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DIVISIBILITY PROPERTIES OF CERTAIN FACTORIALS

J. CHIDAMBARASWAMY

It is well known that multinomial coefficients are integers; i.e., if the integers a_i are nonnegative and $a = \sum_{i=1}^{m} a_i$, then $\prod_{i=1}^{m} (a_i)! \mid a!$. This property may hold good in special cases even though $\sum_{i=1}^{m} a_i > a$. In fact, for each integer $x \geqq 0$, $x!(x+1)! \mid (2x)!$, and it has been asked by Erdos, as a research problem in the 1947 May issue of the Monthly, whether, for a given $c \geqq 1$, there exists an infinity of integers x such that $x!(x+c)! \mid (2x)!$. This problem has been gradually generalized and improved upon by Mordell, Wright, McAndrew, the author, and N. V. Rao. In particular, Rao considers the quotient $Q(x) = ((g(x)+h(x))!)/((g(x)+k)!(h(x))!)$, where k is a positive integer, and $g(x)$ and $h(x)$ are integer coefficient polynomials of positive degree with positive leading coefficients and proves that some multiple of $Q(x)$ is integral infinitely often: a result which includes all the earlier results. In this paper, among other things, this result of Rao has been generalised and improved upon by taking the polynomials over the rationals and by reducing the multiplying factor of $Q(x)$ as obtained by Rao.

Throughout the following i, j, k, r, and n denote positive integral variables and all small letters, unless explicitly mentioned otherwise denote positive integers. As usual, (a, b) and $\{a, b\}$ denote respectively the $G.C.D.$ and $L.C.M.$ of a and b. For any polynomials $X(x)$ and $Y(x)$ (not both zero) over the rationals, $(X(x), Y(x))$ denote their monic $G.C.D.$ over the rationals. m being $\geqq 1$, $t_1, t_2, \cdots, t_m$ are integers each greater than 1. For $1 \leqq i \leqq m$ and $1 \leqq j \leqq t_i$, $f_{ij}(x)$ is a polynomial of positive degree over the rationals with positive leading coefficient; a_{ij} and c_{ij} are nonnegative integers, r_{ij} is a positive rational and k_{ij} is a positive integer. Also, r_i is a nonnegative integer for each i in $1 \leqq i \leqq m$. We use the following symbolism.

$$(1.1) \qquad f_i(x) = \sum_{k=1}^{t_i} f_{ik}(x) \; ; \qquad F_{ij}(x) = \sum_{\substack{k=1 \\ k \neq j}}^{t_i} f_{ik}(x)$$

$$A_i = \sum_{k=1}^{t_i} a_{ik} \; ; \qquad A_{ij} = \sum_{\substack{k=1 \\ k \neq j}}^{t_i} a_{ik}$$

$$R_i = \sum_{k=1}^{t_i} r_{ik} \; ; \qquad R_{ij} = \sum_{\substack{k=1 \\ k \neq j}}^{t_i} r_{ik} \; ;$$

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$$K_i = \sum_{r=1}^{t_t} k_{ir} \; ; \qquad K_{ij} = \sum_{\substack{r=1 \\ r \neq j}}^{t_i} k_{ir}$$

$$C_i = \sum_{r=1}^{t_i} c_{ir} \; ; \qquad C_{ij} = \sum_{\substack{r=1 \\ r \neq j}}^{t_i} c_{ir}$$

$$\begin{aligned} A_{ij}(x) &= (f_i(x) + 1)(f_i(x) + 2) \cdots (f_i(x) + k_{ij}) \\ B_{ij}(x) &= (f_{ij}(x) + 1)(f_{ij}(x) + 2) \cdots (f_{ij}(x) + k_{ij}) \\ C_{ij}(x) &= (F_{ij}(x))(F_{ij}(x) - 1) \cdots (F_{ij}(x) - k_{ij} + 1) \\ G_{ij}(x) &= (A_{ij}(x), B_{ij}(x)) \; ; \\ H_{ij}(x) &= (B_{ij}(x), C_{ij}(x)) \\ L_{ij}(x) &= (G_{ij}(x), H_{ij}(x)) \end{aligned}$$

and

$$Q_{ij}(x) = (f_i(x))!/((f_{ij}(x) + k_{ij})!(F_{ij}(x))!)$$

where $Q_{ij}(x)$ is defined for those values of x for which $f_{ij}(x)$ are all nonnegative integers.

Improving upon the results of Mordell [2], and Wright [7], McAndrew [1], proved that (in our notation) if for a particular i,

$$0 < a_{i1} < A_i \tag{1.2}$$

and

$$c_{i_1} = 0 \; , \tag{1.3}$$

then there exists an infinity of integers x for which

$$(A_i x)! \Big/ \sum_{r=1}^{t_i} (a_{ir} x + c_{ir})! \tag{1.4}$$

is an integer. In [6], the author and N. V. Rao improved upon this result, by proving that, if, together with (1.2), the conditions

$$r_i < A_i/(a_{i1}, A_i) \tag{1.5}$$

and

$$\text{either} \quad c_{i1} < a_{i1}/(a_{i1}, A_i) \quad \text{or} \quad C_i < A_i/(a_{i1}, A_i) \tag{1.6}$$

hold, then there exists an infinity of integers x such that

$$(A_i x - r_i)! x \Big/ \prod_{r=1}^{t_i} (a_{ir} x + c_{ir})! \tag{1.7}$$

is an integer. In [5] and [4] respectively, the author considered the question of existence of an infinity of integers x which make the expressions in (1.4) and (1.7) simultaneously integers for each i. Recently, N. V. Rao, taking the polynomials over the domain of integers and $t_i = 2$ for each i, proved the existence of an infinity of integers x such that

$$Q_{i1}(x)\overline{G_{i1}(x)} \tag{1.8}$$

is an integer for each i in $1 \leqq i \leqq m$, where $\overline{G_{i1}(x)}$ is the integer coefficient G.C.D. with least positive leading coefficient of the integer coefficient polynomials $A_{i1}(x)$ and $B_{i1}(x)$. In fact if, for any rational coefficient $f(x)$, $T(f)$ denotes the l.c.m. of the denominators of the coefficients of $f(x)$, $\overline{G_{i1}(x)} = T(G_{i1})G_{i1}(x)$.

The purpose of this paper is, among other things, to improve upon the above result of Rao, simultaneously 1, by allowing the polynomials to have their coefficients from rationals and 2, by replacing the factor $\overline{G_{i1}(x)}$ in (1.8) by one of its divisors namely $L_{i1}(x)$. That $L_{i1}(x)$ can be a proper divisor of $\overline{G_{i1}(x)}$ is seen if we take $m = 1$, $t_1 = 2$, $f_{11}(x) = f_{12}(x) = x^2 - x$ and $k_{11} = 2$ in which case $\overline{G_{11}(x)} = x^2 - x + 1$ while $L_{11}(x) = 1$. Incidentally, the result in [6] is slightly improved by increasing the possible values of r_i (see Cor. 1) and it turns out that McAndrew's result ((1.4)) in [1] and our result in [5] are particular cases obtainable from a more general result (Theorem IV) by taking x for an arbitrary polynomial $g(x)$ over the rationals with the property that there exists an integer x_0 such that $g(x_0)$ is an integer.

In order to guarantee the existence of integers x for which $f_{ij}(x)$ are integers, we make the following Assumption A: There exist integers y_{ij} such that $f_{ij}(y_{ij})$ are integers and the system of congruences

$$x \equiv y_{ij} \pmod{T(f_{ij})} \qquad 1 \leqq i \leqq m,\ 1 \leqq j \leqq t_i$$

admit a common solution y_0.

We note that all such common solutions are represented by

$$x \equiv y_0(T) \tag{1.9}$$

where

$$T = \{T(f_{11}), T(f_{12}), \cdots, T(f_{mt_m})\} \tag{1.10}$$

and we observe that

$$T(f_i) \mid T \qquad \text{for each } i \text{ in } 1 \leqq i \leqq m\,. \tag{1.11}$$

We need some further notation. Let

$$
\begin{aligned}
(2.1)\qquad \hat{A}_{ij}(x) &= (f_i(x)+1)(f_i(x)+2)\cdots(f_i(x)+K_{ij})\\
\hat{B}_{ij}(x) &= (F_{ij}(x)+1)(F_{ij}(x)+2)\cdots(F_{ij}(x)+K_{ij})\\
\hat{C}_{ij}(x) &= (f_{ij}(x))(f_{ij}(x)-1)\cdots(f_{ij}(x)-K_{ij}+1)\\
\hat{G}_{ij}(x) &= (\hat{A}_{ij}(x), \hat{B}_{ij}(x))\ ; \qquad \hat{H}_{ij}(x) = (\hat{B}_{ij}(x), \hat{C}_{ij}(x))\\
\hat{L}_{ij}(x) &= (\hat{G}_{ij}(x), \hat{H}_{ij}(x))\ ; \qquad d_{ij}(x) = (B_{ij}(x), \hat{B}_{ij}(x))\\
D_{ij}^{tu}(x) &= (f_i(x)+t, f_{ij}(x)+u, F_{ij}(x)-u+t)\\
\hat{D}_{ij}^{tu}(x) &= (f_i(x)+t, F_{ij}(x)+u, f_{ij}(x)-u+t)\\
\hat{Q}_{ij}(x) &= (f_i(x))!/((f_{ij}(x))!(F_{ij}(x)+K_{ij})!)\\
W_{ij}(x) &= (f_i(x))!/((f_{ij}(x)+k_{ij})!(F_{ij}(x)+K_{ij})!)
\end{aligned}
$$

and finally

$$W_i(x) = (f_i(x))! \Big/ \prod_{r=1}^{t_i} (f_{ir}(x)+k_{ir})!$$

Now, we are in a position to state our results

THEOREM I. *Under the Assumption A, there exists an infinity of integers x such that*

(i) $Q_{ij}(x)L_{ij}(x)$,

(ii) $\hat{Q}_{ij}(x)\hat{L}_{ij}(x)$,

(iii) $W_{ij}(x)d_{ij}(x)L_{ij}(x)\hat{L}_{ij}(x)$,

and

(iv) $W_i(x)d_{ij}(x)L_{ij}(x)\hat{L}_{ij}(x)$

are all simultaneously integers for each i in $1 \leqq i \leqq m$ and each j in $1 \leqq j \leqq t_i$.

THEOREM II. *Under the Assumption A, if for each i in $1 \leqq i \leqq m$, there is a $j(i)$ in $1 \leqq j(i) \leqq t_i$ such that for any integers t, u, e, b satisfying*

$$(2.2)\qquad 1 \leqq t,\ u \leqq k_{ij(i)}\ , \qquad 0 \leqq u-t \leqq k_{ij(i)}-1$$

and

$$
\begin{aligned}
(2.3)\qquad & 1 \leqq e\ , \qquad b \leqq K_{ij(i)}\ , \qquad 0 \leqq b-e \leqq K_{ij(i)}-1\ ,\\
& D_{ij(i)}^{tu}(x) = 1 = \hat{D}_{ij(i)}^{eb}(x)\ ,
\end{aligned}
$$

then there exists an infinity of integers x such that

(i) $Q_{ij(i)}(x)$,

(ii) $\hat{Q}_{ij(i)}(x)$,

(iii) $W_{ij(i)}(x)d_{ij(i)}(x)$,

and

(iv) $W_i(x)d_{ij(i)}(x)$

are all simultaneously integers for each i in $1 \leqq i \leqq m$.

In particular, we have:

THEOREM III. (a) *If, for* $1 \leqq i \leqq m$, $g_i(x)$ *is a polynomial over the rationals with positive leading coefficient and if the Asssumption A is satisfied for all the polynomials* $r_{ij}g_i(x)$, *then there exist an infinity of integers* x *such that*

(i) $(R_i g_i(x))!/((r_{ij}g_i(x) + k_{ij})!(R_{ij}g_i(x))!)$,

(ii) $(R_i g_i(x))!/((r_{ij}g_i(x))!(R_{ij}g_i(x) + K_{ij})!)$,

(iii) $(R_i g_i(x))!D_{ij}(x)/((r_{ij}g_i(x) + k_{ij})!(R_{ij}g_i(x) + K_{ij})!)$,

and

(iv) $(R_i g_i(x))!D_{ij}(x)\Big/\prod_{j=1}^{t_i}(r_{ij}g_i(x) + k_{ij})!$

are all simultaneously integers for each i *in* $1 \leqq i \leqq m$ *and each* j *in* $1 \leqq j \leqq t_i$, *where*

$$D_{ij}(x) = ((r_{ij}g_i(x)+1)\cdots(r_{ij}g_i(x)+k_{ij}), (R_{ij}g_i(x)+1)\cdots(R_{ij}g_i(x)+K_{ij})) .$$

(b) *If in* (a) *the integers* k_{ij} *and the rational numbers* r_{ij} *are such that for each* i *in* $1 \leqq i \leqq m$, *there is a* $j(i)$ *in* $1 \leqq j(1) \leqq t_i$ *such that*

$$r_{ij(i)}k - R_{ij(i)}n \neq 0 \tag{2.4}$$

for

$$1 \leqq k \leqq K_{ij(i)}\ , \qquad 1 \leqq n \leqq k_{ij(i)}\ , \tag{2.5}$$

then there exists an infinity of integers x *such that*

$$(R_i g_i(x))!\Big/\prod_{j=1}^{t_i}(r_{ij}g_i(x) + k_{ij})!$$

is an integer for each i *in* $1 \leqq i \leqq m$.

As an immediate consequence of Theorem III we have:

THEOREM IV. *If* a_{ij} *and* c_{ij} *satisfy respectively* (1.2) *and* (1.3) *and if* $g(x)$ *is a polynomial of positive degree over the rationals with the property that there is an integer* x_0 *such that* $g(x_0)$ *is an integer, then there exists an infinity of positive integers* x *such that*

$$(A_i g(x))!\Big/\prod_{j=1}^{t_i}(a_{ij}g(x) + c_{ij})! \tag{2.6}$$

is an integer for each i *in* $1 \leqq i \leqq m$.

Also from Theorem I, we have the following:

COROLLARY 1[1]. *If* a_{ij}, c_{ij} *and* r_i *are such that for each* i *in* $1 \leqq i \leqq m$, *there is a* j *in* $1 \leqq j \leqq t_i$ *satisfying*

(2.7) (i) $0 < a_{ij} < A_i$,
(ii) $r_i \leqq A_i/(a_{ij}, A_i)$,
(iii) *either* $c_{ij} < a_{ij}/(a_{ij}, A_i)$ *or*
$C_{ij} < A_{ij}/(A_{ij}, A_i)$

then there exists an infinity of positive integers x *such that* (1.7) *is an integer for each* i *in* $1 \leqq i \leqq m$.

As remarked earlier, we observe that (1.4) is obtained from (2.6) by taking $g(x) = x$ and Cor. I is an improvement of our result in [6], since, taking $m = 1$, $j = 1$, we are increasing the range of values of r_1 (compare (1.5) and (ii) of (2.7)) and the condition (iii) of (2.7) is a consequence of (1.6) but not conversely; for example, our theorem in [6] does not help us to conclude that

$$(8x)!/((2x + 3)!(4x + 1)!(2x + 1)!)$$

is an integer infinitely often whereas our corollary does. We omit the easy verification of this statement.

LEMMA I. *For each* i *in* $1 \leqq i \leqq m$ *and each* j *in* $1 \leqq j \leqq t_i$, *there exists integer coefficient polynomials* $p_{ij}(x), q_{ij}(x), r_{ij}(x), s_{ij}(x), t_{ij}(x), u_{ij}(x), v_{ij}(x), \hat{p}_{ij}(x), \cdots, \hat{v}_{ij}(x)$ *and positive integers* $\lambda_{ij}, \mu_{ij}, \nu_{ij}, \hat{\lambda}_{ij}, \hat{u}_{ij}, \hat{\nu}_{ij}$, *and* ζ_{ij} *such that*

(i) $A_{ij}(x)p_{ij}(x) + B_{ij}(x)q_{ij}(x) = \lambda_{ij}G_{ij}(x)$
(ii) $B_{ij}(x)r_{ij}(x) + C_{ij}(x)s_{ij}(x) = \mu_{ij}H_{ij}(x)$
(iii) $G_{ij}(x)t_{ij}(x) + H_{ij}(x)u_{ij}(x) = \nu_{ij}L_{ij}(x)$
(iv) $\hat{A}_{ij}(x)\hat{p}_{ij}(x) + \hat{B}_{ij}(x)\hat{q}_{ij}(x) = \hat{\lambda}_{ij}\hat{G}_{ij}(x)$
(v) $\hat{B}_{ij}(x)\hat{r}_{ij}(x) + \hat{C}_{ij}(x)\hat{s}_{ij}(x) = \hat{\mu}_{ij}\hat{H}_{ij}(x)$
(vi) $\hat{G}_{ij}(x)\hat{t}_{ij}(x) + \hat{H}_{ij}(x)\hat{u}_{ij}(x) = \hat{\nu}_{ij}\hat{L}_{ij}(x)$
(vii) $B_{ij}(x)v_{ij}(x) + \hat{B}_{ij}(x)\hat{v}_{ij}(x) = \zeta_{ij}d_{ij}(x)$

Proof. (i) There exist rational coefficient polynomials $\alpha_{ij}(x)$ and $\beta_{ij}(x)$ such that

$$A_{ij}(x)\alpha_{ij}(x) + B_{ij}(x)\beta_{ij}(x) = G_{ij}(x) . \tag{3.1}$$

Multiplying both sides of (3.1) by $\lambda_{ij} = \{T(\alpha_{ij}), T(\beta_{ij})\}$ and writing $p_{ij}(x) = \lambda_{ij}\alpha_{ij}(x)$ and $q_{ij}(x) = \lambda_{ij}\beta_{ij}(x)$ we get (i). The proof of the other parts is similar.

LEMMA 2. *For each* i *in* $1 \leqq i \leqq m$ *and each* j *in* $1 \leqq j \leqq t_i$

[1] This corollary could also be obtained from the result of [**3**] but no mention of this was made in [**3**].

(i) $\lambda_{ij}\mu_{ij}\nu_{ij}Q_{ij}(x)L_{ij}(x)$
$= \mu_{ij}t_{ij}(x)\{A_{ij}(x)Q_{ij}(x)p_{ij}(x) + B_{ij}(x)Q_{ij}(x)q_{ij}(x)\}$
$+ \lambda_{ij}u_{ij}(x)\{B_{ij}(x)Q_{ij}(x)r_{ij}(x) + C_{ij}(x)Q_{ij}(x)s_{ij}(x)\}$

(ii) $\hat{\lambda}_{ij}\hat{\mu}_{ij}\hat{\nu}_{ij}\hat{Q}_{ij}(x)\hat{L}_{ij}(x)$
$= \hat{\mu}_{ij}\hat{t}_{ij}(x)\{\hat{A}_{ij}(x)\hat{Q}_{ij}(x)\hat{p}_{ij}(x) + \hat{B}_{ij}(x)\hat{Q}_{ij}(x)\hat{q}_{ij}(x)\}$
$+ \hat{\lambda}_{ij}\hat{u}_{ij}(x)\{\hat{B}_{ij}(x)\hat{Q}_{ij}(x)\hat{r}_{ij}(x) + \hat{C}_{ij}(x)\hat{Q}_{ij}(x)\hat{s}_{ij}(x)\}$

(iii) $\zeta_{ij}d_{ij}(x)L_{ij}(x)\hat{L}_{ij}(x)W_{ij}(x)$
$= L_{ij}(x)\hat{L}_{ij}(x)\{v_{ij}(x)\hat{Q}_{ij}(x) + \hat{v}_{ij}(x)Q_{ij}(x)\}.$

Proof. (i) follows directly from (i) (ii) (iii) of Lemma I; similarly for (ii) and (iii).

LEMMA 3. *If $f^r(x)$ denotes the rth derivative of the rational coefficient polynomial $f(x)$, then for $r \geqq 0$ $\{T(f)f^r(x)\}/r!$ is a polynomial with integer coefficients.*

Proof. Each coefficient of $f^r(x)$ is a product of a coefficient of $f(x)$ and a product of r consecutive integers.

LEMMA 4. *For each sufficiently large integer x for which each $f_{ij}(x)$ is a positive integer,*

$$A_{ij}(x)Q_{ij}(x),\ B_{ij}(x)Q_{ij}(x),\ C_{ij}(x)Q_{ij}(x)$$
$$\hat{A}_{ij}(x)\hat{Q}_{ij}(x),\ \hat{B}_{ij}(x)\hat{Q}_{ij}(x),\ \hat{C}_{ij}(x)\hat{Q}_{ij}(x)$$

are all positive integers.

Proof. Each of them can be expressed as a binomial coefficient.

Before proceeding to the next lemma, we introduce, for convenience, the following notation: for any positive integers a, b, and c, $h(a, b)$ stands for the exponent of the highest power of b that divides a and $D(a/b, c)$ stands for $h(a, c) - h(b, c)$.

LEMMA 5. *For any positive integer a, and any prime p, $h(a!, p) = (a - S)/(p - 1)$, where S is the sum of the digits of a in the representation of a in the scale of p.*

This is well known and we omit the proof.

LEMMA 6. *Under the assumption A, given any pair of positive integers M and N, there exists an infinity of positive integers x such that for each i in $1 \leqq i \leqq m$ and each j in $1 \leqq j \leqq t_i$ and each prime p dividing M,*

$$D(W_{ij}(x), p) > N. \tag{3.2}$$

Proof. We prove that from among the integers satisfying (1.9), for which by the Assumption A all $f_{ij}(x)$ are integers, we can select an infinite number of them for which (3.2) is satisfied. z_0 being any arbitrary integer, choose x_0 such that

(3.3) (i) $x_0 > z_0$, (ii) $x_0 \equiv y_0(T)$ and (iii) $f_{ij}(x_0)$

and $f'_{ij}(x_0)$ are all positive. Let P be the product of all the distinct prime factors of M and π the smallest of them. Let

$$A = N + 1 + \underset{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq t_i \\ p \mid P}}{\operatorname{Max}} |D(W_{ij}(x_0), p)| . \tag{3.4}$$

Choose β_0 to be the least positive integer such that

$$\pi^{\beta_0} > \underset{1 \leqq i \leqq m}{\operatorname{Max}} (f_i(x_0) + K_i) . \tag{3.5}$$

Observing that any positive integer n in $1 \leqq n \leqq mA$ can be uniquely expressed in the form

$$n = (i-1)A + k, \qquad 1 \leqq i \leqq m , \qquad 1 \leqq k \leqq A ,$$

we define, starting with the integers x_0 and β_0, recurrently the integers γ_n, δ_n, x_n and β_n for $1 \leqq n \leqq mA$ as follows: γ_n is the least positive integer such that

$$\pi^{\gamma_n} > \underset{p \mid P}{\operatorname{Max}} \frac{Tf'_i(x_{n-1})}{p^{h(Tf'_i(x_{n-1}), p)}} ; \tag{3.6}$$

δ_n is the least positive integer so chosen that

(3.7) (i) $\delta_n > \beta_{n-1}$,

(ii) $\delta_n > \underset{p \mid P}{\operatorname{Max}} \dfrac{2h(Tf'_i(x_{n-1}), p) + \gamma_n}{1 + h(Tf'_i(x_{n-1}), p)}$, and

(iii) $\varphi(P^{\gamma_n}) \mid \delta_n$,

φ being Euler's totient function;

$$x_n = x_{n-1} + TP^{\delta_n}(Tf'_i(x_{n-1}))^{\delta_n - 1}. \tag{3.8}$$

And finally β_n is the least positive integer such that

$$\pi^{\beta_n} > \underset{1 \leqq i \leqq m}{\operatorname{Max}} f_i(x_n) - f_i(x_{n-1}) . \tag{3.9}$$

We observe that, by virtue of (3.8), (ii) of (3.3), x_n satisfies (1.9) and so all the $f_{ij}(x_n)$ are positive integers and the proof of lemma will be complete, if it is proved that $Z_1 = x_{mA}$ satisfies (3.2). From now on the proof consists of reformulating the lemmas 2, 3, and 4 of [3] (in our notation) and adjusting their proofs.

For consideration of space, we omit the details.

4. *Proof of Theorem I.* In the first place let us observe that, if $f(x)$ is a polynomial over the rationals, then for any integer x, the denominator of $f(x)$ can contain only primes p in $T(f)$ to a power at most $h(T(f), p)$. Now, taking

$$M = \prod_{i=1}^{m} \prod_{j=1}^{t_i} T(L_{ij})T(\hat{L}_{ij})T(d_{ij})\lambda_{ij}\mu_{ij}\nu_{ij}\hat{\lambda}_{ij}\hat{\mu}_{ij}\hat{\nu}_{ij}\zeta_{ij} \tag{4.1}$$

and

$$N = \sum_{p|M} h(M, p)$$

in Lemma 6, we are guaranteed of the existence of an infinity of integers x ($\equiv y_0 \pmod{T}$) for which (3.2) is satisfied.

For all these integers, by Lemma 4, $A_{ij}(x)Q_{ij}(x)$, $B_{ij}(x)Q_{ij}(x)$, $C_{ij}(x)Q_{ij}(x)$ are all positive integers and so by the first part of Lemma 2, $\lambda_{ij}\mu_{ij}\nu_{ij}Q_{ij}(x)L_{ij}(x)$ is an integer for each i and each j. Since each prime factor of $\lambda_{ij}\mu_{ij}\nu_{ij}$ is necessarily a prime factor of M and since for any prime p, $D(Q_{ij}(x), p) \geqq D(W_{ij}(x), p)$, the remark at the beginning of the proof and the choice of N in (4.1) show that for all these integers $Q_{ij}(x)L_{ij}(x)$ is an integer.

A similar argument, taking into consideration the second and third parts of Lemma 2, shows that for all these integers,

$$\hat{Q}_{ij}(x)\hat{L}_{ij}(x) \qquad \text{and} \qquad W_{ij}(x)d_{ij}(x)L_{ij}(x)\hat{L}_{ij}(x)$$

are also integers. Further, since

$$\begin{aligned} &W_i(x)d_{ij}(x)L_{ij}(x)\hat{L}_{ij}(x) \\ &\quad = W_{ij}(x)d_{ij}(x)L_{ij}(x)\hat{L}_{ij}(x)\left\{\frac{(F_{ij}(x) + K_{ij})!}{\prod_{\substack{r=1 \\ r\neq j}}^{t_i} (f_{ir}(x) + k_{ir})!}\right\} \end{aligned} \tag{4.2}$$

and since for all the integers under consideration, the expression in brackets on the R.H.S. of (4.2) is an integer the L.H.S. of the same is so. Hence Theorem I.

Proof of Theorem II. Theorem II follows from Theorem I and the following lemma:

LEMMA 7. (a) *For each i in $1 \leqq i \leqq m$ and each j in $1 \leqq j \leqq t_i$, $L_{ij}(x) = 1$ if and only if for any t, u satisfying*

$$\begin{aligned} &1 \leqq t, u \leqq k_{i_j}, \qquad 0 \leqq u - t \leqq k_{ij} - 1 \\ &D_{ij}^{tu}(x) = 1 \,. \end{aligned} \tag{4.3}$$

(b) *For each i in $1 \leqq i \leqq m$ and each j in $1 \leqq j \leqq t_i$, $\hat{L}_{ij}(x) = 1$ if and only if for any e, b satisfying*

$$1 \leqq e, b \leqq K_{ij} ; \qquad 0 \leqq b - e \leqq K_{ij} - 1 , \qquad \hat{D}_{ij}^{eb}(x) = 1 .$$

Proof. That $L_{ij}(x)$ cannot be one if for same t, u satisfying (4.3) $D_{ij}^{tu}(x)$ contains as irreducible factor of positive degree, follows from the fact that $D_{ij}^{tu}(x)$ divides $L_{ij}(x)$.

If $L_{ij}(x)$ contains an irreducible factor of positive degree, say $\alpha(x)$, then for some integers t, u, and v satisfying $1 \leqq t, u \leqq k_{ij}$, $0 \leqq v \leqq k_{ij} - 1$, $\alpha(x)$ divides $f_i(x) + t$, $f_{ij}(x) + u$, and $F_{ij}(x) - v$; hence divides $t - u + v$. However, since $\alpha(x)$ is of positive degree, $t - u + v = 0$ and so it divides $D_{ij}^{tu}(x)$.

The proof of (b) is similar.

Proof of Theorem III. (a) It is easily seen that $L_{ij}(x)$ and $\hat{L}_{ij}(x)$ (as related to the notation of this theorem) are 1 for each i and j and hence (a).

(b) The condition (2.4) ensures $D_{ij(i)}(x) = 1$ and so (b) follows from (iv) of (a).

Proof of Theorem IV. If, for a particular i in $1 \leqq i \leqq m$, $C_{i1} = 0$, then (2.6) is an integer for all sufficiently large x for which $g(x)$ is a nonnegative integer. So, there is no loss of generality in assuming $C_{i1} > 0$ for each i in $1 \leqq i \leqq m$.

If, in Theorem III (a), we take 2 for t_i for each i, a_{i1} for r_{i1}, A_{i1} for r_{i2}, C_{i1} for k_{i2}, any positive integer for k_{i1}, and $g(x)$ for $g_i(x)$, the hypothesis of that theorem is satisfied and so by (ii) of that theorem, there exists an infinity of integers x for which

$$(A_i g(x))!/((a_{i1}g(x))!(A_{i1}g(x) + C_{i1})!)$$

is an integer. From this, the theorem follows in the same way as (iv) of Theorem I followed from (iii) of it.

Proof of Corollary I. For each i in $1 \leqq i \leqq m$, fix a j for which (2.7) is satisfied.

CASE (1). Suppose both r_i and C_{ij} are not zero for each i in $1 \leqq i \leqq m$.

In Theorem I, let us take for each i, $t_i = 2$, $f_{i1}(x) = a_{ij}x - r_i$, $f_{i2}(x) = A_{ij}x$, $k_{i1} = r_i + c_{ij}$, and $k_{i2} = C_{ij}$, so that

$$F_{i1}(x) = f_{i2}(x) = A_{ij}(x) \qquad \text{and} \qquad K_{i1} = k_{i2} = C_{ij} .$$

It is easily seen (a proof similar to that of Lemma I [6] works) that (ii) of (2.7) implies $L_{i1}(x) = x$ and (iii) of (2.7) implies $d_{i1}(x) = 1$; further clearly $\hat{L}_{i1}(x) = 1$ and so Corollary I follows from (iv) of Theorem I in this case.

CASE (2). Suppose one or both of r_i and C_{i1} are zero. In this case, the result follows trivially from case (1).

We close with a consideration of sequences of positive integers possibly more general than the sequences of positive integers represented by integer coefficient polynomials for integer values of the variable.

§ 5. Let f_n^{ij}: $1 \leqq i \leqq m$, $1 \leqq j \leqq t_i$, $n \geqq 1$ be a sequence of positive integer satisfying

(5.1) (i) for each i and each j

$$f_n^{ij} \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty$$

(ii) there exist sequences of positive integers

$$\sigma_n^{ij}: \quad 1 \leqq i \leqq m, \; 1 \leqq j \leqq t_i, \; n \geqq 1$$
$$\zeta_{nk}^{ij}: \quad 1 \leqq i \leqq m, \; 1 \leqq j \leqq t_i, \; n \geqq 1, \; k \geqq 1$$

such that $n_1 > n_2$ implies

$$f_{n_1}^{ij} - f_{n_2}^{ij} = (n_1 - n_2)\sigma_{n_2}^{ij} + (n_1 - n_2)^2 \zeta_{n_1 n_2}^{ij} .$$

Defining analogously the various sequences of integers A_n^{ij}, B_n^{ij}, C_n^{ij}, G_n^{ij}, H_n^{ij}, L_n^{ij}, $\hat{A}_n^{ij}$, $\cdots$, $\hat{L}_n^{ij}$ and d_n^{ij} and the sequences of rational numbers W_n^i, W_n^{ij}, Q_n^{ij}, and $\hat{Q}_n^{ij}$, (for example $A_n^{ij} = (f_n^i + 1)(f_n^i + 2) \cdots (f_n^i + k_{ij})$ where $f_n^i = \sum_{j=1}^{t_i} f_n^{ij}$, etc), we can prove the following theorem (Theorem S below) and deduce from that all the theorems of § 2 when the polynomials $f_{ij}(x)$ are taken over the domain of integers.

THEOREM S. *Given any positive integer Z, there exists an infinity of positive integers n, such that*

(i) $Q_n^{ij}[L_n^{ij}, Z]$,

(ii) $\hat{Q}_n^{ij}[\hat{L}_n^{ij}, Z]$,

(iii) $W_n^{ij}[d_n^{ij}, Z][L_n^{ij}, Z][\hat{L}_n^{ij}, Z]$, *and*

(iv) $W_n^i[d_n^{ij}, Z][L_n^{ij}, Z][\hat{L}_n^{ij}, Z]$

are all positive integers simultaneously for each i and each j where the symbol $[a, b]$ denotes the largest divisor prime to b of a.

A natural question in this context is whether, given a sequence f_n^{ij} satisfying (5.1), there exists an integer coefficient polynomial say $f_{ij}(x)$ such that

$$f_n^{ij} = f_{ij}(n) .$$

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