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NILPOTENCE OF THE COMMUTATOR SUBGROUP IN GROUPS ADMITTING FIXED POINT FREE OPERATOR GROUPS

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Let V be a group of operators acting in fixed point free manner on a group G and suppose V has order relatively prime to $|G|$. Work of several authors has shown that if V is cyclic of prime order or has order four, G' is nilpotent. In this paper it is proved that G' is nilpotent if V is non-abelian of order six, but that G' need not be nilpotent for any further groups other than those just mentioned. A side result is that G has nilpotent length at most 2 when V is non-abelian of order pq, p and q primes (non-Fermat, if $|G|$ is even).

A fundamental theorem of Thompson [7] states that if G is a group admitting a fixed free automorphism of prime order, then G is nilpotent. It appears to be well known that if, in this theorem, the group of prime order is replaced by any group of automorphisms of composite order acting in fixed point free manner on G, one can no longer conclude that G is nilpotent. (For the sake of completeness, this fact is proved at the end of § 1.) However, one can frequently draw weaker conclusions concerning G in these cases. For example, D. Gorenstein and I. N. Herstein [4] proved that a group, G, which admits a fixed point free automorphism of order four, has nilpotent length at most two. S. Bauman [1] in 1961 obtained a similar result for the case that the fixed point free operator group was the four-group. Other more general results giving bounds for the nilpotent length of a solvable group, G, admitting various fixed point free operator groups, V, of order prime to $|G|$ can be found in Hoffman [5], Thompson [8] and Shult [6]. In summarizing these results we remark only that the bounds are best possible when V is abelian and subject to a certain restriction on the prime divisors of its order (a restriction which vanishes when $|V|$ and $|G|$ are both odd), but that the bounds are very large otherwise.

In the case that V has order 4, something rather special obtains. Not only does G have nilpotent length 2, but moreover G has a nilpotent commutator subgroup. These findings raise the following question: Let G admit a fixed point free group of operators, V, of order prime to $|G|$. For what groups, V, does this imply nilpotence of the commutator subgroup? From the above-mentioned results of Thompson, Gorenstein

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and Herstein, and Bauman, this situation obtains whenever V is cyclic of prime order, or has order four. In this paper, it is shown that G' is also nilpotent when V is S_3, the symmetric group of degree three, but that this implication does *not* hold for any further groups, V.

In order to address the general question concerning which fixed point free operator groups, V, yield the nilpotence of G', it would seem imperative that one have available general information concerning groups admitting fixed point free groups of operators, particularly information concerning nilpotent length 2. Information of this type can be obtained for the special case that V is abelian (Shult [6]), but so far only bounds on nilpotent length which exceed 5 are available when V is non-abelian (Thompson [8]). In the second section of this paper we produce a special result for the case that V is non-abelian of order pq, p and q primes. Here, if G admits V as a fixed point free operator group, G has nilpotent length at most 2, provided neither p nor q are Fermat primes when G is even. Although this meagre result barely scratches the surface for the case that V is non-abelian, it turns out to be sufficient to answer the central question of this paper: when is G' necessarily nilpotent? In § 2, it is proved that G' is nilpotent when $V \simeq S_3$. Unlike proofs for the case V has order four, this proof does not merely hinge on the fact that a group fixed point free under an automorphism of order two is abelian. Rather, the proof asserts that a group which admits a fixed point free automorphism of order three in a very special way (a special case of condition (3) of (*) in Theorem 1) is abelian. The final section merely consists in showing the existence of groups G, fixed point free under V, for which G' is not nilpotent whenever V is not cyclic of prime order, not of order four and not S_3.

1. Technical preliminaries. The purpose of this section is to standardize notation and to list a few preliminary results which are used repeatedly in the arguments in the main sections of the paper. Throughout all groups considered are finite and E denotes the identity group. The symbol $O_\pi(G)$ denotes the maximal normal π-subgroup of G, where π is a fixed collection of primes.

If V is a group of operators acting on a group, G, the following subgroups are distinguished:

$$G_V = \{g\colon g \in G,\ v(g) = g \text{ for all } v \in V\}$$
$$(V, G) = \text{the subgroup generated by } \{v(g)g^{-1}\colon g \in G,\ v \in V\}$$

V is said to act *fixed point free* on G if $G_V = E$. If N is a normal V-invariant subgroup of G, V will also be regarded as a group of operators acting on N and G/N. The following two lemmas are obvious.

LEMMA 1.1. (a) *(V, G) is always normal in G.*
(b) *If $W \lhd V$, G_W and (W, G) are both V-invariant.*

LEMMA 1.2. *If N is a normal V-invariant subgroup of G, then*
(a) *$(G/N)_V = G/N$ if and only if $(V, G) \subseteq N$*
(b) *if V is fixed point free on G/N, $G_V \subseteq N$.*
From part (b) *it can be seen that if V is fixed point free on both G/N and N, then V is fixed point free on G.*

The following lemma is essentially a special case of a result of Glauberman [3].

LEMMA 1.3. *Let V and G have relatively prime orders and let N be normal and V-invariant. Then*
(a) $G = G_V(V, G)$
(b) $(G/N)_V = G_V N/N$.

Proof. If the coset xN is fixed by V, the theorem of Glauberman asserts that $xN = yN$ for some $y \in G_V$, whence (b). (a) follows from (b) upon setting $(V, G) = N$, and using Lemma 1.1 (a) and Lemma 1.2 (a).

Theorem 1 of § 2 requires a technical theorem which is a special case of Theorem 4.1 proved in [6]:

THEOREM (A). *Let U be cyclic of prime order, p, and suppose U is a group of automorphisms acting on a solvable group G of order relatively prime to p. Let $H = GU$ be the semidirect product and suppose A is a faithful indecomposable KH-module, where K is any field whose characteristic is not p. Then if U acts in fixed point free manner on the module A, U centralizes G provided* (i) *$O_r(G) = E$ when the characteristic of K is r, and* (ii) *p is not a Fermat prime when $|G|$ is even.*

We say that a group theoretic property, P, is *residually complete* if for any group G and any collection of two or more normal subgroups $N_1, \cdots, N_s$ intersecting at E, the fact that G/N_i has property P for $i = 1, \cdots, s$ implies G has property P. In short, P is residually complete if the collection of finite P-groups is closed under taking subdirect products. It is easy to show that (a) having nilpotent commutator subgroup and (b) having nilpotent length $\leqq k$, are residually complete group theoretic properties, and these facts are assumed throughout the remainder of the paper.

We now settle in the negative the question whether there are groups, V, other than those which are cyclic of prime order, which imply the nilpotence of groups, G, admitting V as a fixed point free group of automorphisms.

THEOREM. *Suppose V is a group of composite order. Then there exists a non-nilpotent group, G, admitting V as a fixed point free group of operators.*

Proof. Let T be a proper subgroup of V and suppose $|T|$ is composite. Then by induction there exists a nonnilpotent group, G_1, which admits T as a fixed point free group of automorphisms. Set

$$V = x_1T + x_2T + \cdots + x_kT \text{ with } x_1T = T$$

and let G be a sum of k isomorphic copies of G_1, so $G = G_1 \times \cdots \times G_k$. The action of V on G is defined by letting V permute the components, G_i, as wholes, with T being the subgroup leaving G_1 invariant, and $G_1^{x_i} = G_i$. T acts in fixed point free manner or G_1 and T^{x_i} acts in fixed point free manner on G_i. In effect, G may be regarded as a normal subgroup of the semidirect product, VG, consisting of all k-tuples $(g_1, x_2^{-1}g_2x_2, \cdots, x_k^{-1}g_kx_k)$, $g_i \in G_1$, on which V acts by componentwise conjugation. Then if $u = (u_1, \cdots, u_k)$ was a fixed point, $t^{-1}u_it = u_i$ for all $t \in T^{x_i}$. As T^{x_i} is fixed point free on G_i, each $u_i = 1$ so $u = 1 \in G$. Thus G is a nonnilpotent group admitting V as a fixed point free group of automorphisms. Thus we may suppose that all proper subgroups of V are cyclic of order p. Then V is metacyclic of order pq (p and q primes), and we may suppose that U, the q-subgroup of V, is normal in V. Let R be cyclic of prime order $r \equiv 1 \bmod p$ and let V act on R in such manner that U acts trivially on R. Then in the semidirect product, $X = VR$, U is normal and X/U is the non-abelian group of order pr. Let s be a prime such that $s \equiv 1 \bmod qr$, and let M_1 be the faithful irreducible R-module of dimension 1 over $GF(s)$, and convert M_1 into a UR-module by letting U act by scalar multiplication on M_1. Now let M be the induced $GF(s)X$-module,

$$M = M_1 \otimes_{GF(s)UR} GF(s)X ,$$

affording the representation, ρ, of X. Then $G = MR$, is a subgroup of the semidirect product XM, and V can then be regarded as a group of automorphisms acting on G. Then V is fixed point free on $G/M \simeq R$ since $VG/UM \simeq X/U$ is non-abelian of order pr. Also U is fixed point free on M since M is a sum of conjugate 1-dimensional faithful U-modules. Thus V is fixed point free on G. Also M is a sum of conjugate faithful R-modules and r is prime to s; hence $[R, M] = M$ and so G is not nilpotent.

2. Groups with metacyclic fixed point free operator groups of order pq. Throughout this section, V denotes a non-abelian metacyclic group of order pq where p divides $q-1$. We suppose v and w are elements in V such that $v^p = w^q = 1$, $v^{-1}wv = w^a$, where $a^p \equiv 1 \bmod q$. We set $W = \{w\}$, the subgroup of order q in V.

THEOREM 1. *Let G be a group admitting V as a fixed point free group of operators, where $|V|$ and $|G|$ are coprime and, if $|G|$ is even, p and q are not Fermat Primes. Form the semidirect product $H = GV$ and let A be a faithful KH-module where K is a splitting field for H chosen so that if char $K = r$, r does not divide pq and $O_r(G) = E$. G is assumed to be solvable. Then if the representation, α, afforded by A is such that V acts in fixed point free manner on the non-trivial elements of A, then G has the following properties*

(*) (1) $G = G_1 \times G_2$ *where G_i is V-invariant* ($i = 1, 2$).

(2) G_1 *is fixed elementwise by W.*

(3) G_2 *contains a set of normal subgroups, $N_1(G), \cdots, N_q(G)$, such that*

(i) *the $N_i(G)$ have trivial meet*

(ii) $w(N_i(G)) = N_{i+1(\mathrm{mod}\, q)}(G)$

(iii) *if $v = v_1, v_2, \cdots, v_q$ are the successive conjugates of v under w^{-1} (i.e. $wv_iw^{-1} = v_{i+1(\mathrm{mod}\, q)}$), then v_i leaves $N_i(G)$ invariant and fixes $G/N_i(G)$ elementwise.*

Before proceeding to the proof of Theorem 1, we first establish a number of lemmas. The first few of these concern several aspects of the property (*).

LEMMA 2.1. *Let G be a group admitting V as a fixed point free operator group. Then if G enjoys property* (3) *in* (*), *G is fixed point free under w.*

Proof. Since $W \lhd V$, by Lemma 1.1 (b), G_W is a V-invariant subgroup of G. Since V is fixed point free on G, v is fixed point free on G_W, whence, by Lemma 1.3 (a) $G_W = (v, G_W)$. But since v_i fixes $G/N_i(G)$ elementwise, $(v_i, G) \subseteqq N_i(G)$, by Lemma 1.2 (a). Now $v_{i+1} = w^ivw^{-i}$, and we may write every element in G which is of the form $v_{i+1}(x)x^{-1} = w^ivw^{-i}(x)x^{-1}$ as $w^i(v(y)y^{-1})$ by setting $y = w^{-i}(x)$. Thus

$$(v_{i+1}, G) = (v, G)^{w^i}\,, \qquad i = 0, 1, 2, \cdots, q-1\,.$$

We now have

$$G_W = (v, G_W) \subseteqq (v, G)^{w^{i-1}} = (v_i, G) \subseteqq N_i(G)\,, \qquad i = 1, \cdots, q\,.$$

Since the $N_i(G)$ have trivial meet, $G_W = E$.

LEMMA 2.2. *If a group, G, satisfies condition* (*), *then G is nilpotent.*

Proof. From (*), $G = G_1 \times G_2$. By Lemma 2.1, G_2 is fixed point free under an automorphism, w, of order q and so, by the theorem of Thompson, is nilpotent. Also, G_1 is fixed point free under an automorphism, v, of order p, whence G is also nilpotent.

LEMMA 2.3. *The condition* (*) *is inherited by V-invariant subgroups.*

Proof. Let H be a V-invariant subgroup of G. Then by Lemma 1.3 (a), $H = H_W(W, H)$. From Lemma 1.1 (a), we always have $(W, H) \lhd H$. Since $G_W = G_1$ is normal in G, $H_W = H \cap G_W \lhd H$. Finally $H_W \cap (W, H) \subseteqq G_W \cap (W, G) = E$. Under these circumstances, $H = H_W \times (W, H)$. Since $W \lhd V$, by Lemma 1.1 (b), each of these direct factors are V-invariant. Setting $H_1 = H_w$ and $H_2 = (W, H)$, H satisfies (1) and (2) of (*).

Now set $N_i(H) = H \cap N_i(G)$. Then, because of the v_i-isomorphism $H/N_i(H) \simeq HN_i(G)/N_i(G)$, v_i fixes $H_2/N_i(H)$ elementwise, proving (iii). Now

$$\begin{aligned} w(N_i(H)) &= w(H \cap N_i(G)) \\ &= H^w \cap N_i(G)^w \\ &= H \cap N_{i+1}(G) \\ &= N_{i+1}(H) \end{aligned}$$

where the subscripts are taken $\bmod q$. This proves (ii). Finally the intersection of the $N_i(H)$ is necessarily trivial, so (i) holds.

LEMMA 2.4. *The property* (*) *is preserved under taking direct products.*

Proof. Let G and H be two groups admitting V as a fixed point free group of automorphisms and suppose (*) holds for each group. Set $L = G \times H$. Then L admits V in a natural way and is fixed point free under V. Set $L_i = G_i \times H_i$ $(i = 1, 2)$ so $L = L_1 \times L_2$, each L_i is V-invariant, and $L_W = L_1$, $(W, L) = L_2$. Thus (1) and (2) hold for L. Now define $N_i(L) = N_i(G) \times N_i(H)$. Then $N_i(L)$ is v_i-invariant and normal in L_2, and the $N_i(L)$ have trivial meet.

Consider any left coset of $N_i(L)$ in L_2, say $(x, y)N_i(L)$ where $x \in G_2$ and $y \in H_2$. Since v_i fixes $G/N_i(G)$ and $H/N_i(H)$ elementwise, $v_i(x) = xn$, $v_i(y) = yn'$ where $n \in N_i(G)$ and $n' \in N_i(H)$. Then

$$v_i(x, y)N_i(L) = (x, y)(n, n')N_i(L) = (x, y)N_i(L)$$

since $(n, n') \in N_i(L)$. Thus v_i fixes $L_2/N_i(L)$ elementwise. Clearly, the $N_i(L)$ have trivial meet, and $w(N_i(L)) = N_{i+1}(L)$. Thus (i), (ii), and (iii), and hence (3), hold for L. Thus L satisfies the condition (*).

Proof of Theorem 1. Suppose A is decomposable as a KH-module ($H = GV$): Then $A = A_1 + A_2 + \cdots + A_s$ where each A_i is indecomposable.

Case I. Either $s > 1$ or at least one A_i is reducible.

Let B_i be a proper maximal submodule of A_i and consider the module A_0 defined by the external direct product

$$A_0 = A_1/B_1 + A_2/B_2 + \cdots + A_s/B_s \ , \tag{1}$$

and let α_i and μ be the representations afforded by A_i/B_i and A_0 respectively, $i = 1, 2, \cdots, s$. We now set out to show that μ is faithful. If char $K = 0$, each A_i is irreducible, whence $B_i = 0$ so A_0 coincides with A. Then μ is faithful, since, by hypothesis, A is a faithful KH-module.

On the other hand, if char $K = r$, $O_r(G) = E$. Since G is solvable, we must have, in this case, $C_i = O_{r'}(\ker \alpha_i \cap G) \neq E$. Then C_i fixes A_i/B_i elementwise. Now as an additive group, A_i is a finite elementary abelian r-group, acted on by a group of operators, C_i of order prime to r. Then, since C_i centralizes A_i/B_i, by Lemma 1.3(b), we must have $A_i = (A_i)_{C_i}B_i$, so $(A_i)_{C_i} \neq E$. Since C_i is V-invariant and normal in G, $C_i \lhd H$. Then by Lemma 1.1 (b) $(A_i)_{C_i}$ and (C_i, A_i) are KH-submodules of A_i. Moreover, by Maschke's theorem, $A_i = (A_i)_{C_i} \oplus (C_i, A_i)$. Then, because of the indecomposibility of A_i and the fact that $(A_i)_{C_i} \neq E$, $(C_i, A_i) = E$, whence C_i centralizes all of A_i.

Now suppose $\ker \mu \cap G \neq E$. Then, since G is solvable and $O_r(G) = E$, $O_{r'}(\ker \mu \cap G) \neq E$. Also, $O_{r'}(\ker \mu \cap G)$ is a normal r'-subgroup of $\ker \alpha_i \cap G$ and so $O_{r'}(\ker \mu \cap G) \subseteqq C_i$, $i = 1, \cdots, s$. Then $O_{r'}(\ker \mu \cap G)$ centralizes each A_i and hence all of A. Since A is faithful

$$O_{r'}(\ker \mu \cap G) = E \ ,$$

contrary to our assumption that $\ker \mu \cap G \neq E$. Thus $\mu|_G$ is a faithful representation of G.

Now each A_i/B_i is an irreducible KH-module and $G/(\ker \alpha_i \cap G)$ has no normal r-groups (since any normal r-subgroup of G necessarily acts trivially on A_i/B_i). Further, both A_i/B_i and $G/(\ker \alpha_i \cap G)$ are fixed point free under the action of V. Now if $s > 1$, or $s = 1$ and $B_1 \neq E$ (which will be the case if A_1 is not irreducible), $\dim_K (A_i/B_i) < \dim_K A$, $i = 1, \cdots, s$. Thus, by induction, $G/(\ker \alpha_i \cap G)$ satisfies (*). Then by Lemma 2.3,

$$L = \prod_{i=1}^{s} (G/(\ker \alpha_i \cap G))$$

satisfies (*). But because of the decomposition (1), $\mu(G)$ is isomorphic to a subgroup of L and hence, by Lemma 2.1, also enjoys (*). Since μ is faithful, G itself satisfies (*).

Case II. $s = 1$, $A_1 = A$ *is an irreducible KH-module.*

Here we may apply Clifford's theory [2]. Since $G \lhd H$, A decomposes into homogeneous KG-components, $D_1, \cdots, D_t$, which are permuted transitively by V. Thus t divides pq.

Subcase (a). $t = pq$.

Here, the permutation representation of V afforded by the permutations V effects upon the D_i, is the regular representation. Consequently, $D_i = u_i D_1$ for some unique $u_i \in V$. Thus, selecting $d_1 \in D_1$, $d_1 \neq 0$, $u_i(d_1) \in D_i$, and so

$$d = \sum_{u \in U} u(d_1)$$

is a nonzero element of A, fixed by V. This contradicts our assumption that V is fixed point free on A and so subcase (a) cannot occur.

Subcase (b). $t = q$.

Here, the permutation representation is isomorphic to that induced by multiplication in V on left cosets of a subgroup of index q. Since all such subgroups are conjugate in V, without loss of generality we may take this subgroup to be $\{v\}$. The upshot of this is that w permutes the D_i in a cycle of length q, while v fixes one component, say D_1, and permutes the remaining $q - 1$ components in cycles of length p. If D_1 contained a point d_0 fixed by v, then

$$d = \sum_{i=1}^{q} w^{i-1}(d_0)$$

would be a nonzero point of A fixed by all of V. Thus v acts in fixed point free manner on D_i. Let δ_1 be the representation of G afforded by D_i. Now δ_1 can be extended to a representation of $K(G/\ker \delta_1)\{v\}$, and, by another result of Clifford's is indecomposable since $G\{v\}$ is the stability group in H for the submodule D_1. Set $H_i = O_r(G/\ker \delta_i)$. Since D_i is a sum of equivalent irreducible KG-modules, each D_i is also a sum of conjugate irreducible KH_i-modules. But each of these is trivial since H_i is an r-group and char $K = r$. Thus $H_i = E$ and so none of the groups $G/\ker \delta_i$ have normal r-subgroups. If $[G; \ker \delta_1]$ is even so is $|G|$, and in that case our hypotheses guarantee that p is not a Fermat prime. The groups $\{v\}$, $G/\ker \delta_1$ and module, D_1, now satisfy the conditions of Theorem (A). Thus v fixes $G/\ker \delta_1$ elementwise. Then also, $v_i = w^{i-1}vw^{-(i-1)}$ leaves D_i invariant and fixes $G/\ker \delta_i$ elementwise. Finally, since A is faithful, the groups $\ker \delta_i$, $i = 1, \cdots, 2$, have trivial meet. Thus G satisfies condition (3) of (*), and so (for the case $G_1 = E$) also satisfies (*).

Subcase (c). $t = p$.

Here, the D_i are permuted in a cycle of length p, by v (or any v_i), and w leaves each D_i invariant. Under these circumstances, w must be fixed point free on each D_i since otherwise it would be a simple matter to construct a nonzero point in A fixed by V. Then since q is not a Fermat prime, by Theorem (A), w fixes $G/\ker \delta_i$ elementwise, $i = 1, \cdots, p$. Thus $(w, G) \subseteqq \bigcap_i \ker \delta_i = E$, whence G is fixed elementwise by w. Thus G satisfies condition (*) for the special case that $G_2 = E$.

Subcase (d). $t = 1$.

Here G is homogeneous as a KG-module. At this point we can apply Clifford's theorem relative to any normal subgroup of H lying in G, i.e. any V-invariant normal subgroup of G. Let M be a maximal V-invariant normal subgroup of G. Since G is solvable, G/M is an elementary abelian r_1-group which, as a vector space over the field of r_1 elements, is an irreducible GV-module. Since A is an irreducible KG-module, we may decompose A into its homogeneous KM-components, $E_1, \cdots, E_m$, and, these are permuted transitively by the elements of G alone. Let N denote the subgroup of G which leaves each component invariant. Then if $x \in N$, $x(E_i) = E_i$ and $v(x)E_i = v(x)v(E_j) = v(xE_j) = v(E_j) = E_i$. Thus $v(x) \in N$ whence N is V-invariant. Clearly, $N \supseteqq M$. If $M \subset N$, $N = G$ because of the maximality of M and the fact that $N \lhd H$. Since G/N is abelian, the permutation of the E_i under the action of G is permutation isomorphic to the regular representation of G/N. If $G \neq N$, $[G:N] = [G:M] = r_1^k = m$, the number of homogeneous KM-components. On the other hand, if $N = G$, there is only one component, so A is a homogeneous KM-module. Let us consider the two cases separately.

Subsubcase (i). $N = G$; *A is a homogeneous KM-module.*

Since M is a proper subgroup of G admitting V and A is a KM-module fixed point free (along with M) under the action of V, by induction, M is a subgroup with property (*). By Lemma 2.2, M is nilpotent and so has a nontrivial center, $Z(M)$. Since the hypotheses of the case under investigation demand that A be a homogeneous KM-module, all the irreducible $KZ(M)$-submodules of A are conjugate by an element of M. Since $Z(M)$ is the center, these submodules are even equivalent. Since $Z(M)$ is abelian, A is a homogeneous $KZ(M)$-module and K, being a splitting field for all subgroups of H is certainly a splitting field for $Z(M)$, $Z(M)$ must be represented on A by scalar multiplication by elements of K. Under these circumstances, aside from the fact that $Z(M)$ is cyclic, the matrices representing V commute with those representing $Z(M)$. Since A is a faithful KH-module, this means that the elements of V centralize those of $Z(M)$, contrary to our hypothesis that V acts in fixed point free manner on G.

Subsubcase (ii). $N = M$.

Here there are $[G:M]$ distinct homogeneous KM-components. By applying induction on M and using Lemma 2.2, we have already seen that M is nilpotent, whence $Z(M) \neq E$. The components E_i are permuted by $H = GV$, the resulting permutation representation having kernel, N. Thus the transformation of the E_i can be associated with a faithful transitive permutation representation, π, of the semidirect product, $V(G/N) = VG/N = H/N$, of degree r_1^k. But this is permutation isomorphic to the permutation representation induced by multiplication of the left cosets of some subgroup of index r_1^k, in $V(G/N)$. Such a subgroup necessarily has index r_1^k, and so, since $V(G/N)$ is solvable, is an r_1-complement and is conjugate to V. Thus the representation, π, is permutation isomorphic to that induced by multiplication of left cosets of V in $V(G/N)$ by elements of G/N. In such a representation, V is the subgroup fixing some letter elementwise. Thus, because of the permutation isomorphism, we learn that V leaves some KM-component, say E_1, invariant. Then, by a theorem of Clifford's, since VM is the stability group of E_1 (a consequence of our case division), E_1 is a VM-module. Moreover, V is fixed point free on E_1. Since E_1 is a homogeneous KM-module, $Z(M)$ is represented by scalar multiplication on E_1. Then the matrices representing elements of V commute with the scalar matrices representing $Z(M)$ on E_1. Let β_1 be the representation of VM afforded by E_1. Then if $Z(M) \nsubseteq \ker \beta_1$, V would centralize $Z(M) \ker \beta_1/\ker \beta_1 \neq E$. Since V has order prime to M, by Lemma 1.3 (b), this would imply $C_M(V) \neq E$ contrary to our hypothesis. Thus $Z(M) \subseteqq \ker \beta_1$. But the E_i are conjugate EM-modules, i.e. $E_i = \alpha(x_i)E_1$ for some $x_iM \in G/M$. Under these circumstances, if $\ker \beta_i$ is the kernel in M of the KM-representation afforded by E_i,

$$\ker \beta_i = (\ker \beta)^{x_i^{-1}} ,$$

whence, since $Z(M)$ is normal in G,

$$Z(M) = Z(M)^{x_i^{-1}} \subseteqq (\ker \beta_1)^{x_i^{-1}} = \ker \beta_i , \qquad i = 1, \cdots, r_1^k .$$

Since A is faithful (even when restricted to M) the $\ker \beta_i$ have trivial meet. Thus

$$Z(M) \subseteqq \bigcap_{i=1}^{r_1^k} \ker \beta_i = E .$$

But this is impossible since M is nilpotent. The subsubcase (ii) doesn't arise. This completes the proof.

COROLLARY 1.1. *Theorem 1 still holds when the condition that K be a splitting field for all subgroups of $H = GV$ is dropped.*

Proof. Let G and V satisfy the conditions of Theorem 1. Let K be a field chosen so that if char $K = r$, G has no normal r-groups and r is prime to pq. Let A be a faithful KH-module whose nonzero elements are fixed point free under the action of V. Let L be a splitting field for all subgroups of H, where $[L : K]$ is finite, and form the module $A \otimes_K L = A'$. Then char L is char K. The remainder of the proof simply consists of the observation that A' is faithful and fixed point free under V. An application of Theorem 1 then shows that G satisfies (*).

COROLLARY 1.2. *Let G be a solvable group admitting V as a fixed point free group of operators, where $|V| = pq$ is prime to $|G|$. Then for every prime r dividing $|G|$, $G/O_{r'r}(G)$ is nilpotent.*

Proof. Let F_r be the Frattini factor group of the r-group,

$O_{r'r}(G)/O_r(G)$. Then F_r is a $V(G/O_{r'r}(G))$-module, faithful when restricted to $G/O_{r'r}(G)$. Moreover, from Lemma 1.3 (b), F_r and $G/O_{r'r}(G)$ are both fixed point free under the action of V. By Corollary 1.1, $G/O_{r'r}(G)$ satisfies (*) and so, by Lemma 2.2, is nilpotent.

COROLLARY 1.3. *Let G be a solvable group admitting V as a fixed point free operator group and suppose $|G|$ is prime to $pq = |V|$. Then G has nilpotent length at most two.*

Proof. Let $G \supseteqq M(G) \supseteqq M^2(G) \supseteqq \cdots$, $F(G)$ and $n(G)$ denote the lower nilpotent series, Fitting subgroup, and nilpotent length of G, respectively. By Corollary 1.2, $G/O_{r'r}(G)$ is nilpotent for every r dividing G. Thus $M(G) \subseteqq O_{r'r}(G)$ and in general

$$M(G) \subseteqq \bigcap_{f \mid |G|} O_{r'r}(G) = F(G) ,$$

(where the intersection is taken over all primes, r, dividing $|G|$) whence $M(G)$ is nilpotent. Thus $M^2(G) = E$ and so $n(G) \leqq 2$.

COROLLARY 1.4. *Let G be a solvable group admitting V as a fixed point free group of operators, where $|V| = pq$ is prime to $|G|$. Then G has π-length at most one, where π is any collection of primes dividing $|G|$.*

Proof. Since $F(G) = O_\pi(F(G)) \times O_{\pi'}(F(G))$, $F(G)O_{\pi'}(G)/O_{\pi'}(G)$ is a normal π-subgroup of $G/O_{\pi'}(G)$ whence $F(G) \subseteqq O_{\pi'\pi}(G)$. Since, by Corollary 1.3, $n(G) \leqq 2$, $G/F(G)$ is nilpotent and thus its factor group $G/O_{\pi'\pi}(G)$ is also nilpotent. But in this case, $G/O_{\pi'\pi}(G)$, being a nilpotent group with no normal π-groups, is itself a π'-group. Thus $O_{\pi'\pi\pi'}(G) = G$ and so G has π-length at most one.

3. Nilpotence of the commutator subgroup in groups admitting S_3 as a fixed point free group of operators. Let G be a group of operators, V, isomorphic to S_3, the symmetric group of degree three. Then V is a metacyclic group of the type disscussed in the previous section, with $p = 2$ and $q = 3$. Our object is to show that if V acts in fixed point free manner on G and G is solvable of order prime to 6, then G' is nilpotent. This property is almost entirely the consequence of

THEOREM 2. *Let G be a group of order prime to six admitting $V = S_3$ as a fixed point free group of operators. Let V be generated by elements w and v such that $v^2 = w^3 = 1$, $vw^2 = wv$. Set $v_1 = v$, $v_2 = vw$ and $v_3 = vw^2$ (all conjugates in V). Suppose G contains three normal subgroups, N_1, N_2, and N_3 such that*

(i) $N_1 \cap N_2 \cap N_3 = E$
(ii) N_i *is* v_i*-invariant,* $i = 1, 2, 3,$
(iii) $w^2(N_i) = N_{i+1 (\mathrm{mod}\, 3)}$ $i = 1, 2, 3,$
(iv) G/N_i *is fixed elementwise by* v_i, $i = 1, 2, 3.$
Then G *is abelian.*

Proof. The reader will recognize that (i)—(iv) is the condition (3) in (*) imposed on the subgroup G_2 in Theorem 1, for the case that $p = 2$ and $q = 3$. Then by Lemma 2.2, G is nilpotent, since it is fixed point free under the automorphism, w, of order three. Then, by a theorem of B. H. Neumann G has nilpotent class 2, i.e. $G' \subseteqq Z(G)$.

Now let H be an arbitrary V-invariant subgroup of G. Then V is fixed point free on H. We now show that the hypotheses (i)—(iv) inherit to H. Set $N_i' = H \cap N_i$, $i = 1, 2, 3$. Then $N_1' \cap N_2' \cap N_3' = H \cap (N_1 \cap N_2 \cap N_3) = E$, proving (i). Clearly N_i' is v_i-invariant, being the intersection of two v_i-invariant subgroups of G. Also, $w^2(N_i') = w^2(H \cap N_i) = w^2(H) \cap w^2(N_i) = H \cap N_{i+1} = N_{i+1}'$ (indices taken mod 3). Thus (ii) and (iii) hold. Finally, H/N_i' is v_i-isomorphic to HN_i/N_i, a subgroup of G/N_i. Since the latter is fixed elementwise by v_i, so is the former, proving (iv).

Since G is nilpotent, each of its Sylow subgroups admit V and satisfy (i)—(iv). If G is not a p-group, each of these is proper and, by induction, is abelian. Then their direct product, G, is also abelian. Thus, without loss of generality, we may assume that G is a p-group.

Let F denote the Frattini factor group, $G/\varnothing(G)$. Then F is a V-module, and since $p \nmid |V|$, by Maschke's theorem, F is a direct sum of irreducible V-modules: $F = F_1 \oplus F_2 \oplus \cdots \oplus F_t$. Let G_i be chosen so that $G_i/\varnothing(G) = F_i$. If $t > 1$, each G_i is a proper V-invariant subgroup of G and hence is abelian. In that case $G \supseteqq C_G(\varnothing(G)) \supseteqq \{G_1, \cdots, G_t\} = G$ whence $\varnothing(G) \subseteqq Z(G)$, the center of G. If, moreover, $t > 2$, each of the subgroups G_iG_j is a proper V-invariant subgroup of G and hence is also abelian. In that case,

$$G \supseteqq C_G(G_i) \supseteqq \{G_1, \cdots, G_i, \cdots, G_t\} = G$$

so each G_i lies in the center of G, whence G, which is generated by the G_i, is itself abelian, and we are done. Thus without loss of generality we may assume $t \leqq 2$.

Let us take a closer look at the irreducible V-modules, F_i. These are modules over the field of p elements. The kernel of the representation of V which each affords, is a normal subgroup of V and so is either the identity (in which case each module is faithful) or contains W, the normal subgroup of order 3. In the latter case, F contains points fixed by W. Then by Lemma 1.3 (b), $G_W \neq E$, a contradiction. Thus each F_i is faithful. In this case, we can show that each F_i is, indeed, 2-dimensional.

First, we observe that if $p \equiv 2 \bmod 3$.

$$\text{(A)} \qquad v \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

is a faithful irreducible representation of V. Second, if $p \equiv 1 \bmod 3$, there exists an integer $a \not\equiv 1 \bmod p$ such that $a^3 \equiv 1 \bmod p$. Then

$$\text{(B)} \qquad v \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w \longrightarrow \begin{pmatrix} a & 0 \\ 0 & a^2 \end{pmatrix}$$

(where $a < 3$ is taken as an element of $GF(p) = Z/(p)$ is also a faithful irreducible representation of V of dimension two. Now each F_i is isomorphic, as a $GF(p)V$-module, to a minimal left ideal of the semisimple group algebra $GF(p)V$ of dimension 6. But the modules corresponding to the trivial representation, and to the representation having W as its kernel are both 1-dimensional and thus account for two one-dimensional minimal left ideals in the direct decomposition of $GF(p)V$. This leaves a four-dimensional complement which must contain a two-dimensional minimal left ideal affording one or the other of the representations (A) and (B) given above. Since there are only three conjugate classes in V, these exhaust the nonisomorphic $GF(p)V$-modules. Thus each of the F_i afford representations equivalent to one of the two matrix representations (A) and (B) given above.

Since $G' \subseteqq Z(G)$, commutators in G obey the following laws:

$$\text{(1)} \qquad \begin{aligned} (x, yz) &= (x, y)(x, z) \\ (xy, z) &= (x, z)(y, z) \\ (x^i, y^j) &= (x, y)^{i+j} \\ (x, y^{-1}) &= (x, y)^{-1} = (x, y) = (x^{-1}, y) \, . \end{aligned}$$

Now suppose $t = 1$. We can no longer assert that $\varnothing(G)$ lies in the center of G, although $\varnothing(G)$ is certainly abelian. Here $G/\varnothing(G) = F$ is two-dimensional, and so G is generated by two elements, say x_1 and x_2. Thus if g and h are arbitrary elements in G, each can be expressed as "words" in x_1 and x_2, i.e.,

$$g = x_1^{a_1} x_2^{b_1} x_1^{a_2} x_2^{b_2} \cdots x_1^{a_n} x_2^{b_n}$$

$$h = x_1^{c_1} x_2^{d_1} \cdots x_1^{c_m} x_2^{d_m} .$$

Then from (1)

$$\begin{aligned}(g, h) &= (x_1, h)^{\Sigma a_i}(x_2, h)^{\Sigma b_i} \\ &= (x_1, x_2)^{(\Sigma d_i)(\Sigma a_i)-(\Sigma c_i)(\Sigma b_i)}\end{aligned}$$

whence G' is cyclic. But in that case $G'/\varnothing(G')$ is a one-dimensional $GF(p)V$-module, and hence is fixed by w. Since w is fixed point free on G, it is also on $G'/\varnothing(G')$ whence $G'/\varnothing(G') = E$, i.e. $G' = \varnothing(G') = E$. Thus G is abelian.

We are left with the case that $t = 2$. Here $F = F_1 \oplus F_2$, and $\varnothing(G)$ lies in the center of G. Then the commutators (x, y) all have order p, for $(x, y)^p = (x, y^p) = 1$ since $y^p \in \varnothing(G) \subseteq Z(G)$. Thus G' is elementary abelian and can also be regarded as a V-module. Commutation now defines a V-homomorphism: $F \times F \rightarrow G'$, which, being bilinear in each component, can be factored through $F \otimes_V F$. Thus if x and y belong to the same left coset of $\varnothing(G)$ in G, $x = yz$ for some z in the center. Then $(x, g) = (yz, g) = (y, g)$ and similarly, $(g, x) = (g, yz) = (g, y)$, so the map is well defined in the sense that $F \times F$ can be regarded as its domain. Since, for any $u \in V$, $u(x, y) = (u(x), u(y))$, the map is a V-homomorphism. For convenience we write the elements of the modules, F and G', additively so that $(x, y + z) = (x, y) + (x, z)$ and $(x + y, z) = (x, z) + (y, z)$.

Now suppose $p \equiv 2 \bmod 3$. Then both of the modules F_1 and F_2 afford representations equivalent to (A). Thus we may select a basis $\{x_1, x_2, x_3, x_4\}$ for F such that

$$\begin{aligned} v(x_1) &= x_2 & v(x_3) &= x_4 \\ w(x_1) &= x_2 & w(x_3) &= x_4 \\ w(x_2) &= -x_1 - x_2 & w(x_4) &= -x_3 - x_4 . \end{aligned}$$

Let $\hat{x}_1$ and $\hat{x}_3$ be elements of G such that under the homomorphism $f\colon G \rightarrow G/\varnothing(G)$, $f(\hat{x}_1) = x_1$, $f(x_3) = \hat{x}_3$. Then $f(w(\hat{x}_1)) = x_2$ and $f(w(\hat{x}_3)) = x_4$. Then G is itself generated by $\hat{x}_1$, $w(\hat{x}_1)$, $\hat{x}_3$ and $w(\hat{x}_3)$. The groups G_i, chosen so that $f(G_i) = F_i$, $i = 1, 2$, are abelian, whence $(\hat{x}_1 w(\hat{x}_1)) = 1$ and $(\hat{x}_3, w(\hat{x}_3)) = 1$. Thus in module notation

$$(x_1, x_2) = (x_3, x_4) = 0$$

and G' is generated by the four elements

$$(x_i, x_j) , \qquad i = 1, 2,; j = 3, 4 .$$

Now

$$(2) \qquad w^2(x_1, x_3) = w(x_2, x_4) = (-x_1 - x_2, -x_3 - x_4) = \Sigma(x_i, x_j)\,, \quad i = 1, 2; j = 3, 4.$$

Since w is fixed point free on G', for any element $c \in G'$,

$$(1 + w + w^2)c = c + w(c) + w^2(c) = 0\,.$$

Setting $c = (x_1, x_3)$, we have from (2)

$$(3) \qquad 2(x_1, x_3) + 2(x_2, x_4) + (x_1, x_4) = (x_2, x_3) = 0\,.$$

Similarly, setting $c = (x_1, x_4)$, we obtain

$$(4) \qquad \begin{aligned}(1 + w + w^2)(x_1, x_4) &= 0 \\ &= -(x_1, x_3) - (x_2, x_4) + (x_1, x_4) - 2(x_2, x_3)\,.\end{aligned}$$

Solving for (x_1, x_4) in (4) and substituting for (x_1, x_4) in (3), we obtain

$$3(x_1, x_3) + 3(x_2, x_4) + 3(x_2, x_3) = 0$$

or

$$(5) \qquad (x_1, x_3) + (x_2, x_4) + (x_2, x_3) = 0\,.$$

Adding (4) to (5) yields

$$(6) \qquad (x_1, x_4) = (x_2, x_3)\,.$$

Thus G' is at most two-dimensional, and from (5) and (6) is generated by (x_1, x_3) and (x_2, x_4).

Now N_1 is a normal subgroup of G and contains (v, G). Then $v(x_i)x_i^{-1} \in N_1$ for $i = 1, 2, 3, 4$. Since N_1 is normal, $(h, g) \in N_1$ for any $h \in N_1$ and $g \in G$. Thus the commutators $(v(x_1)x_1^{-1}, x_3)$ and $(v(x_1)x_1^{-1}, w(x_3))$ lie in $N_1 \cap G'$. Thus

$$(x_2 - x_1, x_3) = (x_2, x_3) - (x_1, x_3)$$

and

$$\begin{aligned}(x_2 - x_1, x_4) &= (x_2, x_4) - (x_1, x_4) \\ &= (x_2, x_4) - (x_2, x_3)\,,\end{aligned}$$

by (6), all belong to $N_1 \cap G'$. Thus

$$(x_2, x_3) \equiv (x_2, x_4) \equiv (x_1, x_4) \equiv (x_1, x_3) \bmod N_1 \cap G'\,.$$

Then from (5) $3(x_1, x_3) \equiv 3(x_2, x_4) \equiv 3(x_2, x_3) \equiv 0 \bmod N_1 \cap G'$. Since $p \nmid 3$, (x_1, x_3) and (x_2, x_4), the generators of G', both lie in N_1. Thus $N_1 \supseteqq G'$. Then $N_2 = w^2(N_1) \supseteqq G'$ and $N_3 = w(N_1) \supseteqq G'$. Since the N_i have trivial meet, $G' = E$ and G is abelian.

Now suppose $p \equiv 1 \bmod 3$. Then the two irreducible V-modules, F_1 and F_2, afford representations of V equivalent to that given in (B). Thus we may select a basis, $\{x_1, x_2, x_3, x_4\}$ for which

$$(7) \qquad \begin{array}{ll} v(x_1) = x_2 & v(x_3) = x_4 \\ w(x_1) = ax_1 & w(x_3) = bx_3 \\ w(x_2) = a^2x_1 & w(x_4) = b^2x_4 \end{array}$$

where a and b are scalars in $GF(p)$, different from 1, and satisfying $a^3 = b^3 = 1$. Now the multiplicative group of nonzero elements in $GF(p)$ is cyclic and so has a unique subgroup of order 3. Thus, since a and b both belong to this subgroup, either $a = b$ or $a = b^2$.

Let $\hat{x}_1$ and $\hat{x}_3$ be chosen so that $f(\hat{x}_i) = x_i$, $i = 1, 3$. Then, setting $\hat{x}_2 = v(\hat{x}_1)$ and $\hat{x}_4 = v(\hat{x}_3)$, $f(\hat{x}_j) = x_j$ for $j = 2, 4$. If G_i is chosen so that $G_i/\varnothing(G) = F_i$, $i = 1, 2$, then, by induction, the G_i are abelian. From this and (7) we have

$$(8) \qquad \begin{aligned} (x_1, x_2) &= (x_3, x_4) = 0 \\ w(x_1, x_3) &= ab(x_1, x_3) \\ w(x_1, x_4) &= ab^2(x_1, x_4) \\ w(x_2, x_3) &= a^2b(x_2, x_3) \\ w(x_2, x_4) &= a^2b^2(x_2, x_4) \, . \end{aligned}$$

If $a = b$, (x_1, x_4) and (x_2, x_3), being fixed by w, must be zero. If $a = b^2$, (x_1, x_3) and (x_2, x_4) are zero. In either case, G' is generated by two elements. By interchanging the symbols representing x_3 and x_4 if necessary, we can, without loss of generality assume that $a = b$ so that $(x_1, x_4) = (x_2, x_3) = 0$.

Since $v(x_1)x_1^{-1} \in N_1$, $(x_2x_1^{-1}, x_j) \in N_1 \cap G'$ for $j = 3, 4$. Thus

$$\begin{aligned} &(x_2 - x_1, x_3) = (x_2, x_3) - (x_1, x_3) = (x_2, x_3) \equiv 0 \\ &(x_2 - x_1, x_4) = (x_2, x_4) = (x_1, x_4) = -(x_1, x_4) \equiv 0 \bmod G' \cap N_1 \, . \end{aligned}$$

Since $G' = \{(x_1, x_3), (x_1, x_4)\}$, $N_1 \supseteqq G'$ whence $G' \subseteqq N_1 \cap w(N_1) \cap w^2(N_1) = E$. Thus G is abelian.

COROLLARY 2.1. *Let G be a group admitting $V = S_3$ as a fixed point free group of operators and suppose G has order prime to $|V| = 6$. Then the commutator subgroup of G is nilpotent.*

Proof. The property that G' is nilpotent is residually complete, and so, since V is fixed point free on each factor group, we obtain immediate reduction to the case that G has a unique minimal normal V-invariant subgroup, M. Since G is solvable, $M \subseteqq O_p(G)$ for some

p dividing G, and $O_{p'}(G) = E$. Thus $O_{p'}(G) = O_{p'p}(G)$. Now V is metacyclic of order 6. Since G is odd, the restriction on Fermat primes does not apply, and hence we may use Corollaries 1.1 and 1.2 to obtain that $\bar{G} = G/O_{p'p}(G)$ satisfies (*). Then $\bar{G} = G_1 \times G_2$ where G_1 is fixed point free under v, an automorphism of order 2, and G_2 is a group satisfying conditions (i)—(iv) of Theorem 2. Thus both G_1 and G_2 are abelian, whence $\bar{G}$ is abelian. Thus $G' \subseteqq O_{p'p}(G) = O_p(G)$, which shows that G' is a p-group and hence is nilpotent.

4. Nilpotence of the commutator subgroup in groups admitting fixed point free operator groups. In this section we prove the impossibility of extending the results of Corollary 2.1 to solvable groups, V, other than those already considered. We begin with

THEOREM 3. *Let V be a solvable group satisfying one of the following properties*

(a) *V contains a normal subgroup $W \neq E$ such that $[V: W]$ is an odd prime, p.*

(b) *V has a factor group of order 4, $|V| \neq 4$.*

(c) *V has a dihedral factor group of order $2p$, $p \geqq 5$.*

Then there exists a group, G, having order prime to $|V|$ which admits V as a fixed point free group of operators and for which the commutator subgroup is not nilpotent.

Proof. *Case I.* (*V satisfies* (a)).

Since V is solvable, W' is a proper subgroup of W, normal in G. Select U maximal with respect to the properties: $W' \subseteqq U \subset W$, and $U \lhd V$. Then V/U is either abelian of order p^2 or pq or it is metabelian of order pq^e where e is the exponent of $p \bmod q$ defined by letting an element of order p act irreducibly on the elementary abelian group (W/U) of order q^e.

Let G be a group of order r^3s^{rp} having a normal elementary abelian subgroup, A, of order s^{rp} and factor group G/A isomorphic to the extraspecial group of order r^3. The primes, r and s are chosen so that $r \equiv 1 \bmod p$ and $s \equiv 1 \bmod rq$ (or rp if $[V : U] = p^2$). Since $r \neq s$, G splits over A and we may write $G = AR$ where R is generated by two elements x and y such that $x^r = y^r = 1 = z^r$, where $z = (x, y)$ generates the center of R.

V acts on G as follows: First U acts trivially on G, and W acts trivially on R. If v generates $V \bmod W$, set $v(x) = x^a$, $v(y) = y^a$, $v(z) =$

z^{a^2} where a is a primitive pth root mod r. (Such a root exists since $r \equiv 1 \bmod p$.) The action of V and R on A is defined by writing the elements of A additively, selecting a basis $a_{11}, a_{12}, \cdots, a_{ij}, \cdots, a_{rp,rp}$ for A, and letting $\{a_{11}, a_{12}, \cdots, a_{1r}\}$ afford the representation ρ_1 of $(W/U)R$, defined by

$$\rho_1(z) = \beta I_r \,, \qquad \rho_1(w_1) = \gamma I_r \,, \qquad \rho_1(w_i) = I_r \,, \qquad i > 1$$

where $w_1, \cdots, w_e$ are a basis for W/U ($e = 1$, and the last matrix is not involved if $[W : U] = q$ or p.) I_r is the r by r identity matrix, and β and γ are respectively primitive rth roots and qth roots (or pth if $[W : U] = p$) modulo s. Also,

$$\rho_1(x) = diag(1, \beta^{-1}, \cdots, \beta^{-r+1})$$

and

$$\rho_1(y) = \begin{pmatrix} 0 & 1 & & 0 \\ & & 1 & \ddots \\ & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

If H denotes the semidirect product $(V/U)R$, set

$$\rho(h) = diag(\rho_1(h), \rho_1(vhv^{-1}), \cdots, \rho_1(v^{p-1}hv^{1-p}))$$

for $h \in (W/U)R$, and

$$\rho(v) = \begin{pmatrix} 0 & I_r & 0 & \cdots & 0 \\ & & I_r & \ddots & \\ & & & \ddots & I_r \\ uI_r & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $u = \gamma$ if V/U is cyclic of order p^2 and 1 otherwise. This completely defines the action of VR on A.

W acts in fixed point free manner on A since, on each component $\{a_{i1}, a_{i2}, \cdots, a_{ir}\}$ it is represented as scalar multiplication by γ, (the kernel of the representation of W/U may differ on each component, of course.) Since p is odd, $a^2 \equiv 1 \bmod r$, and so v acts in fixed point free manner on R. Summing up, then, V is fixed point free on $G/A \simeq R$ whence $G_v \subseteqq A$. Again, $G_v \subseteqq A_v = E$, whence V is fixed point free on G.

Note that $G' = A\{z\}$ is not nilpotent.

Case II. V satisfies (b).

Let W be the nontrivial subgroup of order 4, and select U so that $W' \subseteqq U \subset W, U \lhd V$ and U is maximal in this respect. We shall define a group, G, admitting V as a group of operators in such manner that U acts trivially on G. G will have the form QM, where M is an elementary abelian normal subgroup of G and Q is a Hall complement of M in G, such that M becomes a sum of faithful irreducible Q-modules, so that $[Q, M] = M$. Moreover, M will be fixed point free under the action of W alone, while at the same time Q will be V-invariant and centralized by W. In this way, V induces the group V/W of order 4 on the group Q. Now V/U has the normal series, $E = U/U \lhd W/U \lhd V_0/U \lhd V/U$, and we select elements v_1 and v_2 in V so that v_1 generates $V_0 \bmod W$ and v_2 generates $V \bmod V_0$ and $v_2^2 \equiv 1$ or $v_1 \pmod W$ according as V/W is the 4-group or is cyclic. Now we define Q as follows. Let r and s be odd primes, such that $(rs, [V:U]) = 1$ and $s \equiv 1 \bmod r$. Then Q is a group of order rs^2 defined by

$$x^r = 1 = y_1^s = y_2^s$$
$$x^{-1}y_1x = y_1^a,\ \overset{-1}{x}\,y_2x = y_2^{a-1}$$

where a and a^{-1} are primitive rth roots mod s, such that $aa^{-1} \equiv 1 \bmod s$. Then the action of V/W on Q is defined by

$$v_2(x) = x^{-1},\ v(x) = x \text{ for all } v \in V_0\,.$$
$$v_1(y_i) = y_i^{-1},\ i = 1, 2$$
$$v_2(y_1) = y_2 \text{ and } v_2(y_2) = \begin{matrix} y_1 \text{ if } v_2^2 \equiv 1 \bmod W \\ y_1^{-1} \text{ if } v_2^2 \equiv v_1 \bmod W \end{matrix}$$

W acts trivially on Q. Then it is easily seen that Q is fixed point free under V/W and has nilpotent length 2.

Now W/U has order 2, p or p^2, where p is an odd prime, and acts as an irreducible V/W-module. We can then find a factor system, $m_{ij} \in W/U$, such that $v_iv_j \equiv um_{ij}, \bmod U$ where u is the appropriate coset representative, $1, v_1, v_2, v_1v_2$, of W/U in V/U. Now let t be a prime different from 2, r and s such that $t \equiv 1 \bmod p$ if W/U has order p or p^2. Let M_1 be a faithful irreducible Q-module over $GF(t)$. We make M_1 into an irreducible $(U/W)Q$-module by letting W/U act nontrivially by scalar multiplication by -1 or by a pth root according as $[W:U] = 2$ or is odd. Then, as W acts trivially on Q, $WQ \lhd VQ$. Set

$$M = M_1^{VQ} \simeq M_1 \oplus v_1M_1 \oplus v_2M_1 \oplus v_1v_2M_1 \simeq M_1 \otimes_{WQ} VQ$$

the induced module. Then W has a conjugate representation which is also scalar multiplication on each component xM_1, $x = 1, v_1, v_2$, or v_1v_2. Now set G to be the semidirect product QM, where M is a normal abelian subgroup on which VQ acts in a manner prescribed by the module construction of M. Then M has order prime to $|VQ|$, and is a sum of conjugate faithful irreducible Q-modules, whence $[Q, M] = M$. Thus M is the Fitting subgroup of G and G has nilpotent length three. V/U has order prime to G, V is fixed point free on both $Q \simeq G/M$ and M and hence is fixed point free on G, by the remark following Lemma 1.2. Evidently G' is not nilpotent.

Case III. V satisfies (c).

V contains a normal subgroup U such that V/U is dihedral of order $2p$, where $p \geqq 5$.

In this example we let $G = RQ$ where R is a normal elementary abelian r-group and Q is a special q-group of order q^8. We select the primes q and r so that $r \equiv 1 \bmod q$ and $q \equiv 1 \bmod p$, both q and r odd. Q is generated by four elements x_1, x_2, x_3, x_4 subject to the rules:

$$\begin{gathered} x_i^q = 1,\ i = 1, 2, 3, 4\ , \qquad (x_1, x_2) = (x_3, x_4) = 1 \\ z_1 = (x_1, x_3),\ z_2 = (x_1, x_4),\ z_3 = (x_2, x_3),\ z_4 = (x_2, x_4) \\ z_i^q = 1,\ i = 1, 2, 3, 4\ , \qquad \{z_1, z_2, z_3, z_4\} = Z(Q)\ . \end{gathered}$$

R will consist of p irreducible Q-modules, each of which has dimension q. Thus R has order r^{pq}.

The action of V on G is defined as follows: First, U is assumed to act trivially on G so that G admits $\bar{V} = V/U$, the dihedral group of order $2p$, as a group of operators. Let V be generated by elements v and w such that $v^2 = 1$, $w^p = 1$, and $vw = w^{-1}v$. Since $p \geqq 5$, we can find four primitive pth roots modulo q, a, a^{-1}, b, b^{-1}, such that $aa^{-1} \equiv bb^{-1} \equiv 1 \bmod q$ and b is incongruent to both a and $a^{-1} \bmod q$. Then we set

$$w(x_1) = x_1^a,\ w(x_2) = x_2^{a^{-1}},\ w(x_3) = x_3^b,\ w(x_4) = x_4^{b^{-1}}\ .$$

Then w acts in fixed point free manner on $Q/Z(Q)$. We must also have

$$w(z_1) = z_1^{ab},\ w(z_2) = z_2^{ab^{-1}},\ w(z_3) = z_3^{a^{-1}b},\ w(z_4) = z_4^{a^{-1}b^{-1}}\ .$$

The action of v is given by

$$\begin{gathered} v(x_1) = x_2,\ v(x_2) = x_1,\ v(x_3) = x_4,\ v(x_4) = x_3 \\ v(z_1) = z_4,\ v(z_2) = z_3,\ v(z_3) = z_2,\ v(z_4) = z_1 \end{gathered}$$

so that both $Q/Z(Q)$ and $Z(Q)$ are the direct sum of two irreducible V-modules. Form the subgroup $Q_1 = (v, Q)$. This subgroup is generated by the elements $x_1x_2^{-1}$, $x_3x_4^{-1}$, $z_1z_4^{-1}$, $z_2z_3^{-1}$, and $(x_1x_2^{-1}, x_3) = z_1z_3^{-1}$, and has order q^5. Then $Q/Z(Q)$ is an extra special q-group of order q^3, generated $(\bmod Q_1)$ by the elements x_1x_2 and x_3x_4, with center generated $(\bmod Q_1)$ by $z_1z_2z_3z_4$. Q/Q_1 is fixed elementwise by v.

Writing R additively we may regard R as a QV-module. We take R to be the direct sum of p irreducible Q-modules, $R_1, \cdots, R_p$, which are permuted by V according to the rules

$$w^{-1}R_jw = R_{j+1} \qquad \text{(indices take mod } p\text{)}$$
$$v^{-1}R_1v = R_1,\ v^{-1}R_2v = R_p,\ v^{-1}R_3v = R_{p-1},\ \cdots,\ v^{-1}R_{(p-1)/2}v = R_{(p+1)/2}$$

so that the manner in which the R_i are permuted by V provides a faithful transitive permutation representation of V of degree p. v is assumed to act on R_1 by scalar multiplication by -1. Q is represented irreducibly on R_1 with kernel Q_1, that is R_1 represents Q/Q_1 faithfully. The matrices are

$$\rho(x_1) = \rho(x_2) = \operatorname{diag}(1, c^{-1}, \cdots, c^{-q+1})$$
$$\rho(x_3) = \rho(x_4) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$
$$\rho(z_1) = \rho(z_2) = \rho(z_3) = \rho(z_4) = cI_q$$

where c is a primitive qth root modulo r, and I_q denotes the q by q identity matrix. By defining the representation of Q on R_i as the conjugate representation under w^{i-1}, the representation of QV on R is completely defined.

We next observe that V is fixed point free on G. First $w^{-i}vw^i = vw^{2i}$ leaves $R_{i+1} = w^{-i}R_1w^i$ invariant and for any element, e, in R_{i+1}, we have $e = w^{-i}e, w^i$ for some e' in R_1. Then

$$\begin{aligned}(w^{-i}vw^i)^{-1}e(w^{-i}vw^i) &= (w^{-i}v^{-1}w^i)w^{-i}e'w^i(w^{-i}vw^i) \\ &= w^{-i}v^{-1}e'vw^i = w^{-i}(-e')w^i \\ &= -w^{-i}e'w^i = -e\ .\end{aligned}$$

Thus vw^{2i} acts on R_{i+1} by scalar multiplication by -1. Now suppose h is an element of R fixed by V. Then we may write h uniquely in the form $h = h_1 + h_2 + \cdots + h_p$, where $h_i \in R_i$. Then $w^{-i}vw^{-i}$ sends each $h_j \in R_j$ into some element of R_k where $k \neq j$ unless $j = i + 1$. But since it fixes h it must fix h_{i+1}. On the other hand it acts on

R_{i+1} by scalar multiplication by -1. Since $r \neq 2$, this implies that $h_{i+1} = 0$. Repeating this argument for each $i = 1, 2, 3, \cdots, p$, we have that $h = 0$. Thus V is fixed point free on R. But V is also fixed point free on $Q/Z(Q)$ and $Z(Q)$ whence it must be fixed point free on all of G.

In each case $Z(Q)$ is represented on R_i by scalar multiplication by c; thus $(Z(Q), R) = R \subseteqq G'$ and $Z(Q) = Q' \subseteqq G'$, whence G' contains the subgroup $RZ(Q)$, which is not nilpotent.

COROLLARY 3.1. *Let V be a solvable group containing a nontrivial subgroup W such that W is normal in V and V/W is the symmetric group of degree three. Then there exists a group G having order prime to $|V|$, admitting V as a fixed point free group of operators, such that G' is not nilpotent.*

Proof. This case is not directly subsumed under those cases listed in Theorem 3, but the required example is easily provided by that theorem. Let V^* be the unique subgroup of index 2 in V containing W. Then V^* is a solvable group containing a nontrivial normal subgroup, namely W, of index 3, which is an odd prime. Thus V^* satisfies (a) of Theorem 3. Accordingly, there exists a group G_1 having order prime to V^* which admits V^* as a fixed point free group of operators and which has a commutator subgroup which is not nilpotent. Let the element v generate V module V^*, so that v^2 is an element of V^*. Also, let G_2 be an isomorphic copy of G_1 and let f be the isomorphism $f: G_1 \rightarrow G_2$. Then if $H = G_1 \times G_2$, the action of V on H can be defined as follows: G_1 already admits V^*. Let $v(g) = f(g)$ for every $g \in G_1$, and $v(g) = v^2(k)$, where $g = f(k)$, for every $g \in G_2$. The latter is well defined since f is onto and one to one (making k unique) and v^2 is an element of V^* whose action on G_1 is already known. For any $u \in V^*$, and $g \in G_2$ we define $u(g)$ to be $f[(v^{-1}uv)(k)]$ where $f(k) = g$; thus, writing v for f when the domain of v is in G_1, this becomes $v[v^{-1}uv(v^{-1}(g))] = u(g)$ so that V^* can in this way be regarded as a group of operators applying to H. Clearly V^* acts in fixed point free manner on the subgroup G_2 since, if $f(k) = g$ and V^* fixes g, $v^{-1}V^*v = V^*$ must fix k, which is impossible unless k (and hence g) is the identity. Thus V acts in fixed point free manner on $H = G_1 \times G_2$. Now by hypothesis G_1' is not nilpotent, and hence its isomorphic copy, G_2' is also not nilpotent. But it is obvious that both of these subgroups lie in H', whence H' is not nilpotent.

THEOREM 4. *Let V be a solvable group with the property that* *if G is any group admitting V as a fixed point free group of operators and G has order prime to $|V|$, then G' is always nilpotent. Then V is one of the following groups:*

(i) *V is cyclic of prime order*

(ii) *V is one of the groups of order 4*

(iii) *V is the symmetric group of degree three.*

Proof. Case I. V has a factor group of odd prime order.

If $W \lhd V$ and V/W has odd prime order, then by Theorem 3, since V satisfies (a) if $W \neq E$, we must suppose that V is cyclic of prime order.

Case II. V has no factor groups of odd prime order.

Since V is solvable, it contains a normal subgroup V_1 of prime index, and because of the case division the prime must be 2. If $V_1 = E$, V is cyclic of order 2, and so V enjoys (i). Thus we may take $V_1 \neq E$. Select V_2 maximal with respect to being a proper subgroup of V_1 and normal in V. Then, since V is solvable, V_1/V_2 is elementary abelian, and is irreducible as a V/V_1-module. If $[V_1 : V_2]$ is a power of 2 it is in fact equal to 2, so V/V_2 is a group of order 4. Then if $V_2 \neq E$, V satisfies (b) and so by Theorem 3, we would be confronted with a counter example to the hypothesis of this theorem. Thus we must suppose, in this case, that $V_2 = E$, whence (ii) holds. If $[V_1 : V_2]$ is not a power of 2, it is an odd prime, p. If $p \geqq 5$, by Theorem 3, the hypothesis would be denied. Thus $p = 3$. Then if $V_2 \neq E$, by Corollary 3.1, the hypothesis is once more denied. Consequently, $V_2 = E$ and V is the symmetric group of degree three.

COROLLARY 4.1. *The condition that V is solvable can be dropped in Theorem 4.*

Proof. Let T be a proper subgroup of V and suppose that T is not cyclic of prime order, does not have order 4 and is not isomorphic to S_3, the symmetric group of degree 3. Let $1 = x_1, x_2, \cdots, x_k$ be a full set of right coset representatives of T in V. By induction on the order of T, there exists a group, G_1, fixed point free under T such that G_1' is not nilpotent. Let G be the formal set of k-tuples $(g_1, x_2^{-1}g_2x_2, \cdots, x_k^{-1}g_kx_k)$, where the g_i lie in G_1. Under the rule that $x_ix_i^{-1} = 1, G$ becomes a group (under component-wise multiplication) isomorphic to a direct product of k copies of C_1. By defining $t^{-1}gt = g^t$ for all $t \in T$ and $g \in G_1$ (the exponential notation indicating that t

acts as an operator in the manner given by the induction hypothesis), the action of V on G is defined by componentwise conjugation. Then it is easy to verify that V is fixed point free on G, and that G' is not nilpotent. We may thus suppose that any proper subgroup of V is either cyclic of prime order, has order 4, or is S_3.

By Theorem 4, it now suffices to show that V is solvable. Assume V is not solvable. Then a 2-Sylow subgroup, S_2, of V, being proper, has order 2 or 4. If $N_V(S_2) = V$, V/S_2 is metacyclic and hence V is solvable. If $N_V(S_2) < V$, it is clear that since A_4 is not a proper subgroup of V, S_2 lies in the center of its normalizer and so V has a normal 2-complement, K, which is also metacyclic and hence is solvable. Thus V is solvable, contrary to our assumption.

REFERENCES

1. S. Bauman, *The Klein group as a fixed point free automorphism group*, Ph. D. Thesis, University of Illinois, 1962.
2. A. H. Clifford, *Representations induced in an invariance subgroup*, Ann. of Math. **38** (1937), 533-550.
3. G. Glauberman, *Fixed point in groups with operator groups*, Math. Zeit. **84** (1964), 120-125.
4. D. Gorenstein and I. N. Herstein, *Finite groups admitting a fixed point free automorphism of order 4*. Amer. J. Math. **83** (1961), 71-78.
5. F. Hoffman, *Nilpotent height of finite groups admitting fixed-point free automorphisms*, Math. Zeit. **85** (1964), 260-267.
6. E. Shult, *On groups admitting fixed point free abelian operator groups*, Ill. J. Math. **9** (1965), 701-720.
7. J. G. Thompson, *Finite groups with fixed point free automorphisms of prime order*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578-581.
8. ———, *Automorphisms of solvable groups*, J. of Alg. **1** (1964), 259-267.

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