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GENERALIZED CONVEXITY CONES

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We consider in this paper generalized convexity cones $C(\phi_1,\cdots,\phi_n)$ with respect to an Extended Complete Tchebycheffian system $\{\phi_1(x),\cdots,\phi_n(x)\}$. These cones play a significant role in various areas of mathematics, such as moment theory, theory of approximation and interpolation, and theory of differential inequalities.

The properties of the cone $C(\phi_1,\cdots,\phi_n)$ are investigated. In particular, the extreme ray problem is solved explicitly for this cone, and for the intersection of such cones. Several structural properties of the cones are then determined.

The cone dual to $C(\phi_1,\cdots,\phi_n)$, which was introduced by S. Karlin and A. Novikoff is examined and a characterization of the cones which are dual to intersections of generalized convex cones is given. Some extensions of known theorems are given as applications.

We start by recalling some definitions and properties of functions of the generalized convexity cones. The concept of generalized convexity, which can be viewed as a generalization of the classical concept of convex functions of order n, was introduced by T. Popoviciu, [6]. The investigation of the dual cones was started by S. Karlin and A. Novikoff [4].

Let $\{\psi_i(x)\}_{i=1}^n$ be continuous functions on a finite interval $[a, b]$. This system is called a *Tchebycheffian system*, or a T-*system*, provided that the nth order determinants

$$D\begin{pmatrix}\psi_1, & \cdots, & \psi_n\\ x_1, & \cdots, & x_n\end{pmatrix}=\begin{vmatrix}\psi_1(x_1) & \cdots & \psi_1(x_n)\\ \vdots & & \vdots\\ \psi_n(x_1) & \cdots & \psi_n(x_n)\end{vmatrix} \tag{1}$$

maintain the same sign for every choice of $a\leqq x_1<\cdots<x_n\leqq b$. Without loss of generality we may assume that they are positive.

If the functions $\{\psi_i(t)\}_{i=1}^n$ are of class $C^{n-1}[a, b]$, we can extend the definition of $D\begin{pmatrix}\psi_1, & \cdots, & \psi_n\\ x_1, & \cdots, & x_n\end{pmatrix}$, as given by (1), to allow for equalities amongst the x_i. If $a\leqq x_1\leqq\cdots\leqq x_n\leqq b$, then $D\begin{pmatrix}\psi_1, & \cdots, & \psi_n\\ x_1, & \cdots, & x_n\end{pmatrix}$ is

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defined to be the determinant in the right-hand side of (1) where for each set of equal x_i's the corresponding rows are replaced by the successive derivatives evaluated at the point.

With this definition, the system $\{\psi_i(x)\}_{i=1}^n$ will be called an *Extended Tchebycheffian system* (ET-*system*) *provided*

$$D\begin{pmatrix}\psi_1, \cdots, \psi_n \\ x_1, \cdots, x_n\end{pmatrix} > 0 \text{ for all } a \leqq x_1 \leqq \cdots \leqq x_n \leqq b . \tag{2}$$

A system of functions $\{\psi_i\}_{i=1}^n$ such that for every k, $1 \leqq k \leqq n$, $\{\psi_i\}_{i=1}^k$ is an ET-system, will be called an *Extended Complete Tchebycheffian system* (ECT-*system*).

DEFINITION 1. A function $\varphi(x)$ defined on (a, b) is said to be convex with respect to $\{\psi_i\}_{i=1}^n$ provided

$$D\begin{pmatrix}\psi_1, \cdots, \psi_n, \varphi \\ x_1, \cdots, x_n, x_{n+1}\end{pmatrix} \geqq 0 , \quad a < x_1 < \cdots < x_{n+1} < b . \tag{3}$$

The set of functions satisfying (3) is evidently a cone. This cone is referred to as a "generalized convexity cone" and is denoted by $C(\psi_1, \cdots, \psi_n)$.

We consider in this paper only convexity with respect to ECT-systems. It can be proved (see [3]), that for a given ECT-system $\{\theta_i\}_{i=1}^n$ there exist functions $\{\psi_i(x)\}_{i=1}^n$ such that

1) $\psi_i(x) = \sum_{j=1}^i a_{ij}\theta_j(x)$, $i = 1, \cdots, n$.
2) $\{\psi_i\}_{i=1}^n$ is an ECT-system.
3) The cone $C(\theta_1, \cdots, \theta_n)$ is identical with $C(\psi_1, \cdots, \psi_n)$.
4) The functions $\{\psi_i(x)\}_{i=1}^n$ admit of the representation

$$\begin{aligned} \psi_1(x) &= u_1(x) \\ &\vdots \\ \psi_n(x) &= u_1(x)\int_a^x u_2(\xi_2)\int_a^{\xi_2} \cdots \int_a^{\xi_{n-1}} u_n(\xi_n)\, d\xi_n \cdots d\xi_2 \end{aligned} \tag{4}$$

where $u_1(x), \cdots, u_n(x)$ are continuous positive functions on $[a, b]$.

Henceforth, we shall assume that $\{\psi_i\}_1^n$, the basic functions of the cone, admit of the representation (4). The functions $\{u_i\}_1^n$ will be called the generating set of the ECT-system $\{\psi_i\}_1^n$. Note that the ordinary convexity of order n is convexity with respect to the system $(1, x, \cdots, x^{n-1})$ which is generated by $\{u_i \equiv 1, i = 1, \cdots, n\}$.

We define the differential operators

$$D_j f = \frac{d}{dx}\,\frac{1}{u_j} f , \quad E^j f = D_j \cdots D_1 f . \tag{5}$$

and note that

$$(6) \qquad \begin{aligned} E^j\psi_k &= 0 \qquad\qquad 1 \leqq k \leqq j \\ E^j\psi_{j+1} &= u_{j+1} > 0 \,. \end{aligned}$$

We list some properties of functions belonging to $C(\psi_1, \cdots, \psi_n)$. These properties are noted for ready reference as they are used extensively throughout the sequel. The first three are proved in [3], while the others are simple observations:

(1) If $\varphi(x) \in C^n(a, b) \cap C(\psi_1, \cdots, \psi_n)$, then $E^n\varphi(x) \geqq 0$.

(2) If $\varphi(x) \in C(\psi_1, \cdots, \psi_n)$, then $\varphi(x) \in C^{n-2}(a, b)$. Furthermore, $\varphi^{(n-2)}(x)$ has a right derivative $\varphi_R^{(n-1)}$ which is right continuous and a left derivative $\varphi_L^{(n-1)}$ which is left continuous and $\varphi_L^{(n-1)} = \varphi_R^{(n-1)}$ a.e.

(3) The set $\{\varphi(x) \mid \varphi(x) \in C^n(a, b) \cap C(\psi_1, \cdots, \psi_n)\}$ is weakly dense in $C(\psi_1, \cdots, \psi_n)$, in the topology of pointwise convergence.

(4) Let $\varphi(x) \in C(\psi_1)$ and let it be right continuous. Then

$$d\rho(x) = d\left[\frac{\varphi(x)}{\psi_1(x)}\right]$$

is a completely additive positive measure, which is determined uniquely by the requirement that it have no mass at a.

(5) We next introduce the concept of a "reduced" system. The system $(E^i\psi_{i+1}, \cdots, E^i\psi_n)$ is called the ith "reduced" system associated with $(\psi_1, \cdots, \psi_n)$. It is clearly seen that this is the ECT-system generated by $(u_{i+1}, \cdots, u_n)$, so that we may consider the "reduced" cone $C(E^i\psi_{i+1}, \cdots, E^i\psi_n)$. From the definitions, it follows that

$$\varphi(x) \in C(\psi_1, \cdots, \psi_n) \Longrightarrow E^i\varphi(x) \in C(E^i\psi_{i+1}, \cdots, E^i\psi_n)$$

for $n \geqq 2$ and $i \leqq n-2$. For $i = n-1$, in order that the relation persist we have to replace

$$E^{n-1}\varphi(x) \text{ by } E_R^{n-1}\varphi(x) = \left[\frac{E^{n-2}\varphi(x)}{u_{n-1}(x)}\right]_R' .$$

(6) Let $\varphi(x) \in C(\psi_1, \cdots, \psi_n)$, $n \geqq 2$. Then, by property (5), $E_R^{n-1}\varphi(x) \in C(u_n)$. Furthermore, by property (2) it is right continuous. Hence, by property (4), there exists a positive measure $d\rho = d[E_R^{n-1}\varphi(x)/u_n(x)]$ without mass at a, which is uniquely determined by $\varphi(x)$.

2. Extreme rays and the representation theorem. A great deal of information about a cone can be derived from the structure of its extreme rays (if it has any). Fortunately, our cone has extreme rays; moreover, there is available a representation for functions of the cone in terms of the extreme rays. A similar structure is also inherited by the finite intersections of generalized convexity cones, of the type $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$, where C^+ denotes the cone of positive functions.

THEOREM 1. *The extreme rays of* $C(\psi_1, \cdots, \psi_n)$, $n \geqq 2$, *modulo the linear combinations* $\sum_{i=1}^n a_i \psi_i(x)$, *are the solutions of*

$$d[E_R^{n-1}\varphi(x)/u_n(x)] = \delta_t(x) \qquad a < t < b . \tag{7}$$

where $\delta_t(x)$ is the delta distribution with jump at t.

The case $n = 1$ is exceptional, as will be explained later.

Proof. We note first that two solutions of the equation

$$d[E_R^{n-1}\varphi(x)/u_n(x)] = d\mu(x) \tag{*}$$

for a given $d\mu(x)$, can differ only by a linear combination of the $\{\psi_i(x)\}_{i=1}^n$. Indeed, the difference of two such solutions has to satisfy $E^n\varphi(x) \equiv 0$. By (6), the functions $\{\psi_1(x)\}_1^n$ are solutions of this equation, and since their Wronskian is different from zero, they form a fundamental system of solutions, so that every solution is a linear combination of $\{\psi_i(x)\}_{i=1}^n$. Hence, the equation (*) with n prescribed initial conditions admits *a unique solution.*

(a) Let $\varphi^*(x)$ be a solution of (7), and let $\varphi^*(x) = \varphi_1(x) + \varphi_2(x)$ where $\varphi_1(x), \varphi_2(x) \in C(\psi_1, \cdots, \psi_n)$. We have

$$d[E_R^{n-1}\varphi^*(x)/u_n(x)] = \delta_t(x) = d[E_R^{n-1}\varphi_1(x)/u_n(x)] + d[E_R^{n-1}\varphi_2(x)/u_n(x)] ,$$

where the terms on the right hand side are nonnegative measures. Thus, their supports have to be concentrated at the point t. But the only nonnegative measures whose support consists of one point are nonnegative multiples of the δ-functions. Hence,

$$d[E_R^{n-1}\varphi_i(x)/u_n(x)] = k_i\delta_t(x) \qquad \begin{matrix} 0 \leqq k_i \leqq 1 , \\ k_1 + k_2 = 1 \end{matrix}$$

and by the uniqueness property which was mentioned above

$$\varphi_i(x) = k_i\varphi^*(x) + \sum_{j=1}^n a_j^i \psi_j(x) , \qquad i = 1, 2 .$$

Thus, each solution of (7) is an extreme ray.

(b) Let now $\bar{\varphi}(x)$ be a solution of

$$d[E_R^{n-1}\varphi(x)/u_n(x)] = d\mu(x) \geqq 0$$

where the support of $d\mu(x)$ contains at least two distinct points x_1 and x_2. Let I_1 and I_2 be two disjoint intervals such that $x_i \in I_i$, $i = 1, 2$. Then, we have $d\mu(x) = d\mu_1(x) + d\mu_2(x)$ where

$$d\mu_1(x) = \begin{cases} d\mu(x) & x \in I_1 \\ 0 & \text{otherwise} \end{cases}$$

$$d\mu_2(x) = \begin{cases} 0 & x \in I_1 \\ d\mu(x) & \text{otherwise} . \end{cases}$$

If we denote by $\varphi_i(x)$ a solution of

$$d[E_R^{n-1}\varphi(x)/u_n(x)] = d\mu_i(x) \geqq 0 \qquad i = 1, 2 ,$$

then $\varphi_i(x) \in C(\psi_1, \cdots, \psi_n)$, $i = 1, 2$, and

$$\bar{\varphi}(x) = \varphi_1(x) + \varphi_2(x) + \sum_{i=1}^{n} a_i \psi_i(x) .$$

Hence, $\bar{\varphi}(x)$ is not an extreme ray.

REMARK. The case $n = 1$ requires special attention, since in this case we are not assured of the right continuity of $\varphi(x)$ so that $d[\varphi(x)/u_1(x)]$ is not well defined. Nevertheless, the modifications are slight.

The regularization $\tilde{\varphi}(x)$ of a function $\varphi(x) \in C(\psi_1)$ is defined to be the function obtained by changing the values at the points of discontinuity (a denumerable set), so as to render the function right continuous. With this definition, we have:

The extreme rays of $C(\psi_1)$ are the functions $\varphi(x)$ whose regularization satisfies

$$d[\tilde{\varphi}(x)/u_1(x)] = \delta_t(x) \qquad a < t < b . \tag{8}$$

Henceforth, every statement about the cones is to be understood, when $n = 1$, to apply to regularized functions only. The modifications necessary in order to obtain the general case follow the ideas indicated above.

DEFINITION 2. Let $\varphi_n(x; t)$ denote the extreme ray of $C(\psi_1, \cdots, \psi_n)$ corresponding to $\delta_t(x)$, (for $n = 1$, it is determined only a.e.), which satisfies

$$E^{i-1}\varphi(a) = 0 , \qquad i = 1, \cdots, n . \tag{9}$$

where $E^0\varphi(a) = \varphi(a)$.

These conditions determine the function completely because of the uniqueness property mentioned in the proof of Theorem 1 and the following lemma, which will be used several times.

LEMMA 1. *The set of initial data* $\{E^{i-1}f(a)\}_1^n$ *determines completely the data* $\{f^{(i-1)}(a)\}_1^n$ *and conversely.*

Proof. The relations that express one set of data in terms of the other, are explicitly of the form

$$E^0 f(a) = f^{(0)}(a)$$
$$E^{i-1} f(a) = \sum_{j=1}^{i} f^{(j-1)}(a) c_j^i \qquad 2 \leqq i \leqq n .$$

We clearly have, for $2 \leqq i \leqq n$, $c_i^i = 1/u_1(a) \cdots u_{i-1}(a) > 0$. Hence, the matrix of these relations is triangular and nonsingular, and the lemma follows.

A further insight into the structure of the extreme rays is offered by the following theorem:

THEOREM 2. *Let the functions* $\chi_i(x; t)$, $i = 1, \cdots, n$, *be defined recursively by the rules*

$$\chi_1(x; t) = \begin{cases} 0 & a \leqq x < t \\ \psi_1(x) & t \leqq x \leqq b \end{cases} \tag{10}$$

$$\chi_k(x; t) = \begin{cases} 0 & a \leqq x < t \\ \psi_k(x) - \sum_{i=1}^{k-1} \dfrac{E^{i-1}\psi_k(t)}{u_i(t)} \chi_i(x; t) & t \leqq x \leqq b , \\ & 2 \leqq k \leqq n \end{cases} \tag{11}$$

Then $\chi_n(x; t) = \varphi_n(x; t)$ (*as defined in Definition* 2).

Proof. The following formulas can be proved by induction

$$E^j \chi_n(x; t) = \begin{cases} 0 & a \leqq x < t \\ E^j \psi_n(x) & \\ \quad - \sum_{i=2}^{n-j-1} \dfrac{E^{j+i-1}\psi_n(t)}{u_{j+i}(t)} E^j \chi_{i+j}(x; t) & \\ \quad - \dfrac{E^j \psi_n(t)}{u_{j+1}(t)} E_R^j \chi_{j+1}(x; t) , & j \leqq n-2 \\ & t \leqq x \leqq b \end{cases} \tag{12}$$

$$E_R^{n-1} \chi_n(x; t) = \begin{cases} 0 & a \leqq x < t \\ u_n(x) & t \leqq x \leqq b \end{cases} \tag{13}$$

$$d\left[\frac{E_R^{n-1} \chi_n(x; t)}{u_n(x)}\right] = \delta_t(x) . \tag{14}$$

The proof proceeds by double induction. First, we apply induction on n, and then, for fixed n we apply induction on j. We shall not bother the reader with the computational details.

Furthermore, for the point $x = a$, the conditions

$$E^{i-1}\chi_n(a; t) = 0 \qquad i = 1, \cdots, n$$

are evidently satisfied.

Using these formulas, we note that

$$E^j\varphi_n(x; x) = 0 \text{ for } x \in (a, b),\ 0 \leqq j \leqq n-2\ . \tag{15}$$

This result follows by simple induction from (12), since by using (12) and (13) we deduce its validity for $j = n - 2$. Thus, the functions $\varphi_n(x; t)$, $n \geqq 2$, are continuous in t, $a < t < b$, for any fixed x. (Moreover, they possess a continuous derivative in (a, x), (x, b)). For $n = 1$, there is one point of discontinuity, namely, $t = x$. In order to extend the continuity property to the closed interval $a \leqq t \leqq b$, we define

$$\varphi_n(x; a) \equiv \psi_n(x),\ \ \psi_n(x, b) \equiv 0\ . \tag{16}$$

COROLLARY 2.1. *The functions $\varphi_n(x; t)$ belong to*

$$C^+ \cap [\bigcap_{j=1}^{n} C(\psi_1, \cdots, \psi_j)]\ .$$

Proof. We observe first that, for $j \leqq n - 1$, the relation $E^{j-1}\varphi_n(t; t) = 0$ together with $E^j\varphi_n(x; t) \geqq 0$ for $x \geqq t$, imply that $E^{j-1}\varphi_n(x; t) \geqq 0$ for $x \geqq t$. Indeed, from the inequality $E^j\varphi_n(x; t) \geqq 0$ we deduce that $E^{j-1}\varphi_n(x; t)/u_j(x)$ is monotone increasing for $x \geqq t$. Since its value at t is zero and $u_j(x) \geqq 0$, it follows that $E^{j-1}\varphi_n(x; t) \geqq 0$ for $x \geqq t$. By (13), (15) and this observation, we have

$$E^{j-1}\varphi_n(x; t) \geqq 0 \text{ for every } x, t; j = 1, \cdots, n-1\ .$$

Furthermore, by (13) and (14), $\varphi_n(x; t)$ belongs also to

$$C(\psi_1, \cdots, \psi_{n-1}) \cap C(\psi_1, \cdots, \psi_n)\ .$$

Having obtained the extreme rays and some of their structural properties, we can now state the representation theorem.

THEOREM 3. *Let $f(x)$ be any function of $C(\psi_1, \cdots, \psi_n)$, and let*

$$d\left[\frac{E_R^{n-1}f(x)}{u_n(x)}\right] = d\rho(x) \geqq 0\ . \tag{17}$$

Then $f(x)$ admits of the representation

$$f(x) = \int_a^x \varphi_n(x; t)\, d\rho(t) + \sum_{i=1}^{n} \frac{E^{i-1}f(a)}{u_i(a)}\, \psi_i(x)\ . \tag{18}$$

REMARK. For $n = 1$, the representation (18) is valid only for the regularized $\tilde{f}(x)$, but the adaptation for general $f(x)$ is simple.

Proof. We shall prove the theorem for $n \geqq 2$. Denote the right hand side of (18) by $f^*(x)$. We shall show that

$$
\begin{aligned}
&d\left[\frac{E_R^{n-1} f^*(x)}{u_n(x)}\right] = d\rho(x) \\
&E^{i-1} f^*(a) = E^{i-1} f(a)\ , \qquad\qquad i = 1, \cdots, n\ .
\end{aligned} \tag{19}
$$

Then, by Lemma 1 and the uniqueness property, $f^*(x)$ will have to be identical with $f(x)$.

We shall make use of the following lemma.

LEMMA 2. *The operators* $\{E^i\}_1^n$ *satisfy*

$$
\begin{aligned}
&E^{i-1}\left[\int_a^x \varphi_n(x; t)\, d\rho(t)\right] = \int_a^x E^{i-1}\varphi_n(x; t)\, d\rho(t)\ , \quad i = 1, \cdots, n-1\ , \\
&E_R^{n-1}\left[\int_a^x \varphi_n(x; t)\, d\rho(t)\right] = \int_a^{x+} E_R^{n-1}\varphi_n(x; t)\, d\rho(t)\ .
\end{aligned} \tag{20}
$$

Proof. The first equation is proved by induction on i. The second equation follows by the same argument as the one used in the induction. For $i = 1$, (20) holds trivially. Suppose now that it holds for $i = k \leqq n - 2$. Then

$$
\begin{aligned}
E^k\left[\int_a^x \varphi_n(x; t)\, d\rho(t)\right] &= D_k E^{k-1}\left[\int_a^x \varphi_n(x; t)\, d\rho(t)\right] \\
&= D_k\left[\int_a^x E^{k-1}\varphi_n(x; t)\, d\rho(t)\right] \quad \text{(by the induction hypothesis)} \\
&= \frac{d}{dx}\left[\int_a^x \frac{E^{k-1}\varphi_n(x; t)}{u_k(x)}\, d\rho(t)\right] \\
&= \int_a^x E^k \varphi_n(x, t)\, d\rho(t)\ .
\end{aligned}
$$

The last equation holds because of the properties of $\varphi_n(x; t)$ expressed in Theorem 2 and equation (15). For the case $k = n - 1$, we have to include in the integral the possible jump of $\rho(t)$ at $t = x$. In the other cases, by (15), this inclusion makes no difference.

Using Lemma 2, (19) follows easily. Indeed, by (6)

$$
E^{k-1}\left[\sum_{i=1}^n \frac{E^{i-1} f(a)}{u_i(a)}\, \psi_i(x)\right]_{x=a} = E^{k-1} f(a)\ , \qquad 1 \leqq k \leqq n\ .
$$

Furthermore, the integral term vanishes for $x = a$, since $d\rho(t)$ is normalized so that no mass occurs at a. This establishes the second part of (19).

By Lemma 2,

$$E_R^{n-1}f^*(x) = \int_a^{x+} E_R^{n-1}\varphi_n(x;\, t)\, d\rho(t) + \frac{E^{n-1}f(a)}{u_n(a)}\, u_n(x)$$

which by (13) is equivalent to

$$\frac{E_R^{n-1}f^*(x)}{u_n(x)} = \int_a^{x+} d\rho(t) + \frac{E^{n-1}f(a)}{u_n(a)}\,,$$

or $$d\left[\frac{E_R^{n-1}f^*(x)}{u_n(x)}\right] = d\rho(x)\,, \qquad \text{as was to be proved.}$$

THEOREM 4. *The functions $\{\varphi_n(x;\, t)\}$ together with $\{\psi_i(x)\}_1^n$ form the complete system of extreme rays of $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$.*

Proof. Note that here it is no longer modulo a linear combination, but a bona fide system of extreme rays.

We observe that $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)] \subset C(\psi_1, \cdots, \psi_n)$. Furthermore, by Corollary 2.1, the functions $\{\varphi_n(x;\, t)\}$ belong to $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$. Hence, since $\{\varphi_n(x;\, t)\}$ are extreme rays for $C(\psi_1, \cdots, \psi_n)$ and belong to the subcone $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$, they have to be extreme rays for the subcone as well.

Next we show that $\{\psi_i(x)\}_1^n$ are also extreme rays of $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$. By (6), we have

$$\psi_i(x) \in C^+ \cap [\bigcap_{j=1}^n C(\psi_1, \cdots, \psi_j)]\,, \qquad i = 1, \cdots, n\,. \tag{21}$$

Furthermore, we assert that the following is true,

$$\theta(x) = \sum_{j=1}^n b_j\, \psi_j(x) \in C^+ \cap [\bigcap_{j=1}^n C(\psi_1, \cdots, \psi_j)] \quad \text{if and only if } b_j \geqq 0,\ j = 1, \cdots, n\,. \tag{22}$$

Indeed, sufficiency of the conditions $b_j \geqq 0$, $j = 1, \cdots, n$ is obvious by (21). The necessity is easily proved by contraposition: If b_{j_0} is the first negative coefficient, then by applying the operator E^{j_0-1} to $\sum_{j=1}^n b_j \psi_j(x)$ and substituting $x = a$, we ascertain that $\sum_{j=1}^n b_j \psi_j(x)$ cannot belong to $C(\psi_1, \cdots, \psi_{j_0-1})$.

Let i be any number between 1 and n. Suppose that there exists a decomposition of $\psi_i(x)$ of the form

$$\psi_i(x) = \varphi_1(x) + \varphi_2(x)$$

where $$\varphi_1(x),\, \varphi_2(x) \in C^+ \cap [\bigcap_{j=1}^n C(\psi_1, \cdots, \psi_j)]\,.$$

Applying the operator $d[E_R^{n-1} \cdot / u_n(x)]$ to both sides and using the nonnegativity of the resulting summands, we deduce

$$E^n \varphi_1(x) \equiv E^n \varphi_2(x) \equiv 0\,.$$

Hence, both $\varphi_1(x)$ and $\varphi_2(x)$ are linear combinations of $\{\psi_i(x)\}_1^n$. Using (22) and the linear independence of $\{\psi_i(x)\}_1^n$, we find

$$\varphi_1(x) = a\psi_i(x) \quad \varphi_2(x) = (1-a)\psi_i(x) \qquad 0 \leqq a \leqq 1 ,$$

thus proving that $\psi_i(x)$ is an extreme ray.

The same argument as in Theorem 1 part (*b*) shows that there can be no extreme ray other than $\{\varphi_n(x; t)\}$ and $\{\psi_i(x)\}_1^n$.

REMARK. The representation theorem for the cone $C(\psi_1, \cdots, \psi_n)$ gives rise to a similar one for the subcone $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$. The additional requirement is that the coefficients $E^{i-1}f(a)/u_i(a)$, $i = 1, \cdots, n$, in (18) be nonnegative.

Consider now the jth "reduced" system $\{E^j\psi_{j+1}(x), \cdots, E^j\psi_n(x)\}$ and the corresponding cone $C\{E^j\psi_{j+1}, \cdots, E^j\psi_n\}$. Let

$$\mathscr{D}_{k;j}, \quad \mathscr{E}^{k;j} \qquad k = 1, \cdots, n-j$$

be the differential operators associated with this cone. (They are defined with respect to the new system in the same way that D_j, E^j were defined in (5) with respect to the original system.) Denote by $\{\varphi_{n;j}(x; t)\}$ the extreme rays of this cone which satisfy

$$\mathscr{E}^{k-1;j}\varphi(a) = 0 \qquad k = 1, \cdots, n-j . \tag{23}$$

LEMMA 3. *The extreme rays $\varphi_{n;j}(x; t)$ are related to the extreme rays of the original cone by means of the formula*

$$E^j\varphi_n(x; t) = \varphi_{n;j}(x; t) . \tag{24}$$

Proof. Direct application of the definitions yields.

$$\begin{cases} \mathscr{D}_{k;j}f(x) \equiv D_{j+k}f(x) \\ \mathscr{E}^{k;j}E^jf(x) = E^{j+k}f(x) . \end{cases}$$

Hence, by using the properties of $\varphi_n(x; t)$, we obtain

$$d\left[\frac{\mathscr{E}_R^{n-j-1;j}E^j\varphi_n(x; t)}{u_n(x)}\right] = d\left[\frac{E_R^{n-1}\varphi_n(x; t)}{u_n(x)}\right] = \delta_t(x)$$

and

$$[\mathscr{E}^{k-1;j}E^j\,\varphi_n(x; t)]_{x=a} = E^{k+j-1}\varphi_n(a; t) = 0 , \qquad k = 1, \cdots, n-j$$

and the uniqueness property implies (24).

3. Characterization of the dual cones. Throughout this section $d\mu$ will denote a signed measure of bounded variation on (a, b). The dual cone to a cone C of functions is the set of signed measures $d\mu$

such that

$$\int_a^b \varphi(x)\, d\mu(x) \geqq 0 \text{ for all } \varphi \in C\,. \tag{25}$$

The dual cone is denoted by C^*.

Necessary and sufficient conditions for a signed measure to belong to $C^*(\psi_1, \cdots, \psi_n)$—the cone dual to $C(\psi_1, \cdots, \psi_n)$—were obtained in [4]. We wish to characterize the dual cone of an intersection of generalized convexity cones. We will obtain, for example, necessary and sufficient conditions for a measure to belong to the cone dual to the cone of positive, increasing and convex functions.

We first investigate finite intersections of generalized convexity cones, of the type $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$.

THEOREM 5. *Necessary and sufficient conditions for $d\mu$ to belong to the dual to $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$ are:*

$$(1) \quad \int_a^b \psi_i(x)\, d\mu(x) \geqq 0 \qquad\qquad i = 1, \cdots, n \tag{26}$$

$$(2) \quad \int_a^b \varphi_n(x; t)\, d\mu(x) \geqq 0 \qquad\qquad \textit{for } a \leqq t \leqq b\,. \tag{27}$$

Proof. (a) *Necessity.* By (6), the functions $\{\psi_i(x)\}_{i=1}^n$ belong to $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$. Hence, (26) is a necessary condition. Furthermore, by Corollary 2.1, $\varphi_n(x; t) \in C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$ for every t. Hence, (27) is necessary too.

(b) *Sufficiency.* Let $g(x)$ be an arbitrary function belonging to $C^+ \cap [\cap_{j=1}^n C(\psi_1, \cdots, \psi_j)]$. Then, by Theorem 3 and the Remark following Theorem 4, $g(x)$ admits of the representation

$$g(x) = \int_a^x \varphi_n(x; t)\, d\rho(t) + \sum_{i=1}^n \frac{E^{i-1}g(a)}{u_i(a)}\, \psi_i(x)$$

where $d\rho(t)$ is a nonnegative measure and

$$\frac{E^{i-1}g(a)}{u_i(a)} \geqq 0 \qquad\qquad \text{for } i = 1, \cdots, n\,.$$

Thus,

$$\begin{aligned}
\int_a^b g(x)\, d\mu(x) &= \int_a^b \left[\int_a^x \varphi_n(x; t) d\rho(t)\right] d\mu(x) + \sum_{i=1}^n \frac{E^{i-1}g(a)}{u_i(a)} \int_a^b \psi_i(x)\, d\mu(x) \\
&\geqq \int_a^b \left[\int_t^b \varphi_n(x; t)\, d\mu(x)\right] d\rho(t) \\
&= \int_a^b \left[\int_a^b \varphi_n(x; t)\, d\mu(x)\right] d\rho(t) \qquad\qquad \text{(by Theorem 2)} \\
&\geqq 0\,.
\end{aligned} \tag{28}$$

An examination of (28) yields also:

COROLLARY 5.1. *Necessary and sufficient conditions for* $d\mu$ *to belong to the dual to* $\bigcap_{j=k}^{n} C(\psi_1, \cdots, \psi_j)$, $1 \leqq k \leqq n$, *are*

(1) $\int_a^b \psi_i(x)\, d\mu(x) = 0 \qquad i = 1, \cdots, k$

(2) $\int_a^b \psi_i(x)\, d\mu(x) \geqq 0 \qquad k+1 \leqq i \leqq n$

(3) $\int_a^b \varphi_n(x; t)\, d\mu(x) \geqq 0 \qquad a \leqq t \leqq b\,.$

REMARK. For $k = n$, Corollary 5.1 expresses the characterization of the cone dual to $C(\psi_1, \cdots, \psi_n)$. Thus it subsumes as a special case the results for $C^*(\psi_1, \cdots, \psi_n)$ which were obtained, in a different way, by S. Karlin and A. Novikoff [4].

The dual cones of other types of intersections can also be characterized, but the conditions are more complicated. One might be led to believe that replacing the suitable inequalities in (26) by equalities, as was done in Corollary 5.1, would provide necessary and sufficient conditions. Unfortunately, as we show later by a counter example, this is not the case. It is true, however, that the conditions obtained in this way are sufficient. The sufficiency proof proceeds in exactly the same way as in Theorem 5.

We defer the discussion of the counter example and divert now our attention to the case $n = 2$, which is important in applications.

Let $C(\psi_1, \cdots, \psi_n)^-$ denote the cone of functions $\varphi(x)$ such that $-\varphi(x) \in C(\psi_1, \cdots, \psi_n)$. For example, $C(1)^-$ is the cone of nonincreasing functions and $C(1, x)^-$ is the cone of concave functions (in the standard sense).

THEOREM 6. *Necessary and sufficient conditions for* $d\mu$ *to belong to the dual to* $C(\psi_1)^- \cap C(\psi_1, \psi_2)$ *are*

$$\int_a^b \psi_1(x)\, d\mu(x) = 0 \tag{29}$$

$$\int_a^b \psi(t; x)\, d\mu(x) \geqq 0 \qquad a \leqq t \leqq b \tag{30}$$

where $\psi(t; x)$ is defined by

$$\psi(t; x) = \begin{cases} \dfrac{\psi_2(t)}{u_1(t)}\, \psi_1(x) - \psi_2(x) & a \leqq x \leqq t \\ 0 & t \leqq x \leqq b\,. \end{cases}$$

Proof: (a) *Necessity*. Since $\pm\,\psi_1(x) \in C(\psi_1)^- \cap C(\psi_1, \psi_2)$, we find that (29) is necessary. It can be easily verified that $\psi(t; x) \in C(\psi_1)^- \cap C(\psi_1, \psi_2)$, so that (30) is necessary as well.

(b) *Sufficiency*. According to property 3) of Section 1, it would be sufficient to consider only functions with continuous second derivatives. Let $\varphi(x)$ be an arbitrary function of $C^2(a, b) \cap C(\psi_1)^- \cap C(\psi_1, \psi_2)$ and introduce the notation

$$I_1(x) = -\int_a^x \psi_1(t)\, d\mu(t)\ ,\quad I_2I_1(x) = -\int_a^x I_1(t)\, u_2(t)\, dt\ .$$

Repeated integration by parts together with relation (29) yields

$$\int_a^b \psi(t; x)\, d\mu(x) = I_2I_1(t) \tag{31}$$

and

$$\int_a^b \varphi(x)\, d\mu(x) = -\, I_2I_1(b)\, \frac{D_1\varphi(b)}{u_2(b)} + \int_a^b I_2I_1(x)\, E^2\varphi(x)\, dx\ . \tag{32}$$

Using (31), (30) and taking into consideration the fact that $\varphi(x) \in C(\psi_1)^- \cap C(\psi_1, \psi_2)$, the right hand side of (32) is seen to be nonnegative.

Similar arguments establish also:

THEOREM 7. *Necessary and sufficient conditions for $d\mu$ to belong to the dual to $C^+ \cap C(\psi_1, \psi_2)$ are*

(1) $\int_a^b \psi_1(x)\, d\mu(x) \geqq 0$

(2) $\int_a^b \psi(t; x)\, d\mu(x) \geqq 0 \qquad a \leqq t \leqq b$

(3) $\int_a^b \varphi_2(x; t)\, d\mu(x) \geqq 0 \qquad a \leqq t \leqq b\ .$

REMARK. We give now the counter example whose existence was asserted before Theorem 6. Let $a = 0$, $b = 1$ and $u_1(x) \equiv u_2(x) \equiv 1$. The necessary and sufficient conditions for $d\mu$ to belong to the dual of $C^+ \cap C(1, x)$ take the simple form

$$\int_0^1 d\mu(x) \geqq 0\ ,$$

and

$$\int_0^t (t - x)\, d\mu(x) \geqq 0\ ,\quad \int_t^1 (x - t)\, d\mu(x) \geqq 0\ , \qquad 0 \leqq t \leqq 1\ .$$

Considering $d\mu(x) = (x + \alpha)dx$, we find that a necessary and sufficient condition for this $d\mu$ to belong to the dual is that $\alpha \geqq 0$.

On the other hand, if the methods of Corollary 5.1 were applicable, a necessary condition would be

$$\int_0^1 x \, d\mu(x) = 0 ,$$

which, for $d\mu(x) = (x + \alpha)dx$, is equivalent to $3\alpha + 2 = 0$. Since this is incompatible with $\alpha \geqq 0$, it follows that it is not a necessary condition, as was to be shown.

4. Miscellaneous properties and applications. In this section we wish to demonstrate the possibilities inherent in our approach. Classical inequalities can be obtained in a straightforward manner, and some further results can be discovered.

(a) Let $\{u_i(x)\}_{i=1}^k$ and $\{v_j(x)\}_{j=1}^n$ be two sets of positive functions. Let $\{\psi_i(x)\}_1^k$ and $\{\theta_j(x)\}_1^n$ be the ECT-systems generated by $\{u_i(x)\}_1^k$ and $\{v_j(x)\}_1^n$ respectively. Define $\{u_{k+i}(x)\}_{i=1}^n$ by

$$u_{k+i}(x) = v_i(x) \qquad i = 1, \cdots, n ,$$

and let $\{\chi_i(x)\}_{i=1}^{k+n}$ be the ECT-system generated by $\{u_i(x)\}_{i=1}^{k+n}$. Denote by $\{\varphi_k(x; t)\}$ the extreme rays of $C(\psi_1, \cdots, \psi_k)$ which satisfy

$$E^{i-1}\varphi(a) = 0 \qquad i = 1, \cdots, k ,$$

where D_i, E^i, $i = 1, \cdots, k + n$ are the differential operators associated with $\{\chi_i(x)\}_1^{k+n}$. With these definitions, we have

THEOREM 8. *Let $f(x) \in C(\theta_1, \cdots, \theta_n)$; then*

$$g(x) = \int_a^x \varphi_k(x; t) \, f(t) \, dt \tag{33}$$

belongs to $C(\chi_1, \cdots, \chi_{k+n})$.

Proof. By Lemma 2, we have

$$E^{k-2}g(x) = \int_a^x E^{k-2}\varphi_k(x; t) \, f(t)dt .$$

Substituting the value of $E^{k-2}\varphi_k(x; t)$ from (12) results in

$$E^{k-2}g(x) = \int_a^x \left[E^{k-2}\psi_k(x) - \frac{E^{k-2}\psi_k(t)}{u_{k-1}(t)} \, u_{k-1}(x) \right] f(t)dt .$$

Hence, $E^{k-1}g(x)$ exists in this case, and it satisfies

$$E^{k-1}g(x) = \int_a^x u_k(x) \, f(t)dt = u_k(x) \int_a^x f(t)dt$$

so that

$$E^k g(x) = f(x) \ .$$

We observe that from the definitions it follows that $\{\theta_i(x)\}_1^n$ is the kth reduced system associated with $\{\chi_i(x)\}_1^{k+n}$. We denote by $\mathscr{D}_{i;k}$, $\mathscr{E}^{i;k}$, $i = 1, \cdots, n$, the differential operators associated with $\{\theta_i(x)\}_1^n$. Then

$$d\left[\frac{E_R^{k+n-1}g(x)}{u_{n+k}(x)}\right] = d\left[\frac{\mathscr{E}_R^{n-1;k}E^k g(x)}{u_{n+k}(x)}\right] = d\left[\frac{\mathscr{E}_R^{n-1;k}f(x)}{v_n(x)}\right]$$

which is nonnegative since $f(x) \in C(\theta_1, \cdots, \theta_n)$.

REMARK. Let us consider the case of ordinary convexity of order n, i.e., let $u_i(x) \equiv v_j(x) \equiv 1$ for every i and j. We then have explicit expressions for the extreme rays of the cone $C(1, x, \cdots, x^{k-1})$. In fact, as can easily be verified by induction (using relation (12)), we have

$$\varphi_k(x; t) = \begin{cases} 0 & a \leqq x < t \\ \dfrac{(x-t)^{k-1}}{(k-1)!} & t \leqq x \leqq b \ . \end{cases} \tag{34}$$

Using this result, Theorem 8 yields:

COROLLARY 8.1. *If $f(x) \in C(1, \cdots, x^{n-1})$, then the function*

$$g(x) = \int_a^x (x-t)^{k-1} f(t) dt$$

belongs to $C(1, \cdots, x^{n+k-1})$.

The result stated in Corollary 8.1 was mentioned by Popoviciu [6] and is a generalization of a result due to Montel [5].

(b) A second application is related to the following interesting problem: Under what conditions on the sequences $\{a_i\}$, $\{b_i\}$, $i = 0, \cdots, n-1$, can we construct a function $f(x)$ belonging to $C(\psi_1, \cdots, \psi_n)$ on $[a, b]$ and satisfying

$$f^{i-1}(a) = a_{i-1} \ , \qquad f^{i-1}(b) = b_{i-1} \ , \qquad i = 1, \cdots, n \ . \tag{35}$$

Furthermore, what are the conditions under which the solution is unique?

For the case of ordinary convexity of order n, the question was solved by S. Kakeya [2] and T. Popoviciu [6].

We note first that, by Lemma 1, prescribing the data (35) is equivalent to specifying the values

$$\frac{E^{i-1}f(a)}{u_i(a)} = a_{i-1}^* , \qquad E^{i-1}f(b) = \bar{b}_{i-1} , \qquad i = 1, \cdots, n .$$

THEOREM 9. *Let the sets* $\{a_i\}, \{b_i\}$, $i = 0, \cdots, n-1$ *be given. We define*

$$c_{i-1}^* = \bar{b}_{n-i} - \sum_{j=n-i+1}^{n} a_{j-1}^* E^{n-i}\psi_j(b) , \qquad i = 1, \cdots, n . \tag{36}$$

There are four possible cases

(1) *If* $c_0^* = 0$, *there exists a solution if and only if* $c_{i-1}^* = 0$ *for* $i = 2, \cdots, n$. *If a solution exists it is unique.*

If $c_0^* \neq 0$, *we have three alternatives:*

(2) *If the numbers* $(c_i^*/c_0^*)u_n(b)$, $i = 1, \cdots, n-1$ *lie in the interior of the cone spanned by the functions* $\{E^j\varphi_n(b; t), j = 0, \cdots, n-2, a \leqq t \leqq b\}$, *then there exists an infinite number of solutions.*

(3) *If the numbers lie on some parts of the boundary of this cone, there exists a unique solution.*

(4) *If the numbers are exterior to the cone, there is no solution.*

Proof. By Theorem 3, every $f(x)$ belonging to $C(\psi_1, \cdots, \psi_n)$ can be written as

$$f(x) = \int_a^x \varphi_n(x; t)\, d\rho(t) + \sum_{j=1}^{n} \frac{E^{j-1}f(a)}{u_j(a)} \psi_j(x) .$$

Using Lemma 2 and equations (6), we obtain

$$\bar{b}_{n-i} = \int_a^b E^{n-i}\varphi_n(b; t)\, d\rho(t) + \sum_{j=n-i+1}^{n} a_{j-1}^* E^{n-i}\psi_j(b) \qquad \text{for } i = 2, \cdots, n$$

$$\bar{b}_{n-1} = \int_a^b E_R^{n-1}\varphi_n(b; t)\, d\rho(t) + a_{n-1}^* u_n(b) .$$

With the aid of (36) and (13) we deduce

$$\begin{aligned} c_{i-1}^* &= \int_a^b E^{n-i}\varphi_n(b; t)\, d\rho(t) \qquad i = 2, \cdots, n \\ c_0^* &= u_n(b) \int_a^b d\rho(t) . \end{aligned} \tag{37}$$

Hence, if $c_0^* = 0$, then $d\rho(t) \equiv 0$ since it is a nonnegative measure. Thus, $f(x)$ must satisfy

$$E^n f(x) \equiv 0$$

and it is evident that there exists a solution if and only if $c_{i-1}^* = 0$, $i = 1, \cdots, n$. The linear combination $\sum_{j=1}^n a_{j-1}^* \psi_j(x)$ is then the unique solution of the problem.

In the other case, i.e., when $c_0^* \neq 0$, the relations (37) imply that the existence of a nonnegative measure $d\sigma(t)$ such that

$$\int_a^b d\sigma(t) = 1$$

$$\int_a^b E^{n-i}\varphi_n(b; t)\, d\sigma(t) = \frac{c_{i-1}^*}{c_0^*}\, u_n(b) \qquad i = 2, \cdots, n$$

is the necessary and sufficient condition for a solution. This in turn implies that we have only the three alternatives mentioned in the theorem.

(c) It is of importance to be able to construct classes of measures of the dual cones, since every such measure induces an inequality which might be useful in various contexts. We shall present some results in this direction.

THEOREM 10. *No nontrivial linear combination of the form* $d\mu = \sum_{i=1}^n a_i \psi_i(x) dx$, *belongs to* $C^*(\psi_1, \cdots, \psi_n)$.

Proof. By Corollary 5.1, a necessary condition for $d\mu$ to belong to $C^*(\psi_1, \cdots, \psi_n)$ is that

$$\int_a^b \Big(\sum_{i=1}^n a_i\, \psi_i(x)\Big)\psi_j(x)\, dx = 0 \qquad j = 1, \cdots, n\,. \tag{38}$$

The determinant of this system of n equations in n unknowns is the "Gram determinant" which does not vanish since $\{\psi_i(x)\}_1^n$ are linearly independent.

We next construct a simple measure belonging to the dual cone of $C^+ \cap C(1) \cap C(1, t)$ and deduce a new inequality. We let $a = 0$ in order to simplify calculations. Let $f(t) = t - \alpha$. By using the necessary and sufficient conditions stated in Theorem 5, we find that $d\mu = (t - \alpha)\, dt \in [C^+ \cap C(1) \cap C(1, t)]^*$ if and only if $\alpha \leqq b/2$. We shall use here the discrete analogue of this result, viz.,

LEMMA 3. *The sequence* $\{a_i = i - c,\ i = 1, \cdots, n\}$ *belongs to the cone dual to the cone of positive, increasing and convex n-sequences, if and only if* $c \leqq (n + 1)/2$.

Consider now the generalized mean function

$$M_t(\alpha; x) = \Big(\sum_{i=1}^n \alpha_i x_i^t\Big)^{1/t},\ \alpha_i \geqq 0,\ \sum_{i=1}^n \alpha_i = 1\,.$$

It is well known that $t \log M_t(\alpha, x)$ is a convex function [1]. If $x_i \geqq 1$, $i = 1, \cdots, n$, then $t \log M_t(\alpha, x)$ is also a positive increasing function of t, for $t \geqq 0$. Hence, by Lemma 3, we have

$$(39) \qquad \sum_{i=1}^{n}(i-c)\,t_i \log M_{t_i}(\alpha, x) \geqq 0\ , \qquad \begin{cases} x_i \geqq 1,\ i=1,\cdots,n\ ; \\ c \leqq \dfrac{n+1}{2}\ ; \\ 0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \end{cases}$$

or,

$$\prod_{i=1}^{n} M_{t_i}^{it_i} \geqq \prod_{i=1}^{n} (M_{t_i}^{t_i})^c\ .$$

The same analysis applies to the function

$$S_t(x) = \left(\sum_{i=1}^{n} x_i^t\right)^{1/t}$$

Here, we have

$$\prod_{j=1}^{n}\left(\sum_{i=1}^{n} x_i^{t_j}\right)^j \geqq \prod_{j=1}^{n}\left(\sum_{i=1}^{n} x_i^{t_j}\right)^c \qquad \begin{cases} x_i \geqq 1,\ i=1,\cdots,n\ ; \\ 0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n\ ; \\ c \leqq \dfrac{n+1}{2}\ . \end{cases}$$

(d) As a final application, we derive by our methods a result which is ascribed to Fejer, and is stated and proved in [7].

Let $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$; $0 = y_0 < \cdots y_{n+1} < 1$, and let these numbers satisfy

$$(40) \qquad x_\nu < y_{\nu+1} < x_{\nu+1} \qquad \nu = 0, \cdots, n\ .$$

We introduce the notation

$$y_{\nu+1} - x_\nu = p_\nu,\ x_{\nu+1} - y_{\nu+1} = q_\nu\ , \qquad \nu = 0, \cdots, n\ ,$$
$$p_{n+1} = 0\ ,$$

and we form the Riemann sums corresponding to these subdivisions.

$$S_n^{(l)}(f) = \sum_{\nu=0}^{n}(x_{\nu+1} - x_\nu) f(x_\nu)\ ,$$
$$S_{n+1}^{(l)}(f) = \sum_{\nu=0}^{n+1}(y_{\nu+1} - y_\nu) f(y_\nu)\ ,$$
$$S_n^{(r)}(f) = \sum_{\nu=0}^{n}(x_{\nu+1} - x_\nu) f(x_{\nu+1})\ .$$

The result we prove is:

THEOREM (Fejer). *Necessary and sufficient conditions for*

$$(41) \qquad S_n^{(r)}(f) \leqq S_{n+1}^{(r)}(f);\ S_n^{(l)}(f) \geqq S_{n+1}^{(l)}(f)$$

to hold for every $f(x)$ decreasing and convex, are

$$(42) \qquad \sum_{\nu=0}^{\mu} q_\nu(p_\nu - p_{\nu+1} - p_\nu) \geqq 0 \qquad \mu = 0, \cdots, n\ .$$

Proof. We prove the result for $S_n^{(l)}$, but this is essentially the proof for $S_n^{(r)}$ too. We have

$$S_n^{(l)}(f) - S_{n+1}^{(l)}(f) = \sum_{\nu=1}^{2n+2} a_\nu f(z_\nu) ,$$

where the $\{z_\nu\}$ are the numbers $\{x_\nu\}$, $\{y_\nu\}$ ordered by magnitude, while the coefficients $\{a_\nu\}$ are given by

$$a_1 = q_0; \quad a_{2k} = -(p_k + q_{k-1}) \qquad k = 1, \cdots, n+1$$
$$a_{2k+1} = q_k + p_k \qquad k = 1, \cdots, n .$$

We see easily that Theorem 6 is applicable with $u_1(t) \equiv u_2(t) \equiv 1$, and a discrete measure $d\mu(t)$ with mass a_ν at z_ν, $\nu = 1, \cdots, 2n+2$. The necessary and sufficient conditions in this case reduce to:

$$\begin{aligned} &\int_0^1 d\mu(x) = 0 \\ &\int_0^x \left[\int_0^z d\mu(t)\right] dz \geqq 0 , \qquad 0 \leqq x \leqq 1 . \end{aligned} \tag{43}$$

The function $\int_0^z d\mu(t)$ takes the values

$$\int_0^z d\mu(t) = \begin{cases} q_i & x_i \leqq z < y_{i+1} \qquad i = 0, \cdots, n \\ -p_i & y_i \leqq z < x_i \qquad i = 1, \cdots, n \\ 0 & y_{n+1} \leqq z \leqq 1 . \end{cases} \tag{44}$$

Hence, the first condition is automatically satisfied. Since the p_i, q_i are nonnegative, it is clear that if the second inequality holds for $x = x_k$, $k = 1, \cdots, n$, it will hold for every x. Hence,

$$\begin{aligned} \int_0^{x_\nu}\left[\int_0^z d\mu(t)\right]dz &= \sum_{i=0}^{\nu-1} q_i p_i - \sum_{i=1}^{\nu} p_i q_{i-1} \\ &= \sum_{i=0}^{\nu-1} q_i(p_i - p_{i+1}) \geqq 0 , \qquad \nu = 1, \cdots, n \end{aligned}$$

are the desired necessary and sufficient conditions.

REMARKS. We can derive some further results by our methods.

(1) From Corollary 5.1 we deduce that a necessary condition for $d\mu \in C(1)^*$ is that $-\int_0^x d\mu(t) \geqq 0$ for $0 \leqq x \leqq 1$. Hence, (44) implies that there do not exist $\{a_\nu\}$ such that (41) holds for every increasing function.

(2) Necessary and sufficient conditions for (41) to hold for every convex function are

$$\begin{aligned} &\sum_{\nu=0}^{\mu-1} q_\nu(p_\nu - p_{\nu+1}) \geqq 0 , \qquad \mu = 1, \cdots, n \\ &\sum_{\nu=0}^{n} q_\nu(p_\nu - p_{\nu+1}) = 0 . \end{aligned} \tag{45}$$

These conditions are incompatible since $q_n > 0$, $p_n > 0$ while $p_{n+1} = 0$. Thus, there do not exists $\{a_\nu\}$ such that (41) holds for every convex function.

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