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ON THE MULTIPLICATIVE EXTENSION PROPERTY

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A subspace M of a Banach algebra B is said to have the multiplicative extension property (abbr. m.e.p.) if whenever L is a linear functional on M of norm not greater than one, L is the restriction to M of a multiplicative linear functional on B. This property is considered in two settings—the measure algebra $M(G)$ of a suitable group, and the disc algebra $A(D)$ of functions analytic in the unit disc with continuous boundary values. The following theorems are proved.

THEOREM 2. **If Q is a compact subset of G such that $M_c(Q)$ has the m.e.p., then (i) for every nonzero $t \in G$, the set $Q \cap (Q - t)$ has μ-measure zero for every continuous measure μ on G, and (ii) $m(Q) = 0$, where m is the Haar measure for G.**

THEOREM 3. **Suppose G contains an independent Cantor set. Then there exists a compact subset Q of G such that for infinitely many $t \neq 0$, $Q \cap (Q - t)$ is countably infinite, and $M_c(Q)$ has the m.e.p.**

THEOREM 4. **There exist infinite dimensional subspaces of $A(D)$ with the m.e.p.**

These last two theorems are proved by constructing examples using a special decomposition of the Cantor set. This decomposition is presented in a separate section to simplify notation.

The multiplicative extension property was formulated by Hewitt in [1] after Hewitt and Kakutani [2] had given examples of such subspaces of the measure algebra of certain locally compact groups. In [1] Hewitt poses the problem of characterizing the subspaces with the m.e.p. in a general Banach algebra, and points out that the question was open even for the algebra $C(X)$ of all continuous complex valued functions on a compact Hausdorff space X. Later Phelps [4], who calls the m.e.p. "property (H)", announced such a characterization for $C(X)$. Phelps has shown that a closed subspace A of $C(X)$ has the m.e.p. if and only if X is homeomorphic to a symmetric compact convex subset of a locally convex space and A is the space of linear functions on X. If B is any commutative Banach algebra with unit and maximal ideal space X, and if M is a closed subspace of B with the m.e.p., then the Gelfand transform is an isometric isomorphism of M onto a closed subspace of $C(X)$ which has the m.e.p. Thus Phelps' result characterizes arbitrary subspaces with the m.e.p., in a sense. However, this result gives no information on whether a given algebra has subspaces

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with the m.e.p., or how to find such subspaces if they do exist.

The purpose of this paper is to give some further examples of subspaces with the m.e.p. We consider two algebras—the measure algebra of a suitable group G, and the disc algebra A of all functions analytic in the unit disc with continuous boundary values. In $M(G)$ we consider the following question: what conditions must be satisfied by a compact set Q in order that the space $M_c(Q)$ of all continuous measures supported on Q have the m.e.p.? This question seems to be very difficult, and we cannot give a complete answer, but we do arrive at some necessary conditions on Q. Roughly speaking, such a set Q must be fairly independent from a measure theoretic point of view, while an example is constructed to show that it can be rather dependent from an algebraic point of view. This is done in Section 3. In Section 4 we consider the disc algebra A where we construct an infinite dimensional subspace M with the multiplicative extension property. This subspace M has the interesting property that if f and g are linearly independent elements of norm 1 in M, then g assumes every value in some closed disc on the set of zeroes of f.

While the two examples of this paper are very different in character, they are both constructed by means of a special decomposition of the Cantor set into an uncountable family of disjoint perfect subsets. While this decomposition is not new, it is presented in Section 2 in order to provide a common notation for the later constructions.

2. Decomposition of the Cantor Set. Let C be the Cantor ternary set. If we let $x(n)$ denote the nth ternary digit of $x \in C$, where $x(n) = 0$ or $x(n) = 2$, then each x has a unique ternary expansion. For each $x \in C$, let

$$E_x = \{y \in C\colon y(2n) = x(n) \quad \text{for all} \quad n \in N\} .$$

The following statements are easily verified:

(1) *For each $x \in C$, E_x is a nonempty perfect subset of C.*

(2) *If $x \neq y$ then $E_x \cap E_y = \emptyset$, and $C = \bigcup_{x \in C} E_x$.*

(3) *If $t_n \in E_{x_n}$ $(n = 1, 2, \cdots,)$, and $t_n \to t_0$ where $t_0 \in E_{x_0}$, then $x_n \to x_0$.*

For each $x \in C$ we define a map h_x from E_0 to E_x as follows: for $y \in E_0$ and $k \in N$,

$$[h_x(y)](k) = \begin{cases} x(k/2), & \text{if } k \text{ is even} \\ y(k), & \text{if } k \text{ is odd.} \end{cases}$$

Then it is easy to check that each h_x is a homeomorphism of E_0 onto E_x, and h_0 is the identity map. Moreover, we have

(4) *For $x \in C$, $y \in E_0$ let $H(x, y) = h_x(y)$. Then H is continuous*

from $C \times E_0$ *onto* C.
Indeed, if $\{x_n: n \in N\} \subset C$, $\{y_n: n \in N\} \subset E_0$, $x_n \to x_0$, and $y_n \to y_0$, and $k \in N$, then we may choose ν so that for $n \geqq \nu$, $y_n(j) = y_0(j)$ for $j \leqq 2k+1$, while $x_n(j) = x_0(j)$ for $j \leqq k$. Thus, for $j \leqq 2k+1$ we have $H(x_n, y_n)(j) = H(x_0, y_0)(j)$, $n \geqq \nu$. Since k is arbitrary, we see that

$$\lim H(x_n, y_n) = H(x_0, y_0) .$$

3. Subspaces of $M(G)$.

In this section we consider the algebra $M(G)$ of measures on a nondiscrete locally compact abelian group G. We are interested in what properties a compact $Q \subset G$ must have in order that $M_c(Q)$, the subspace of continuous measures carried on Q, have the m.e.p. Hewitt and Kakutani show in [2] that if P is an independent Cantor set in G then $M_c(P \cup (-P))$ has the m.e.p. The proof of this relies heavily on the independence of P. Our results show that in general if $M_c(Q)$ has the m.e.p., then Q must be "almost independent" in a sense to be made more precise. Then we give an example that shows that the set Q can be, at the same time, rather dependent.

THEOREM 1 (Hewitt and Kakutani). *Suppose Q is a compact subset of G such that $M_c(Q)$ has the m.e.p. Then if k, l are distinct positive integers and $\lambda_1, \cdots, \lambda_k$ and $\mu_1, \cdots, \mu_l$ are nonnegative measures in $M_c(Q)$ and if π, ρ are arbitrary invertible elements of norm 1 in $M(G)$, then the measures $\lambda = \lambda_1 * \cdots * \lambda_k * \pi$ and $\mu = \mu_1 * \cdots * \mu_l * \rho$ are mutually singular.*

Hewitt and Kakutani prove this in [2] (Theorem 4.8) in the case that $Q = P \cup (-P)$. But their proof only uses the fact that $M_c(Q)$ has the m.e.p. and goes over to the general case without change.

THEOREM 2. *If Q is a compact subset of G such that $M_c(Q)$ has the m.e.p., then*

(i) *for every nonzero $t \in G$, the set $Q \cap (Q - t)$ has μ-measure zero for every continuous measure μ on G.*

(ii) *$m(Q) = 0$, where m is the Haar measure for G.*

Proof. (ii) is a simple consequence of (i) since $m(Q \cap (Q - x))$ is continuous as a function of x. To see (i), suppose on the contrary that $\mu(Q \cap (Q - t)) \neq 0$ for some $t \neq 0$ and $\mu \in M_c(G)$. Let ν be the restriction of μ to subsets of $Q \cap (Q - t)$. Then both ν and $\nu_t = \nu * \varepsilon_t$ are in $M_c(Q)$. Now ν and ν_t cannot be linearly independent, since then there would be a multiplicative linear functional h on $M(G)$ such that

$h(\nu) = 0$, $h(\nu_t) \neq 0$, by the m.e.p. But $h(\nu_t) = h(\nu)h(\varepsilon_t)$. Hence $\nu_t = \alpha \cdot \nu$ for some $\alpha \neq 0$. Now let V be a symmetric neighborhood of 0 such that $t \notin V + V$. Then choose $x \in Q \cap (Q - t)$ so that $\nu(x + V) \neq 0$. Let λ be the restriction of ν to subsets of $U = x + V$. Since $U \cap (U + t) = \emptyset$ and since λ_t is carried on $U + t$, we see that $\lambda_t \neq \alpha \cdot \lambda$, for any α. But λ and λ_t are both in $M_c(Q)$, and by the same argument as above with λ in place of ν, $\lambda_t = \alpha \cdot \lambda$, for some α. This contradiction establishes the result.

In particular, we see from (i) that if Q is metrizable then $Q \cap (Q - t)$ must be at most countable for every $t \neq 0$. This is the sense of "almost independence" mentioned above.

We give an example to show that $Q \cap (Q - t)$ can actually be countably infinite infinitely often. For this example we suppose G contains an independent set P homeomorphic to Cantor's set. This is always true, for example, if G contains arbitrarily small elements of infinite order [5, p. 100].

THEOREM 3. *Suppose G contains an independent Cantor set. Then there exists a compact subset Q of G such that for (countably) infinitely many $t \neq 0$, $Q \cap (Q - t)$ is countably infinite, and $M_c(Q)$ has the m.e.p.*

Proof. Let P be an independent Cantor set in G and let $P = S \cup T$ where S and T are disjoint Cantor sets. Let σ and τ be homeomorphisms of C onto S and T, respectively, and for $x \in C$, let $S_x = \sigma(E_x)$ and $T_x = \tau(E_x)$. Let $c_0 < c_1 < c_2 < \cdots$ be a monotone sequence in E_0 and denote the limit by c_∞. Choose a sequence $\{U_n: n = 0, 1, 2, \cdots\}$ of disjoint clopen (relative to E_0) sets such that $c_n \in U_n$. Write $U_\infty = \{c_\infty\}$. For $0 \leqq n \leqq \infty$, $x \in C$, let

$$
\begin{aligned}
a_n(x) &= \sigma(h_x(c_n)) \\
b_n(x) &= \tau(h_x(c_n)) \\
B_n(x) &= \tau(h_x(U_n)) \quad \text{if} \quad n \neq 0 \\
B_0(x) &= \sigma(E_x) = \sigma(h_x(E_0)) \, . \\
A_n(x) &= B_n(x) - b_n(x) + a_n(x) \, .
\end{aligned}
$$

Let K be any countable closed subset of C and let

$$Q = \bigcup_{x \in K} \bigcup_{0 \leqq n \leqq \infty} A_n(x)$$

We break the rest of the proof into several steps.

LEMMA 1. *Q is closed.*

Proof. Indeed, consider the map γ on $\bigcup \{U_n: 0 \leqq n \leqq \infty\}$ to

$\{c_n: 0 \leqq n \leqq \infty\}$ defined by

$$\gamma(t) = \textstyle\sum_{0 \leqq n \leqq \infty} c_n 1_{U_n}(t) ,$$

where 1_A denotes the characteristic function of A. Then γ is continuous. By Remark (4) of Section 2, the mapping

$$F(x, t) = \tau(h_x(t)) - \tau(h_x(\gamma(t))) + \sigma(h_x(\gamma(t)))$$

is a continuous mapping of the compact set $K \times \bigcup\{U_n: 0 \leqq n \leqq \infty\}$ onto Q. Thus Q is closed.

LEMMA 2. $m(Q) = 0$.

Proof. Each element of Q is of the form $p - b + a$, where p, b, a are in P. Hence Q is contained in the subgroup generated by P, and, since P is independent, the Haar measure of this subgroup is zero. (C.f. [5], p. 108).

LEMMA 3. *$Q \cap (Q - t)$ is infinite for infinitely many $t \neq 0$.*

Proof. For each $n \geqq 1$ and $x \in K$,

$$\begin{aligned} a_n(x) \in A_n(x) &\cap [A_0(x) + b_0(x) - a_0(x)] \\ &\subset Q \cap (Q - t_x) \end{aligned}$$

where $t_x = a_0(x) - b_0(x) \neq 0$.

We also note here that $Q \cap (Q - t)$ is at most countable for any $t \neq 0$.

LEMMA 4. *Suppose $\mu_1, \cdots, \mu_r$ are nonnegative continuous measures concentrated on disjoint subsets $D_1, \cdots, D_r$ of Q. Let $\lambda = \mu_1^{n1} * \cdots * \mu_r^{nr}$ and $\nu = \mu_1^{m1} * \cdots * \mu_r^{mr}$. Then λ and μ are mutually singular unless $n_i = m_i$ for $1 \leqq i \leqq r$.*

Proof. There are only a countable number of elements of Q of the form

$$\begin{aligned} b_n(x) - b_n(x) + a_n(x) \ , &\quad n \geqq 1 \\ \text{or} \quad a_k(x) - b_0(x) + a_0(x) \ , &\quad k \geqq 0 \ . \end{aligned}$$

Since the μ_i are continuous, we may assume without loss of generality that no D_i contains a point of this type. λ is concentrated on $D_\lambda = n_1 D_1 + \cdots + n_r D_r$ and ν is concentrated on $D_\nu = m_1 D_1 + \cdots + m_r D_r$. We shall show that $D_\lambda \cap D_\nu = \emptyset$ unless $n_i = m_i$ for $1 \leqq i \leqq r$.

Let $D = D_\lambda \cap D_\nu$. Each element of D has two representations

$$(*) \qquad (y_1^1 + \cdots + y_{n_1}^1) + \cdots + (y_1^r + \cdots + y_{n_r}^r) \\ = (z_1^1 + \cdots + z_{m_1}^1) + \cdots + (z_1^r + \cdots + z_{m_1}^r) ,$$

where y_k^i and z_k^i are in D_i. Each element of D_i is of the form

$$\beta_n(x) - b_n(x) + a_n(x)$$

where $\beta_n(x) \in B_n(x)$, and $\beta_n(x) \neq b_n(x)$, $n \geqq 1$, and $\beta_0(x) \neq a_k(x)$, $k \geqq 0$. Thus (*) is an equation between two combinations of elements of P. Since P is independent, the coefficients of distinct terms on either side must be equal. For fixed n and x suppose $a_n(x)$ appears k times on the left hand side of (*), i.e., the coefficient of $a_n(x)$ on the left is k. This means that there are k elements on the left of the form

$$\beta_n^i(x) - b_n(x) + a_n(x)$$

and k elements on the right of the form

$$\gamma_n^i(x) - b_n(x) + a_n(x)$$

where $\beta_n^i(x)$ and $\gamma_n^i(x)$ are in $B_n(x)$, since $a_n(x)$ can occur only in this way. Since none of the β's or γ's can occur in any other way on either side of the equation, we must have

$$\sum_{i=1}^{k} \beta_n^i(x) = \sum_{i=1}^{k} \gamma_n^i(x)$$

and so the $\beta_n^i(x)$ are just a permutation of the $\gamma_n^i(x)$. Thus the y_j^i that belong to a fixed $A_n(x)$ are just a permutation of the z_j^i that belong to $A_n(x)$. It follows that the y's are a permutation of the z's and, by the disjointedness of $D_1, \cdots, D_r$, $n_i = m_i$ for $1 \leqq i \leqq r$.

LEMMA 5. *$M_c(Q)$ has the m.e.p.*

The rest of the proof is exactly the same as the proof of Theorem 5.4.1 in [5], so we will not duplicate that proof here.

By a slight modification of the proof of Theorem 6.2 in [2], one can show that if P is, in addition, a Kronecker set, then the multiplicative extensions of linear functionals are in the closure of the dual of G.

It is natural to ask whether the above construction of the set Q could be modified in such a way that $Q \cap (Q - t)$ is infinite for uncountably many t. The following example shows that $M_c(Q)$ may fail to have the m.e.p. even though $Q \cap (Q - t)$ has at most *two* elements for every $t \neq 0$.

Consider the set

$$Q = \bigcup_{x \in \sigma} [A_0(x) \cup A_1(x) \cup A_2(x)] ,$$

where the notation is as above, except for Q. This Q is a compact set of Haar measure zero, and $Q \cap (Q - t)$ has at most two points for $t \neq 0$, but $M_c(Q)$ does not have the m.e.p. Indeed, suppose λ is the Cantor measure on C. Let $\mu_1, \mu_2, \mu_3, \mu_4$ and ρ be the measures obtained respectively as images of λ under the (bicontinuous) maps $x \to a_1(x)$, $x \to a_2(x)$, $x \to a_1(x) - b_0(x) + a_0(x)$, $x \to a_2(x) - b_0(x) + a_0(x)$, and $x \to a_1(x) + a_2(x) - b_0(x) + a_0(x)$. Then $\mu_1, \cdots, \mu_4$ are mutually singular measures in $M_c(Q)$ and we have

$$\rho \ll \mu_1 * \mu_4 \quad \text{and} \quad \rho \ll \mu_2 * \mu_3 .$$

But there exists a linear functional L of norm 1 on $M_c(Q)$ such that $L(\sigma) = 0$ for $\sigma \ll \mu_1$ and $L(\sigma) = \sigma(G)$ for $\sigma \ll \mu_j$, $j = 2, 3, 4$. If $M_c(Q)$ has the m.e.p. then there exists a multiplicative linear functional M on $M(G)$ that extends L. By the results of Šreĭder [6], M is represented by a generalized character, i.e., a function χ on $M(G) \times G$ such that for each $\sigma \in M(G)$, $\chi(\sigma, .)$ is Borel measurable and $M(\sigma) = \int \chi(\sigma, x)\sigma(dx)$. Combining a basic property of generalized characters ([6], equation (35)), namely

$$\chi(\sigma * \psi, s + t) = \chi(\sigma, s)\chi(\psi, t) \quad \text{a.e.} \quad (\sigma \times \psi) ,$$

with the fact that

$$\int \alpha(z)\sigma * \psi(dz) = \iint \alpha(s + t)\sigma(ds)\psi(dt) ,$$

we see that $\chi(\mu_1 * \mu_4, t) = 0$ a.e. $(\mu_1 * \mu_4)$ and $\chi(\mu_2 * \mu_3, t) = 1$ a.e. $(\mu_2 * \mu_3)$. Since $\chi(\rho, t) = \chi(\mu_1 * \mu_4, t) = \chi(\mu_2 * \mu_3, t)$ a.e. (ρ) ([6], Definition 1), we have a contradiction.

4. The disc algebra. Suppose B is an arbitrary Banach algebra with unit and maximal ideal space X. Let $\mathfrak{F}$ be a family of linearly independent elements of B of norm 1. Denote by $[\mathfrak{F}]$ ($[\mathfrak{F}]^-$) the linear span (closed linear span) of $\mathfrak{F}$. The following elementary proposition is useful in constructing subspaces of B with the m.e.p.

PROPOSITION. *If for any $\{f_1, \cdots, f_n\} \subset \mathfrak{F}$ and any points $z_1, \cdots, z_n$ in $\bar{D} = \{z: |z| \leqq 1\}$, there exists an $M \in X$ with $M(f_j) = z_j$ $j = 1, \cdots, n$, then $[\mathfrak{F}]^-$ has the m.e.p.*

Proof. If L is a linear functional on $[\mathfrak{F}]^-$ with $\|L\| \leqq 1$, the hypothesis states that the closed sets

$$\{M \in X: M(f) = L(f)\}$$

for $f \in \mathfrak{F}$ have the finite intersection property. The conclusion is then immediate from the compactness of X.

We denote by $A(D)$ the algebra of functions continuous on $\bar{D}$ and analytic in D.

THEOREM 4. *There exist infinite dimensional subspaces of* $A(D)$ *with the m.e.p.*

Proof. By the transformation $x \to e^{2\pi i x}$ of the unit interval we obtain a Cantor set on the boundary of D. By abuse of language we keep all the notation of Section 2, and regard C, E_x, h_x as the objects induced by this transformation.

Let $\bar{D}^\omega$ be the product of $\bar{D}$ with itself a countable number of times. Since this is a compact metric space, there is a continuous map φ of C onto $\bar{D}^\omega$.

For $\sigma \in \bar{D}^\omega$, let $E_\sigma = \cup \{E_x : \varphi(x) = \sigma\}$. Each E_σ is a perfect subset of C. In fact $E_\sigma = H[\varphi^{-1}(\{\sigma\}) \times E_0]$. Also the E_σ are pairwise disjoint and $C = \cup \{E_\sigma : \sigma \in \bar{D}^\omega\}$.

Define a sequence $\{f_n\}$ of functions on C as follows. For $\zeta \in C$, choose σ so that $\zeta \in E_\sigma$, and set $f_n(\zeta) = \sigma(n)$. Then each f_n is continuous on C. For suppose $\zeta_j \to \zeta_0$, $\zeta_j \in C$. Then each $\zeta_j \in E_{x_j}$ for some $x_j \in C$, and $\varphi(x_j) = \sigma_j \in \bar{D}^\omega$, and $\zeta_0 \in E_{x_0}$ with $\varphi(x_0) = \sigma_0$. By 2.3, $x_j \to x_0$. Since φ is continuous $\sigma_j \to \sigma_0$. Thus

$$f_n(\zeta_j) = \sigma_j(n) \to \sigma_0(n) = f_n(\zeta_0) .$$

Also clearly $\|f_n\| = 1$ for each n.

By Rudin's theorem on extension of continuous functions to analytic functions [3, p. 81], for each n there is a function $F_n \in A$ which agrees with f_n on C and $|F_n(z)| \leqq 1$ for $z \in \bar{D}$. If we let

$$\mathfrak{F} = \{F_1, F_2, \cdots\} ,$$

it is clear that the hypotheses of the proposition are satisfied and thus $[\mathfrak{F}]^-$ has the m.e.p., and the theorem is proved.

The following properties of the functions F_n in the above example are noteworthy:

(i) $\|F_n + F_m\| = 2$ *for all* n, m

(ii) $F_n^{-1}(\{w\})$ *is uncountable for every* $w \in \bar{D}$ *and every* n.

(iii) $F_m[F_n^{-1}(\{w\})] = \bar{D}$ *for every* $w \in \bar{D}$ *and every pair* n, m *with* $n \neq m$.

(iv) *If* f *and* g *are any two linearly independent elements of* $[\mathfrak{F}]^-$, *then for some* r, $0 < r \leqq 1$,

$$g[f^{-1}(\{0\})] = \{z : |z| \leqq r\} .$$

Subspaces with the m.e.p. may be constructed in this manner in any algebra B whose maximal ideal space contains a set K homeomorphic to C with the following property. If $f \in C(K)$ there is an $x \in B$ such that $\| x \| = \| f \|_\infty$ and the Gelfand transform of x agrees with f on K. For example, the algebra of continuous functions on the Stone-Čech compactification of the reals has this property.

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