{p_1,\cdots,p_{n-1}}$$

(sum as above) and $\nu\colon \bar{K}^{n-1}_0 \to A^n$ is given on $\bar{K}^{n-1}_{p_1,\cdots,p_{n-1}}$ by

$$(-1)^{**}\nu(p_1, \cdots, p_{n-1})$$

(sign as above). Assuming that x is a well defined element of Mult$_i$ $(A^1, \cdots, A^n)$ it follows [since $\pi = (-1)^{\varepsilon(\bar{p}_j)}\pi(j+1)$] that Fx is the homology class of the cycle z [Choose the identity for $h(1, -)$ and π for $h(r, -)$, $r > 1$.] Hence to show F is an epimorphism we need only show that x is in fact a well defined element of Mult$_i$ $(A^1, \cdots, A^n)$. For n odd this amounts to showing that $\nu\tilde{\partial} = 0$, where $\tilde{\partial}$ is the boundary operator in $\tilde{K}^{n-1}$. (Similarly for n even. we must show that $\nu\tilde{\partial}^* = 0$.). The proof of this fact is tedious but straightforward and we omit most of details. One first computes ∂z and notes that an element of the form

$$\langle 1, \bar{K}^1_{p_1}, \pi(2), \cdots, \bar{K}^{r-1}_{p_1,\ldots,p_{r-1}}, \pi(r), \bar{K}^r_{p_1,\ldots,p_r}, \\ \cdots, \bar{K}^{n-1}_{p_1,\ldots,p_{n-1}}, \nu(p_1, \cdots, p_{n-1})\rangle$$

can be written in the form

$$\pm \langle 1, \bar{K}^1_{p_1}, \pi(2), \cdots, \bar{K}^{r-1}_{p_1,\ldots,p_{r-1}}, \pi, \bar{K}^r_{p_1,\ldots,p_r-1}, \\ \cdots, \bar{K}^{n-1}_{p_1,\ldots,p_r-1,\ldots,p_{n-1}}, \nu(-)\hat{\partial}\rangle ,$$

where (for n odd) $\hat{\partial}$ is the map $1 \otimes \cdots \otimes 1 \otimes \partial \otimes 1 \cdots \otimes 1$ on

$K^1_{p_1} \otimes \cdots \otimes K^r_{p_r-1} \otimes \cdots \otimes K^{n-1}_{p_{n-1}}$. This is a consequence of the definition of the map π, repeated use of the defining relations in Mult_0 and the fact that $\pi = (-1)^{\varepsilon(\bar{p}_j)}\pi(j+1)(j = 1, \cdots, n-2)$. Since $\partial z = 0$ the uniqueness statement of Proposition 3.5 implies that $\nu(p_1, \cdots, p_{n-1})\hat{\partial} = 0$. It then follows that $\nu\tilde{\partial} = 0$ as desired. Hence x is well defined and F is an epimorphism.

In order to prove that F is a monomorphism we need the following lemma.

LEMMA 3.7. *Every generator of* $Mult_i(A^1, \cdots, A^n)$ *can be written in the form*:

$$\langle \varepsilon, \tilde{K}^1, \varepsilon\pi, \tilde{K}^2, \cdots, \varepsilon\pi, \tilde{K}^{n-1}, \nu \rangle ,$$

where the $\tilde{K}^r$ *are formed as above from suitably chosen finitely generated free subcomplexes of free resolutions* $K^r \xrightarrow{\varepsilon} A^r$ *of the* A^r.

The proof is given below; assume the lemma for the present. Suppose $x = \langle \varepsilon, \tilde{K}^1, \varepsilon\pi, \tilde{K}^2, \cdots, \varepsilon\pi, \tilde{K}^{n-1}, \nu \rangle$ is a generator of Mult_i $(A^1, \cdots, A^n)$ and that $Fx = 0$, i.e. Fx is a boundary in

$$K^1 \otimes \cdots \otimes K^{n-1} \otimes A^n .$$

Then there is a chain

$$u = \sum \langle 1, \bar{K}^1_{p_1}, \pi(2), \bar{K}^2_{p_1,p_2}, \cdots, \bar{K}^{n-1}_{p_1,\cdots,p_{n-1}}, \zeta(p_1, \cdots, p_{n-1}) \rangle ,$$

(where the sum is taken over all $(p_1, \cdots, p_{n-1})$ such that $\sum_j p_j = i + 1$) and $\partial u = x$. The remarks above show that ∂u can be written in the form:

$$\sum \langle 1, \bar{K}^1_{p_1}, \pi, \cdots, \bar{K}^{n-1}_{p_1,\cdots,p_{n-1}}, \zeta\tilde{\partial} \rangle ,$$

where the sum is over all $(p_1, \cdots, p_{n-1})$ such that $\sum_j p_j = i$ and $\tilde{\partial}$ is the boundary in $\tilde{K}^{n-1}$ (if n is odd: replace $\tilde{\partial}$ by $\tilde{\partial}^*$ for n even; recall that $\sum \bar{K}^{n-1} \subseteqq \tilde{K}^{n-1}$). It also follows that Fx can be written in the form

$$\sum (-1)^* \langle 1, \tilde{K}^1_{\bar{p}_1}, \pi, \tilde{K}^2_{\bar{p}_2}, \pi, \cdots, \tilde{K}^{n-1}_{\bar{p}_{n-1}}, \nu \rangle$$

(sum over all $(p_1, \cdots, p_{n-1})$ such that $\sum_j p_j = i$). It follows from Proposition 3.5 that

$$\nu = \pm \zeta\tilde{\partial} .$$

Hence,

$$
\begin{aligned}
x &= \langle \varepsilon, \tilde{K}^1, \varepsilon\pi, \tilde{K}^2, \varepsilon\pi, \cdots, \tilde{K}^{n-1}, \nu\rangle \\
&= \pm \langle \varepsilon, \tilde{K}^1, \varepsilon\pi, \cdots, \tilde{K}^{n-1}, \zeta\tilde{\partial}\rangle \\
\text{(i)}\quad &= \pm \langle \varepsilon, \tilde{K}^1, \cdots, \varepsilon\pi(1^* \otimes \tilde{\partial}^*), \tilde{K}^{n-1}, \zeta\rangle \\
\text{(ii)}\quad &= \pm \langle \varepsilon, \tilde{K}^1, \cdots, \varepsilon\pi(\tilde{\partial}^* \otimes 1^*), \tilde{K}^{n-1}, \zeta\rangle \\
&\ \ \vdots \\
\text{(iii)}\quad &= \pm \langle \varepsilon\partial, \tilde{K}^1, \varepsilon\pi, \cdots, \tilde{K}^{n-1}, \zeta\rangle \\
&= 0 .
\end{aligned}
$$

(i) results from applying the defining relations in Mult$_i$; (ii) follows since by the definition of the generators of Mult$_i$,

$$\varepsilon\pi(\tilde{\partial}^* \otimes 1^*) \pm \varepsilon\pi(1^* \otimes \tilde{\partial}^*) = 0$$

(where $\tilde{\partial}$ is used to denote either the boundary in $\tilde{K}^{n-2}$ or $\tilde{K}^{n-1}$). Repeated use of this finally gives (ii); by definition $\varepsilon\partial = 0$ and hence $x = 0$. Thus F is a monomorphism, and therefore an isomorphism. Except for the lemma, this completes the proof of Theorem 3.6, and justifies the use of the notation Mult$_i(A^1, \cdots, A^n)$ to denote unambiguously either the group defined by resolutions (§ 1) or the group defined by generators and relations (§ 2).

Proof of Lemma 3.7. Given a generator

$$x = \langle \mu, E^1, \mu, E^2, \cdots, E^{n-1}, \nu\rangle$$

of Mult$_i(A^2, \cdots, A^n)$, there is a chain map $h: E^1 \to K^1$ lifting the identity map on A^1 (K^1 is a free resolution of A^1 as above). Let $\hat{K}^1$ be a finitely generated free subcomplex of length i of K^1 which includes the image of h. Then

$$
\begin{aligned}
x &= \langle \mu, E^1, \cdots, E^{n-1}, \nu\rangle = \langle \varepsilon h, E^1, \mu, \cdots, E^{n-1}, \nu\rangle \\
&= \langle \varepsilon, \hat{K}^1, \mu(h^* \otimes 1^*), E^2, \cdots, E^{n-1}, \nu\rangle .
\end{aligned}
$$

Note that $\hat{K}^1$ can be taken as $\tilde{K}^1$. We now proceed by induction and assume that x can be written in the form

$$\langle \varepsilon, \tilde{K}^1, \varepsilon\pi, \tilde{K}^2, \cdots, \varepsilon\pi, \tilde{K}^t, \mu, E^{t+1}, \cdots, E^{n-1}, \nu\rangle .$$

For convenience, assume t is even. We wish to define a chain map $\varphi: E^{t+1} \to \tilde{K}^{t+1}$ such that $\mu = \varepsilon\pi(1 \otimes \varphi)$. If we can do this, then

$$x = \langle \varepsilon, \tilde{K}^1, \cdots, \tilde{K}^t, \varepsilon\pi, \tilde{K}^{t+1}, \mu(\varphi^* \otimes 1^*), \cdots, \nu\rangle$$

and the induction is completed.

In order to define φ, note that there is a chain map h:

$$\begin{array}{ccc} (\widetilde{K}^t \otimes E^{t+1}) & \xrightarrow{\ \mu\ } & A^{t+1} \\ \Big\downarrow h & & \Big\| \\ K^{t+1} & \xrightarrow{\ \varepsilon\ } & A^{t+1} \end{array},$$

(where we take $(\widetilde{K}^t \otimes E^{t+1})$ from dimensions $2i$ to i; note that $(\widetilde{K}^t \otimes E^{t+1})$ is finitely generated). Let $\widetilde{K}^{t+1}$ be a finitely generated free subcomplex of length i of K^{t+1} which includes the image of h. Denote the generators of $E^t, E^{t+1}, \widetilde{K}^t, \widetilde{K}^{t+1}$ as above and define $\varphi\colon E_s^{t+1} \longrightarrow \widetilde{K}_s^{t+1}$ on a generator ${}^{t+1}e_s(v)$ by

$$\varphi x = \sum \sum (-1)^{\varepsilon(r)}\, {}^t k_{(r)}(u_1, \cdots, u_t) \otimes h[{}^t k^*_{(r)}(u_1, \cdots, u_t) \otimes {}^{t+1}e_s(v)] \,,$$

where the second sum runs over all $(r) = (r_1, \cdots, r_t)$ such that $\sum r_j = r$ and r takes all values from 0 to s; for each $(r_1, \cdots, r_t)$ the first sum runs over all $(u_1, \cdots, u_t)$, where the generators of $\hat{K}^r_{r_j}$ are indexed by the u_j. Note that ${}^t k_{(r)}(u_1, \cdots, u_t)$ is a generator of $\widetilde{K}^{t*}_{i-r}$ and $h[\widetilde{K}^t_{i-r} \otimes E^{t+1}_s] \subseteqq K^{t+1}_{s-r}$; hence

$$\varphi x \in \widetilde{K}^{t*}_{i-r} \otimes K^{t+1}_{s-r} \,,$$

and therefore $\varphi x \in \widetilde{K}^{t+1}_s$. A tedious calculation shows that φ is in fact a chain map and that $\varepsilon\pi(1 \otimes \varphi) = \mu\colon \widetilde{K}^t \otimes E^{t+1} \longrightarrow A^{t+1}$. The procedure for t odd is similar. This completes the proof of the lemma.

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