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SIMPLE n-ASSOCIATIVE RINGS

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This paper is concerned with certain classes of nonassociative rings. These rings are defined by first extending the associator $(a, b, c) = (ab)c - a(bc)$. The n-associator $(a_1, \cdots, a_n)$ is defined by

$$(1.1)\quad \begin{aligned}&(a_1, a_2) = a_1a_2\ ,\\ &(a_1, \cdots, a_n) = \sum_{k=0}^{n-2}(-1)^k(a_1, \cdots, a_k, a_{k+1}a_{k+2}, \cdots, a_n)\ .\end{aligned}$$

A ring is defined to be n-associative if the n-associator vanishes in the ring. It is shown that simple 4-associative and simple 5-associative rings are associative; simple $2k$-associative rings are $(2k-1)$ associative or have zero center; and simple, commutative n-associative rings, $6 \leqq n \leqq 9$, are associative. The concept of rings which are associative of degree $2k+1$ is defined, and it is shown that simple, commutative rings which are associative of degree $2k+1$ are associative. The characteristic of the ring is slightly restricted in all but one of these results.

The concepts of the n-associator and n-associative rings were defined by A. H. Boers [1; Ch. 3 and Ch. 4]. Our results extend Boers' main result that an n-associative division ring is associative with minor restriction on the characteristic [1; Th. 6]. We do not consider 2-associative rings.

To obtain our results, it is necessary to extend the concept of the n-associator. In a ring R, define $S(2j+1,\ 2k+1)$, $1 \leqq j \leqq k$, by defining $S(2j+1,\ 2j+1)$ to be the set of all finite sums of $(2j+1)$-associators with entries in R, and then by defining $S(2j+1,\ 2k+1)$, $k > j$, to be the set of all finite sums of $(2j+1)$-associators $(a_1, \cdots, a_{2j+1})$ such that $(a_1, \cdots, a_{2j+1}) \in S(2j+1,\ 2k-1)$ and such that at least one of the $2k-1$ entries of $(a_1, \cdots, a_{2j+1})$ is in $S(3, 3)$. For example, $(((a_1, a_2, a_3), a_4, a_5), a_6, (a_7, a_8, a_9)) \in S(3, 9)$.

Clearly, a ring R is $(2n+1)$-associative if and only if $S(2n+1,\ 2n+1) = 0$ in R. This leads us to call a ring $R(2n+1)$-associative of degree $2k+1$ if $S(2n+1,\ 2k+1) = 0$ in R. No mention of degree will be made in case $k = n$.

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Of particular interest are rings which are associative (3-associative) of degree $2k+1$. In the first place, this in itself is an interesting extension of the concept of associativity. Consider the 4-dimensional algebra A over an arbitrary field with basis a_1, a_2, a_3, a_4 such that $a_1^2 = a_2$, $a_1a_2 = a_3 - a_4$, $a_2a_1 = a_3$, and all other products zero. It can be verified that $S(3, 5) = 0$ in A but that A is not associative. Also, it turns out that a ring which is associative of degree $2k+1$ is $(2k+1)$-associative, but not conversely.

2. Preliminaries. We will need the following three identities derived by Boers.

$$(2.1) \qquad (a_1, \cdots, a_n) = \sum_{k=1}^{n-3} (a_1, \cdots, a_k, (a_{k+1}, a_{k+2}, a_{k+3}), a_{k+4}, \cdots, a_n) \ ,$$

$$(2.2) \quad (a_1, \cdots, a_n) = \sum_{k=0}^{(1/2)n-1} \binom{(1/2)n-1}{k} (a_1, \cdots, a_{n-2k-1})\,(a_{n-2k}, \cdots, a_n)$$

for even n where $\binom{r}{s}$ denotes the binomial coefficient [1; Ch. 3]. In a commutative ring, we have

$$(2.3) \qquad (a_1, \cdots, a_n) = (-1)^{[(1/2)n-(1/2)]}(a_n, \cdots, a_1)$$

where $[x]$ denotes the greatest integer $\leqq x$ [2; Th. A].

Next, we will need

LEMMA 2.1. *$S(2j+1,\ 2k+1) \subset S(2m+1,\ 2n+1), 1 \leqq m \leqq j$, $m \leqq n \leqq k$.*

Proof. It is immediate from the definition of $S(2j+1,\ 2k+1)$ that $S(2j+1, 2k+1) \subset S(2j+1, 2n+1), j \leqq n \leqq k$. Hence we need only show that $S(2j+1, 2n+1) \subset S(2m+1, 2n+1), 1 \leqq m \leqq j$. The result is obvious if $j = m$. Assume that $S(2j-1,\ 2n+1) \subset S(2m+1, 2n+1), j > m$. Then by (2.1), $S(2j+1,\ 2n+1) \subset S(2j-1, 2n+1) \subset S(2m+1, 2n+1)$, and we are finished.

Let A and B be subsets of a ring. Define AB to be the set of all finite sums of elements of the form ab such that $a \in A$, $b \in B$.

Let $I(3, 2k+1) = S(3, 2k+1) + S(3, 2k+1)R$. The next lemma is a generalization of the fact that $I(3, 3)$ is an ideal of an arbitrary ring R [4; p. 985].

LEMMA 2.2. *$I(3, 2k+1)$ is a right ideal of an arbitrary ring R for $k = 1, 2, \cdots$.*

Proof. Let $I = I(3, 2k+1), S = S(3, 2k+1)$. We have

$$IR \subset SR + SR \cdot R \subset SR + (S, R, R) .$$

However, by Lemma 2.1, $(S, R, R) \subset S$. Hence $IR \subset I$.

The key to our results is

LEMMA 2.3. *If R is a simple, commutative ring, and if $A = \{a \in R \mid a\, S(3, 2k+1) = 0\}$, then $A = 0$ or $S(3, 2k+1) = 0$.*

Proof. We can assume that $R = I(3, 2k+1)$, for otherwise we are finished by Lemma 2.2. If K is the ideal generated by AR, then $K \subset S(3, 3)$. Indeed, $AR = AI(3, 2k+1) \subset S(3, 3)$. Define $R_x \colon y \to yx$. Assume that we have shown that $aR_{x_1} \cdots R_{x_n} \in S(3, 3)$ for $a \in A$ and for every choice of n elements $x_1, \cdots, x_n \in R$, $n > 1$. Then

$$\begin{aligned} aR_{x_1} \cdots R_{x_n} R_{x_{n+1}} &= ((aR_{x_1} \cdots R_{x_{n-1}})x_n)x_{n+1} \\ &= (aR_{x_1} \cdots R_{x_{n-1}}, x_n, x_{n+1}) + aR_{x_1} \cdots R_{x_{n-1}} R_{x_n x_{n+1}} \in S(3, 3) . \end{aligned}$$

If $K = 0$, then $AR = 0$ which implies $A = 0$ and we are finished. Hence assume $K = R$. Thus $R = S(3, 3)$ which implies $R = S(3, 2k+1)$ by induction since each element can now be replaced by a sum of associators. Therefore $AR = 0$, and hence $A = 0$.

To ease the computations of the proofs which follow, we define

$$T(n, i, j) = \sum_{m=i}^{j} (a_1, \cdots, a_m, (a_{m+1}, a_{m+2}, a_{m+3}), a_{m+4}, \cdots, a_n)$$

where $0 \leqq i \leqq j \leqq n-3$. Note that (2.1) becomes $(a_1, \cdots, a_n) = T(n, 0, n-3)$. The next lemma, whose proof follows easily from the definition of $T(n, i, j)$ (with the use of (2.3) in the case of part (d)), contains all the additional facts about $T(n, i, j)$ that we will need.

LEMMA 2.4.
(a) $T(n, i, j) - T(n, i, k) = T(n, k+1, j)$, $k < j$;
(b) $T(n, i, j) - T(n, k, j) = T(n, i, k-1)$, $i < k$;
(c) *$S(2k+1, 2k+3)$ consists of all finite sums of elements of the form $T(2k+3, i, i)$, $i = 0, 1, \cdots, 2k$; and*
(d) *if $0 = T(n, i, j)a$ is an identity in a commutative ring, then so is $0 = T(n, n-3-j, n-3-i)a$.*

We will not cite Lemma 2.4 when we use it.

Finally, the nucleus N of a ring R is defined by $N = \{u \in R \mid (u, x, y) = (x, u, y) = (x, y, u) = 0$ for all $x, y \in R\}$. The center C of R is defined by $C = \{c \in N \mid cx = xc$ for all $x \in R\}$.

3. n-Associative rings. In what follows, we will use the fact that (1.1) implies that if R is an n-associative ring, then R is k-associative for all $k \geqq n$.

THEOREM 3.1. *If R is a simple 4-associative or 5-associative ring of characteristic not 2, then R is associative.*

Proof. By (2.2) with $n = 6$, we have

$$0 = S(3, 3)^2 . \tag{3.1}$$

Since $I(3, 3)$ is an ideal of R, we can assume that $R = I(3, 3)$, for otherwise $S(3, 3) = 0$. Hence (3.1) yields $S(3, 3)R \subset S(3, 3)$. Therefore $R = S(3, 3)$, but then we have $S(3, 3)R = 0 = RS(3, 3)$ by (3.1). Thus $S(3, 3) = 0$.

Theorem 3.1 extends a result of Boers [3; p. 126] who has also shown it to be false for characteristic 2.

THEOREM 3.2. *Let R be a simple $2k$-associative ring for $k \geqq 2$. Then $C = 0$ or R is $(2k - 1)$-associative where C is the center of R.*

Proof. We first show that if N is the nucleus of R, then any $(2j + 1)$-associator with an entry $u \in N$ vanishes. Indeed, if $j = 1$, the result follows by the definition of N. Assume that we have established the result for $j = i$. By (2.1), $(a_1, \cdots, a_{2i+3}) = T(2i+3, 0, 2i)$. Each n-associator in $T(2i + 3, 0, 2i)$ is a $(2i + 1)$-associator. Hence if $u \in N$ is an entry of $(a_1, \cdots, a_{2i+3})$, then $(a_1, \cdots, a_{2i+3}) = 0$.

Now, use (2.2) with $n = 2k$. In the resulting identity, let $a_{2k} \in C$. Then, since $C \subset N$, we have $S(2k - 1, 2k - 1)\, C = 0$. Therefore $C = 0$ or $S(2k - 1, 2k - 1) = 0$ since R is simple and the annihilators of C may be shown to form an ideal of R.

Theorems 3.1 and 3.2 imply

COROLLARY 3.1. *If R is a simple 6-associative ring of characteristic not 2, then $C = 0$ or R is associative.*

We now turn our attention to commutative rings.

THEOREM 3.3. *If R is a simple, commutative 6-associative or 7-associative ring of characteristic not 2 or 3, then R is associative.*

Proof. By Theorem 3.1, it is sufficient to show that $S(5, 5) = 0$.

Let $n = 8$ and 10 in (2.2) to obtain

$$0 = (a_1, \cdots, a_5)(a_6, a_7, a_8) + (a_1, a_2, a_3)(a_4, \cdots, a_8) \tag{3.2}$$

and

$$0 = S(5, 5)^2. \tag{3.3}$$

Next, use (2.1) and (3.3) to obtain $T(5, 0, 2)S(5, 5) = 0$ which yields $T(7, 0, 2)S(3, 3) = 0$ upon application of (3.2). Hence we may assume that $T(7, 0, 2) = 0$, for otherwise $S(3, 3) = 0$ by Lemma 2.3. Using (2.1), we compute $0 = T(7, 0, 4) - T(7, 0, 2) = T(7, 3, 4)$; hence $T(7, 0, 1) = 0$ by (2.3). Thus $T(7, 2, 2) = 0$ since $T(7, 2, 2) = T(7, 0, 2) - T(2, 0, 1)$. Hence we have $T(5, 2, 2)S(5, 5) = 0$ using (3.2). Thus $S(3, 5)S(5, 5) = 0$ upon using (2.3), (2.1) with $n = 5$, and (3.3), in that order. Application of Lemma 2.3 and then Lemma 2.1 completes the proof.

THEOREM 3.4. *If R is a simple, commutative 8-associative or 9-associative ring of characteristic not 2, 3, or 5, then R is associative.*

Proof. By Theorem 3.3, it is sufficient to show that $S(7,7) = 0$.

If we let $n = 10$, 12, and 14 in (2.2), we get

$$\begin{aligned} 0 = {} & 2(a_1, \cdots, a_7)(a_8, a_9, a_{10}) + 3(a_1, \cdots, a_5)(a_6, \cdots a_{10}) \\ & + 2(a_1, a_2, a_3)(a_4, \cdots, a_{10}), \end{aligned} \tag{3.4}$$

$$0 = (a_1, \cdots, a_7)(a_8, \cdots, a_{12}) + (a_1, \cdots, a_5)(a_6, \cdots, a_{12}), \tag{3.5}$$

and

$$0 = S(7, 7)^2. \tag{3.6}$$

Our first goal is to establish

$$S(7, 9)S(5, 5) = 0 = S(7, 7)S(5, 7). \tag{3.7}$$

Applying (2.1) to (3.6) yields $T(7, 0, 4)S(7, 7) = 0$, to which we apply (3.5) to obtain $T(9, 0, 4)S(5, 5) = 0$. Using (2.1), we compute $0 = (T(9, 0, 6) - T(9, 0, 4))S(5, 5) = T(9, 5, 6)S(5, 5)$ which implies $T(9, 0, 1)S(5, 5) = 0$. Then $0 = (T(9, 0, 4) - T(9, 0, 1))S(5, 5) = T(9, 2, 4)S(5, 5)$, to which we apply (3.5) to obtain $T(7, 2, 4)S(7, 7) = 0$. Hence we have $T(7, 0, 2)S(7, 7) = 0$. Using (3.5) again yields $T(9, 0, 2)S(5, 5) = 0$. Thus we can compute $0 = (T(9, 0, 2) - T(9, 0, 1))S(5, 5)$ to obtain

$$T(9, 2, 2)S(5,5) = 0 = T(9, 4, 4)S(5, 5). \tag{3.8}$$

Computing $0 = (T(9, 2, 4) - T(9, 2, 2) - T(9, 4, 4))S(5, 5)$ yields

$$0 = T(9, 3, 3)\, S(5, 5)\,. \tag{3.9}$$

Applying (3.5) to (3.9), we have $T(7, 3, 3)\, S(7, 7) = 0$, from which we obtain $T(7, 1, 1)\, S(7, 7) = 0$, and hence (3.5) implies

$$0 = T(9, 1, 1)\, S(5, 5) = 0\,. \tag{3.10}$$

Computing $0 = (T(9, 0, 1) - T(9, 1, 1))\, S(5, 5)$ yields $T(9, 0, 0)\, S(5, 5) = 0$ which, along with (3.8), (3.9), and (3.10), implies (3.7) after using (3.5).

Our next goal is to establish

$$0 = S(5, 7)^2\,. \tag{3.11}$$

Let $a_1 = (x_1, x_2, x_3)$, $a_i = x_{i+2}$, $i > 1$ in (3.4); then let $a_1 = x_1$, $a_2 = (x_2, x_3, x_4)$, $a_i = x_{i+2}$, $i > 2$ in (3.4); and then let $a_1 = x_1, a_2 = x_2, a_3 = (x_3, x_4, x_5)$, $a_i = x_{i+2}$, $i > 3$ in (3.4). Add the resulting identities and apply (2.1) to obtain

$$\begin{aligned} 0 = {} & 2T(9, 0, 2)\,(x_{10}, x_{11}, x_{12}) + 3T(7, 0, 2)\,(x_8, \cdots, x_{12}) \\ & + 2(x_1, \cdots, x_5)\,(x_6, \cdots, x_{12}) \end{aligned} \tag{3.12}$$

where the $T's$ are now written in terms of the $x_i's$. Substitute (x_{10}, x_{11}, x_{12}) for x_{10}, x_{13}, for x_{11}, and x_{14} for x_{12} in (3.12); then substitute (x_{11}, x_{12}, x_{13}) for x_{11} and x_{14} for x_{12} in (3.12); and then substitute (x_{12}, x_{13}, x_{14}) for x_{12} in (3.12). Add the resulting identities, and use (2.1) and (3.7) to obtain

$$0 = T(7, 0, 2) \sum_{i=9}^{11} (x_8, \cdots, (x_{i+1}, x_{i+2}, x_{i+3}), \cdots, x_{14})\,. \tag{3.13}$$

Applying (2.1) and (3.7) to (3.13) and then using (2.3), we get, after subtracting the resulting identity from (3.13) with subscripts relabeled,

$$0 = T(7, 0, 2)\,(x_8, x_9, (x_{10}, x_{11}, x_{12}), x_{13}, x_{14})\,. \tag{3.14}$$

If we apply (3.4) to (3.14), then (2.1) with $n = 5$ and (3.7), we obtain $0 = T(9, 0, 2)\,((x_{10}, x_{11}, x_{12}), x_{13}, x_{14})$ which yields

$$0 = T(9, 0, 2)\, S(3, 5) \tag{3.15}$$

upon using (2.3) and then (2.1) and (3.7).

Application of (3.4) to (3.15); then use of (2.1) and (3.7) followed by (2.3) yields $T(7, 0, 2) S(5, 7) = 0$. Hence using (2.1) and (3.7) we compute $0 = (T(7, 0, 4) - T(7, 0, 2))S(5, 7) = T(7, 3, 4)S(5, 7)$ which yields

$$0 = T(7, 0, 1)\, S(5, 7)\,. \tag{3.16}$$

Computing $0=(T(7,0,2)-T(7,0,1))\,S(5,7)$, we obtain

$$0=T(7,2,2)\,S(5,7)\,. \tag{3.17}$$

Returning to (3.15), we can assume that $T(9,0,2)=0$, for otherwise $S(7,7)=0$ by Lemma 2.3 and Lemma 2.1. Hence we have $T(9,4,6)=0$. Computing $0=T(9,0,6)-T(9,0,2)-T(9,4,6)$, we obtain $T(9,3,3)=0$ which we apply to (3.4) to get

$$0=2(x_1,x_2,x_3)\,c+3T(7,3,3)\,(s_1,\cdots,s_5) \tag{3.18}$$

where $c=((x_4,x_5,x_6),x_7,s_1,\cdots,s_5)$ and where at least one of s_1, s_2, s_3, s_4, or $s_5\in S(3,3)$. Let $x_3=z\in S(3,3)$ in (3.18) and use (3.17) to obtain $0=(x_1,x_2,z)c$, to which we apply (2.3) and then (2.1) and (3.7) to get $S(3,5)\,c=0$. Since $c\in S(3,5)$ by Lemma 2.1, Lemma 2.3 implies that $c=0$. Hence (3.18) yields $T(7,3,3)\,S(5,7)=0$, and therefore $T(7,1,1)\,S(5,7)=0$. Now, recalling (3.16), we compute $0=(T(7,0,1)-T(7,1,1))\,S(5,7)$ to obtain $T(7,0,0)\,S(5,7)=0$, but then $T(7,4,4)\,S(5,7)=0$, and we have established (3.11).

Equations (3.4) and (3.11) yield

$$0=((x_4,x_5,x_6),x_7,x_8)\,(x_9,x_{10},s_1,s_2,x_1,x_2,x_3)$$

where s_1 or $s_2\in S(3,3)$ since $c=0$, to which we apply (2.3) and then (2.1) and (3.7) to obtain $0=(x_9,x_{10},s_1,s_2,x_1,x_2,x_3)\,S(3,5)$ which in turn, using (3.4) and (3.11), yields $0=(x_9,x_{10},s)\,T(9,i,i)$ for $i=4,5$, and 6 and where $s\in S(3,3)$. Thus, by (2.3) and then (2.1) and (3.7), we have $S(3,5)\,T(9,i,i)=0$ for $i=4$, 5, and 6. Therefore, we have $S(3,5)\,S(7,9)=0$ since $T(9,3,3)=0$. Lemmas 2.3 and 2.1 then imply that

$$0=S(7,9)\,. \tag{3.19}$$

Equations (3.4) and (3.19) imply $0=S(5,5)\,T(7,i,i)$, $i=0,1$, which yields

$$0=S(5,5)\,T(7,i,i), \qquad i=0,1,3,4\,. \tag{3.20}$$

Using (2.3), (3.4), and (3.19), we compute

$$3T(7,2,2)\,S(5,5)\subset-2T(5,2,2)\,S(7,7)\subset 2T(5,0,0)\,S(7,7)=0\,,$$

the equality by (3.4), (3.19), and (3.20). Hence

$$S(5,7)\,S(5,5)=0\,. \tag{3.21}$$

Finally (3.4), (3.19), and (3.21) imply that $S(3,5)\,S(7,7)=0$. Lemma 2.3 and Lemma 2.1 then imply $S(7,7)=0$, and we are finished.

Theorems 3.2, 3.3, and 3.4 imply

COROLLARY 3.2. *If R is a simple, commutative 10-associative ring of characteristic not 2, 3, or 5, or if R is a simple, commutative 8-associative ring of characteristic 5, then R is associative or $C = 0$ where C is the center of R.*

4. Rings which are n-associative of degree $2k+1$.

An immediate corollary of Lemma 2.1 is

LEMMA 4.1. *If R is $(2m+1)$-associative of degree $2n+1$, then R is $(2k+1)$-associative for all k such that $m \leqq k$ and $n \leqq k$.*

The converse of Lemma 4.1 is false as can be seen by the following example. Let A be the 13-dimensional commutative algebra with basis $u_1, u_2, \cdots, u_{13}$ satisfying $u_1^2 = u_4$, $u_1u_2 = u_3$, $u_1u_3 = u_8$, $u_1u_5 = u_7$, $u_1u_7 = u_{10}$, $u_1u_9 = u_{11}$, $u_2^2 = u_5$, $u_2u_3 = u_9$, $u_2u_4 = u_6$, $u_2u_8 = -u_{10} + u_{11}$, $u_2u_{10} = u_{12}$, $u_2u_{11} = u_{12} - u_{13}$, the commutative law, and all other products zero. It can be verified that A is 5-associative, but that $((u_1, u_1, u_2), u_2, u_2) = u_{13} \neq 0$, and hence A is not associative of degree 5.

THEOREM 4.1. *If R is a simple, commutative ring of characteristic not a prime $\leqq k$ which is associative of degree $2k+1$, then R is associative.*

Proof. Assume that $k > 1$. We have $S(3, 2k+1) = 0$. We will show that this implies that $S(3, 2k-1) = 0$, from which the proof is completed by an obvious induction.

Applying Lemma 2.1 to $S(3, 2k+1) = 0$, we have

$$0 = S(2j+1, 2m+1) \text{ for } j \geqq 1 \text{ and } m \geqq k\,. \tag{4.1}$$

Let $n = 2k+2$ in (2.2). Then by (4.1) with $i = m = k$, we have

$$k(a_1, a_2, a_3)(a_4, \cdots, a_{2k+2}) = -\sum_{i=1}^{k-2}\binom{k}{i}(a_1, \cdots, a_{2k-2i+1})(a_{2k-2i+2}, \cdots, a_{2k+2})\,,$$

to which we apply (4.1) with $j = k - i$ and $m = 2k - i - 2$, $i = 1, \cdots, k-2$, to obtain $0 = S(3, 2k-1)S(2k-1, 2k-1)$. Therefore, by Lemma 2.3, $S(3, 2k-1) = 0$ or $S(2k-1, 2k-1) = 0$. Assume that we have shown that $S(3, 2k-1) = 0$ or $S(2k-2j+1, 2k-2j+1) = 0$. Assume that

$$0 = S(2k-2j+1, 2k-2j+1)\,. \tag{4.2}$$

Let $n = 2k - 2j + 2$ in (2.2). Then, as above, we apply (4.2) and (4.1) with $m = 2k - j - i - 2$, $i = 1, \cdots, k-j-2$, to obtain $0 =$

$S(3, 2k-1)\,S(2k-2j-1, 2k-2j-1)$. Therefore, as before, $S(3, 2k-1)=0$ or $S(2k-2j-1, 2k-2j-1)=0$. Hence, $S(3, 2k-1)=0$ or $S(3, 3)=0$, and we are finished.

In view of Lemma 4.1, Theorem 4.1 is an extension of the results of §3 for a more restricted class of rings.

Finally, define $S(3, 2k+1)^n = S(3, 2k+1)^{n-1} S(3, 2k+1)$, $n > 1$. We have

COROLLARY. *If R is a simple, commutative ring of characteristic not a prime $\leqq k$ in which $S(3, 2k+1)^n = 0$ for some n, then R is associative.*

Proof. Because of Theorem 4.1, we need only show that $S(3, 2k+1)=0$ in R. Assume $n > 1$. Then $0 = S(3, 2k+1)^{n-1} S(3, 2k+1)$. Lemma 2.3 and an easy induction yield $S(3, 2k+1) = 0$.

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