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CONTRACTIVE PROJECTIONS IN L_p SPACES

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Let $(\Omega, \mathfrak{A}, m)$ be a finite measure space and L_p $(1 < p < \infty)$ the Lebesgue space of all complex valued measurable functions whose absolute p-th powers are integrable. Given a closed linear subspace of L_p, the operator which assigns to f the function in the subspace with minimum distance from it is continuous, idempotent, but not linear in general except the case $p = 2$ when the operator is just an orthogonal projection. A problem is to determine when such an operator Q is linear. It is linear if and only if $P = I - Q$ is a contractive projection, i.e., a linear idempotent operator with $\|P\| \leqq 1$, so that the problem takes an equivalent form to give complete description of contractive projections in L_p. In this paper the problem will be settled in the latter form, not only for $1 < p < \infty$ but also for $0 < p \leqq 1$.

Recall an important class of contractive projection; given a Borel subring $\mathfrak{B}$ with maximum element B, consider for each $f \in L_p$ $(1 \leqq p < \infty)$, a general measure $\nu(A) = \int_{A \cap B} f dm$ defined on the Borel subfield generated by $\mathfrak{B}$ and Ω, then the operator $E_{\mathfrak{B}}$ which assigns to f the Radon-Nikodym derivative of ν with respect to the measure m restricted on the subfield is called the *conditional expectation* relative to $\mathfrak{B}$. A conditional expectation is a contractive projection in each L_p $(1 \leqq p < \infty)$. When a Borel subring consists of all measurable sets contained in a fixed B, the conditional expectation operates as the multiplication by the characteristic function of B; in this case P_B is used instead of $E_{\mathfrak{B}}$. The operator P_B can be considered a contractive projection in each L_p $(0 < p < \infty)$.

Recently Douglas [2] gave a characterization of a contractive projection in L_1 to reveal a role of a conditional expectation, while Rota [6] treated L_p $(1 \leqq p \leqq \infty)$ case under an additional condition. A point of this paper is in the reduction of general case $1 < p < \infty$ to the case $p = 1$ (Theorem 1). Roughly speaking, every contractive projection is isometrically equivalent to a conditional expectation in case $1 \leqq p < \infty$ (Theorem 2), and is of P_B type in case $0 < p < 1$ (Theorem 3). A geometric description of a range of a contractive projection can be derived; a closed linear subspace can be a range of a contractive projection if and only if it is of L_p type, i.e., isometrically isomorphic to some L_p space, in case $1 \leqq p < \infty$ and consists of all functions which vanish outside a fixed measurable set in case $0 < p < 1$ (Theorem

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4). When $1 < p < \infty$, transfer to the conjugate space shows that, given a closed linear subspace, the operator which assigns the function in the subspace with minimum distance is linear if and only if the quotient space with respect to the subspace is of L_p type (Theorem 5).

The results in this paper will make it possible to establish pointwise convergence theorems for a sequence of contractive projections and for a sequence of predictions, as treated in [1], under general setting. This will be published elsewhere.

2. Reduction. In what follows, a measurable function and a measurable set are called simply a function and a set respectively. For a set A, χ_A is its characteristic function and A^c denotes its complement. $f \geqq g$ and $A \supseteqq B$ mean $f(\omega) \geqq g(\omega)$ almost everywhere and $m(B - A) = 0$ respectively. $\{f > \alpha\}$ stands for the set $\{\omega: f(\omega) > \alpha\}$ and the *support* of f is the set $\{|f| > 0\}$. For a real number r the power f^r is defined by

$$f^r(\omega) = \operatorname{sgn}(f(\omega)) \cdot |f(\omega)|^r ,$$

and $\|f\|_p$ is the L_p-norm $(0 < p < \infty)$:

$$\|f\|_p = \left\{\int |f|^p \, dm\right\}^{1/p} .$$

Two functions which are equal almost everywhere are identified.

If P is a contractive projection in L_p $(1 < p < \infty)$, its adjoint P^* is a contractive projection in L_q with $p^{-1} + q^{-1} = 1$ and there is nice duality between the range $\mathfrak{R}_P$ of P and $\mathfrak{R}_{P^*}$, that of P^*.

LEMMA 1. *If P is a contractive projection in L_p $(1 < p < \infty)$, then $f \in \mathfrak{R}_P$ is equivalent to $f^r \in \mathfrak{R}_{P^*}$ with $r = p - 1$.*

Proof. By duality, it suffices to prove that $Pf = f$ implies $P^*f^r = f^r$. Suppose that $Pf = f$, then by Hölder's inequality

$$\begin{aligned} \|f\|_p^p &= \int f \cdot \bar{f}^r dm = \int Pf \cdot \bar{f}^r dm \\ &= \int f \cdot \overline{P^* f^r} dm \leqq \|f\|_p \cdot \|P^* f^r\|_q \\ &\leqq \|f\|_p \cdot \|f^r\|_q = \|f\|_p^p , \end{aligned}$$

so that $P^*f^r = f^r$, because there is only one function h (up to scalar) for which $\|f\|_p \|h\|_q = \int f \cdot \bar{h} dm$.

An immediate consequence is that a closed linear subspace of L_p $(1 < p < \infty)$ can be a range of at most one contractive projection; in fact, if P_1, P_2 are contractive projections with $\mathfrak{R}_{P_1} = \mathfrak{R}_{P_2}$, then by Lemma 1 $\mathfrak{R}_{P_1^*} = \mathfrak{R}_{P_2^*}$, hence $\mathfrak{R}_{I-P_1} = \mathfrak{R}_{I-P_2}$ because $\mathfrak{R}_{I-P_1}$, say, is the annihilator of $\mathfrak{R}_{P_1^*}$ by duality theorem, and $P_1 = P_2$ follows. In particular, a contractive projection in L_2 is an orthogonal projection.

A linear operator T is called *positive*, if $g \geqq 0$ implies $Tg \geqq 0$. A consequence of the positivity is that for all f

$$|Tf| \leqq T|f| ,$$

which can be proved through the approximation of f by step-functions. Clearly T is positive, if $T\chi_A \geqq 0$ for all A.

LEMMA 2. *If T is a contraction in L_p $(0 < p \leqq 1)$ which makes constant functions invariant, then it is necessarily positive and for all sets A*

$$\int T\chi_A dm = \int \chi_A dm .$$

Proof. Since T is a contraction and $T1 = 1$ where 1 denotes the constant function with value 1,

$$\begin{aligned}\int 1 dm &= \int 1^p dm \\ &= \int |T1|^p dm \\ &= \int |T\chi_A + T\chi_{A^c}|^p dm \\ &\leqq \int |T\chi_A|^p dm + \int |T\chi_{A^c}|^p dm \\ &\leqq \int \chi_A^p dm + \int \chi_{A^c}^p dm \\ &= \int 1 dm ,\end{aligned}$$

it follows that

$$|T\chi_A + T\chi_{A^c}|^p = |T\chi_A|^p + |T\chi_{A^c}|^p$$

and

$$\int |T\chi_A|^p dm = \int \chi_A dm ,$$

so that, on account of the property of the function $|\xi|^p$, $T\chi_A$ and $T\chi_{A^c}$ have the same signature when $p = 1$, and $T\chi_A \cdot T\chi_{A^c} = 0$ when $0 < p < 1$ (cf. [5]), hence in any case $T\chi_A \geqq 0$, because

$$T\chi_A + T\chi_{A^c} = 1$$

The second statement in the assertion is immediate when $p = 1$, and follows from the observation that $T\chi_A$ is a characteristic function, when $0 < p < 1$.

THEOREM 1. *A contractive projection P in L_p $(1 < p < \infty, p \neq 2)$ which makes constant functions invariant is contractive with respect to L_1-norm, i.e.,*

$$\int |Pg|\,dm \leqq \int |g|\,dm \qquad \text{for } g \in L_p\,.$$

The same conclusion is true, when $p = 2$ and P is positive in addition.

Proof. (1) The case $1 < p < 2$. If $f \in \mathfrak{R}_P$, by Lemma 1 f^r and constant functions are in $\mathfrak{R}_{P^*}$ with $r = p - 1$, so that $1 + \varepsilon f^r$ is in it for all $1 > \varepsilon > 0$. The interchange of role between P and P^* shows that $(1 + \varepsilon f^r)^{1/r}$ is in $\mathfrak{R}_P$. Consider the function

$$h_\varepsilon(\omega) \stackrel{\text{def}}{=\!=} \frac{\{1 + \varepsilon f^r(\omega)\}^{1/r} - 1}{\varepsilon}\,.$$

Since

$$f^r(\omega) = \{a(\omega) + b(\omega)\sqrt{-1}\}\,|f(\omega)|^r$$

where $a(\omega) = \operatorname{Re}(\operatorname{sgn} f(\omega))$ and $b(\omega) = \operatorname{Im}(\operatorname{sgn} f(\omega))$, and

$$\{1 + \varepsilon f^r(\omega)\}^{1/r} = \{1 + \varepsilon f^r(\omega)\}\cdot|1 + \varepsilon f^r(\omega)|^{(1/r)-1}\,,$$

with $c(\omega) = \varepsilon\,|f(\omega)|^r$ and $s = (1 - r)/2r$, it follows

$$\begin{aligned}\{1 + \varepsilon f^r(\omega)\}^{1/r} = \{1 &+ a(\omega)\cdot c(\omega) + b(\omega)\cdot c(\omega)\sqrt{-1}\} \\ &\times \{1 + 2a(\omega)c(\omega) + c^2(\omega)\}^s\,,\end{aligned}$$

so that

$$\begin{aligned}h_\varepsilon(\omega) = f^r(\omega)&\cdot\{1 + 2a(\omega)c(\omega) + c^2(\omega)\}^s \\ &+ |f(\omega)|^r\,\frac{\{1 + 2a(\omega)\cdot c(\omega) + c^2(\omega)\}^s - 1}{c(\omega)}\end{aligned}$$

with the convention $0 \times \infty = 0$. The modulus of the first term of the above formula is majorated by $|f(\omega)|^r\,\{1 + |f(\omega)|^r\}^{2s}$, which is, in turn, majorated by $\alpha\{1 + |f(\omega)|\}$ with a constant α. The modulus of the second term is majorated by $\beta\,|f(\omega)|^r$ with a constant β at every point ω where $c(\omega) > 1/2$, and at a point ω where $0 < c(\omega) \leqq 1/2$ the mean value theorem shows

$$\left|\frac{\{1 + 2a(\omega)\cdot c(\omega) + c^2(\omega)\}^s - 1}{c(\omega)}\right| \leqq \delta \text{ for a constant } \delta ,$$

so that the modulus of the second term is majorated by

$$\gamma\{1 + |f(\omega)| + |f(\omega)|^r\}$$

everywhere with some constant γ. Since $|f|^r$ is in L_p because of $1 < p < 2$, the conclusion is that $h_\varepsilon(\omega)$ converges dominatedly in L_p, as $\varepsilon \to 0$, to $\{a(\omega)/r + b(\omega)\sqrt{-1}\}\cdot|f(\omega)|^r$, which is in $\mathfrak{R}_P$ as a result. In the same way, consider $\{\sqrt{-1} + \varepsilon f^r(\omega)\}^{1/r}$ to prove that

$$\left\{a(\omega) + \frac{b(\omega)\sqrt{-1}}{r}\right\}|f(\omega)|^r$$

is in R_P. These together show that $\{a(\omega) + b(\omega)\sqrt{-1}\}\cdot|f(\omega)|^r$ i.e., $f^r(\omega)$ by definition, is in $\mathfrak{R}_P$ (and, of course, in $\mathfrak{R}_{P^*}$). Prove by induction that f^{r^n} is in $\mathfrak{R}_{P^*}$ for all n, then since $0 < r < 1$, f^{r^n} converges dominatedly by $1 + |f|^r$ in L_q almost everywhere to $\operatorname{sgn}(f)$ where $p^{-1} + q^{-1} = 1$. Now given $g \in L_p$, let $f = Pg$, then

$$\begin{aligned}\int|Pg|\,dm &= \int Pg\cdot\overline{\operatorname{sgn}(f)}dm \\ &= \lim_{n\to\infty}\int Pg\cdot\bar{f}^{r^n}dm \\ &= \lim_{n\to\infty}\int g\cdot\overline{P^*f^{r^n}}dm \\ &= \lim_{n\to\infty}\int g\cdot\bar{f}^{r^n}dm \leqq \int|g|\,dm .\end{aligned}$$

(2) The case $2 < p < \infty$. Since the adjoint P^* makes constant functions invariant by Lemma 1, the preceding proof shows that P^* is contractive with respect to L_1-norm, so that it is positive by Lemma 2. Given $g \in L_p$, let $h = \operatorname{sgn}(Pg)$, then by the positivity

$$\begin{aligned}\int|Pg|\,dm &= \int Pg\cdot\bar{h}dm \\ &= \int g\cdot\overline{P^*h}dm \leqq \int|g|\cdot|P^*h|\,dm \\ &\leqq \int|g|\cdot P^*|h|\,dm \leqq \int|g|\cdot P^*1dm \\ &= \int|g|\,dm .\end{aligned}$$

(3) Of $p = 2$ and P is positive, the adjoint P^* is obviously positive, so that the same arguments as in (2) are valid. This completes the proof.

It should be mentioned that Rota [6] proved an equivalent form of Theorem 1 under an additional hypothesis (the averaging property) that $Pg \cdot Ph = P(g \cdot Ph)$ for bounded functions g, h.

Theorem 1 will make it possible to reduce the study of contractive projections in L_p $(1 < p < \infty, p \neq 2)$ to that of the case $p = 1$. For this purpose some preliminaries are necessary.

LEMMA 3. *A closed linear subspace $\mathfrak{M}$ of L_p $(0 < p < \infty)$ contains a function with maximum support, that is, there is $f \in \mathfrak{M}$ such that $S_f \supseteqq S_g$ for all $g \in \mathfrak{M}$, where S_f and S_g are supports of f and g respectively.*

Proof. Obviously there is a sequence $\{f_j\} \subseteqq \mathfrak{M}$ such that $\|f_j\|_p = 1$ and $\bigcup_{j=1}^{\infty} S_{f_j} \supseteqq S_g$ for all $g \in \mathfrak{M}$. Starting with $\alpha_1 = 1, A_0 = \varnothing$, and $\gamma_{0,k} = 1$ $(k = 1, 2, \cdots)$, construct by induction the sequences $\{\alpha_j\}, \{\gamma_{j,k}\}$, and $\{A_j\}$ $(j = 1, 2, \cdots; k = j + 1, \cdots)$ which together obey the requirements:

(a) $2^{-k} \geqq \gamma_{j,k} \geqq \gamma_{j+1,k} > 0$ for $k \geqq j + 2$,

(b) $S_{g_j} \supseteqq A_j$ and $m(S_{g_j} - A_j) \leqq 2^{-j+2}$ for $j \geqq 1$,

(c) $\gamma_{j-1,j} \geqq \alpha_j > 0$ and $S_{g_j} = \bigcup_{k=1}^{j} S_{f_k}$ for $j \geqq 1$,

where $g_j = \sum_{k=1}^{j} \alpha_k f_k$,

(d) $|g_j(\omega)| > \sum_{k=j+1}^{\infty} \gamma_{j,k} |f_k(\omega)|$ almost everywhere on A_j for $j \geqq 1$.

Suppose that $A_j, \gamma_{j,k}$ for $j \leqq n - 1$ and $k \geqq j + 1$, and α_j for $j \leqq n$ have been found and obey the requirements. Take $1 > \varepsilon > 0$ so small that

$$m(S_{g_n} - \{|g_n| > \varepsilon\}) < 2^{-n},$$

then since

$$m(|f_k| > 2^{kn/p}) \leqq 2^{-nk} \int |f_k|^p \, dm = 2^{-nk},$$

let

$$A_n = \bigcap_{k=n+1}^{\infty} \{|f_k| \leqq 2^{kn/p}\} \cap \{|g_n| > \varepsilon\}$$

and

$$\gamma_{n,k} = \min \{\gamma_{n-1,k}, 2^{-k-(nk/p)} \varepsilon\}$$

to get (a) and (b) for n; in fact

$$m(S_{g_n} - A_n) \leqq m(S_{g_n} - \{|g_n| > \varepsilon\}) + \sum_{k=n+1}^{\infty} m(|f_k| > 2^{nk/p}) \leqq 2^{-n+2}.$$

Since there is at least one α such that $0 < \alpha < \gamma_{n,n+1}$ and

$$m\Big(S_{f_{n+1}} \cap \Big\{\frac{g_n}{f_{n+1}} = -\alpha\Big\}\Big) = 0 ,$$

for otherwise there arise uncountably many disjoint sets with positive measure, take one of such α's as α_{n+1} to get (c) for n, that is, $S_{g_{n+1}} = S_{f_{n+1}} \cup S_{g_n}$ where $g_{n+1} = g_n + \alpha_{n+1} f_{n+1}$. From the construction it follows that for $j \geqq n+1$

$$\Big\| \sum_{k=n+1}^{j} \gamma_{n,k} |f_k| \Big\|_p \leqq \begin{cases} \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} & \text{if } p \geqq 1 , \\ \Big\{\sum_{k=n+1}^{\infty} 2^{-kn}\Big\}^{1/p} = 2^{-n/p} & \text{if } 0 < p < 1 , \end{cases}$$

so that $\sum_{k=n+1}^{\infty} \gamma_{n,k} |f_k|$ converges almost everywhere and by (a) for $\omega \in A_n$

$$|g_n(\omega)| - \sum_{k=n+1}^{\infty} \gamma_{n,k} |f_k(\omega)| \geqq \varepsilon - \sum_{k=n+1}^{\infty} 2^{-k}\varepsilon > 0 ,$$

thus (d) is satisfied for n. Let finally $f = \sum_{k=1}^{\infty} \alpha_k f_k$ which converges in $\mathfrak{M}$ because of (a) and (c), then on account of (d)

$$f(\omega) = g_n(\omega) + \sum_{k=n+1}^{\infty} \alpha_k f_k(\omega) \neq 0 \text{ on } A_n,$$

so that by (b) and (c)

$$\begin{aligned} m\Big(\bigcup_{n=1}^{\infty} S_{f_n} - S_f\Big) &= \lim_{n\to\infty} m\Big(S_{g_n} - S_f\Big) \\ &\leqq \lim_{n\to\infty} m(S_{g_n} - A_n) \leqq \lim_{n\to\infty} 2^{-n+2} = 0 \end{aligned}$$

and f meets the requirement of maximum support.

COROLLARY. *The range of a positive contractive projection P in L_p $(0 < p < \infty)$ contains a nonnegative function with maximum support.*

Proof. Let f be a function with maximum support in the range by Lemma 3, then the positivity implies

$$|f| = |Pf| \leqq P|f| .$$

On the other hand, $\| P|f| \|_p \leqq \| |f| \|_p$ by the contractive property, so that $P|f| = |f|$ follows and $|f|$ meets the requirement.

3. Contractive projections. Before entering a basic proposition on representation of a contractive projection, recall the characteristic properties of the conditional expectation relative to a Borel subring $\mathfrak{B}$ with the maximum element B (cf. [4]). In what follows a function f is said to be measurable with respect to $\mathfrak{B}$, if $f\cdot\chi_B$ is measurable with respect to the Borel subfield generated by $\mathfrak{B}$ and Ω.

$$(1)\qquad E_{\mathfrak{B}}P_B = P_B E_{\mathfrak{B}} = E_{\mathfrak{B}}\,,$$

$$(2)\qquad E_{\mathfrak{B}}f \text{ is measurable with respect to } \mathfrak{B}\,,$$

$$(3)\qquad \int_A E_{\mathfrak{B}} f dm = \int_A f dm \qquad \text{for } A\in\mathfrak{B}\,.$$

The following are consequences:

$$(4)\qquad E_{\mathfrak{B}} \text{ is positive}\,,$$

$$(5)\qquad E_{\mathfrak{B}}g = P_B g \quad\text{and}\quad E_{\mathfrak{B}}(g\cdot f) = g\cdot E_{\mathfrak{B}}f\,,$$

whenever g is measurable with respect to $\mathfrak{B}$ and $g\cdot f\in L_1$. The positivity guarantees the generalized Hölder's inequality (cf. [7]):

$$(6)\qquad |E_{\mathfrak{B}}(f\cdot h)| \leqq \{E_{\mathfrak{B}}|f|^p\}^{1/p}\{E_{\mathfrak{B}}|h|^q\}^{1/q}$$

with $p^{-1}+q^{-1}=1$, whenever $f\in L_p$ and $h\in L_q$.

LEMMA 4. *A contractive projection P in L_1 which makes constant functions invariant is a conditional expectation.*

This is a basic result of Douglas [2] and is contained implicitly in Rota [7]. Here is a sketch of a quick proof. Let $\mathfrak{B}$ be the least Borel subfield with respect to which all Ph are measurable. Since P is positive by Lemma 2, $f\in\mathfrak{R}_P$, the range of P, implies $\mathrm{Re}(f), \mathrm{Im}(f)\in\mathfrak{R}_P$. If a real valued function h is in $\mathfrak{R}_P$, its positive part, i.e., $h^+ = \max(h, 0)$ is also in it; this is proved just as in the proof of Corollary of Lemma 3. Thus for any α, the characteristic function of $\{h>\alpha\}$ is in $\mathfrak{R}_P$, because it is the limit of $1-\{1-n(h-\alpha)^+\}^+$ as $n\to\infty$, so that it immediately follows that $\mathfrak{R}_P$ is just the collection of all functions in L_p which are measurable with respect to $\mathfrak{B}$. Take an arbitrary $A\in\mathfrak{B}$, and consider the contractions $P_APP_A+P_{A^c}$ and $P_{A^c}PP_{A^c}+P_A$ to get, on the basis of Lemma 2,

$$\int_A P(f\cdot\chi_A)dm = \int_A f dm$$

and

$$\int_{A^c} P(f\cdot\chi_{A^c})dm = \int_{A^c} f dm\,.$$

When $f\geqq 0$,

$$\int_{A^c} P(f\cdot\chi_{A^c})dm \leqq \int P(f\cdot\chi_{A^c})dm \leqq \int_{A^c} fdm ,$$

so that

$$\int_A P(f\cdot\chi_{A^c})dm = 0$$

consequently

$$\int_A Pfdm = \int_A fdm .$$

The last equality holds without positivity assumption on f; this shows that P is the conditional expectation relative to $\mathfrak{B}$.

Suppose that P is a contractive projection in L_p $(1 \leqq p < \infty, p \neq 2)$. According to Lemma 3, take a function f with maximum support in the range of P and consider the measure space $(B, \mathfrak{D}, m_p)$ where B is the support of f, $\mathfrak{D}$ is the Borel field consisting of all measurable subsets in B, and $dm_p = |f|^p dm$. Use a convention that a function on Ω with its support contained in B is identified, in the natural way, with a function on B and conversely. The operator T in $L_p(B, \mathfrak{D}, m_p)$, defined by

$$Th = \frac{P(f\cdot h)}{f}$$

is a contractive projection; in fact, the idempotency is a consequence of that of P, combined with the maximum property of B, and

$$\begin{aligned}\int |Th|^p dm_p &= \int \left|\frac{P(f\cdot h)}{f}\right|^p |f|^p dm \\ &= \int |P(f\cdot h)|^p dm \leqq \int |f\cdot h|^p dm \\ &= \int |h|^p dm_p ,\end{aligned}$$

because P is contractive in $L_p(m)$ by assumption. Furthermore T makes constant functions invariant, because $Pf = f$ by assumption. Then Lemma 4 together with Theorem 1 shows that T is a conditional expectation relative to some Borel subfield $\mathfrak{B}$ of $\mathfrak{D}$ (with respect to the measure m_p), so that by (3)

$$\int_A Th\,|f|^p dm = \int_A h\,|f|^p dm \qquad \text{for } A \in \mathfrak{B} .$$

$\mathfrak{B}$ can be identified with a Borel subring of $\mathfrak{A}$ with the maximum element B, and T can be considered as an operator acting on function on Ω by the relation

$$Th = T(h \cdot \chi_B) .$$

Apply the conditional expectation $E_{\mathfrak{B}}$ (with respect to the measure m), with (2), (3), and (5) in mind, to get

$$E_{\mathfrak{B}}(Th \cdot |f|^p) = E_{\mathfrak{B}}(h \cdot |f|^p) ,$$

then since Th is measurable with respect to $\mathfrak{B}$ it follows from (5)

$$Th \cdot E_{\mathfrak{B}} |f|^p = E_{\mathfrak{B}}(h \cdot |f|^p) .$$

The original projection P is reproduced from T through the relation

$$Ph = f \cdot T\left(\frac{h}{f}\right) + P(h \cdot \chi_{B^c}) ,$$

hence finally

$$Ph = \frac{f \cdot E_{\mathfrak{B}}(h \cdot \bar{f}^{p-1})}{E_{\mathfrak{B}} |f|^p} + P(h \cdot \chi_{B^c}) .$$

When $p \neq 1$, the last term Pg with $g = h \cdot \chi_{B^c}$ disappears, for

$$\begin{aligned} (1 + \varepsilon)^p \int |Pg|^p \, dm &= \int |P(Pg + \varepsilon g)|^p \, dm \\ &\leqq \int |Pg + \varepsilon g|^p \, dm \\ &= \int |Pg|^p \, dm + \varepsilon^p \int |g|^p \, dm , \end{aligned}$$

because P is a contractive projection and the support of Pg is contained in B, disjoint from the support of g, but the above inequality is possible, for fixed p and all $\varepsilon > 0$, only when $\int |Pg|^p \, dm = 0$. These observation give a half of the proof of the main theorem.

THEOREM 2. *P is a contractive projection in $(1 \leqq p < \infty, p \neq 2)$ if and only if there is a Borel subring $\mathfrak{B}$ and a function $f \in L_p$ such that the support B of f is the maximum element of $\mathfrak{B}$ and P is represented in the form*

$$Ph = \frac{f \cdot E_{\mathfrak{B}}(h \cdot \bar{f}^{p-1})}{E_{\mathfrak{B}} |f|^p} + Vh$$

where, when $p = 1$, V is a contraction such that $V = P_B V$, $VP_B = 0$, and Vh/f is measurable with respect to $\mathfrak{B}$, and, when $p \neq 1$, $V = 0$.

Proof. Suppose that P admits such a representation. The case $p = 1$ is observed by Douglas [2] (cf. the proof of Theorem 3). When $p \neq 1$ (p may be equal to 2), by (4) and (6)

$$\begin{aligned} |E_{\mathfrak{B}}(h\cdot\bar{f}^{p-1})| &\leqq E_{\mathfrak{B}}(|h|\cdot|f|^{p-1}) \\ &\leqq \{E_{\mathfrak{B}}|h|^p\}^{1/p}\cdot\{E_{\mathfrak{B}}|f|^p\}^{1/q}, \end{aligned}$$

where $p^{-1}+q^{-1}=1$, so that from (2), (3), and (5) it follows that

$$\begin{aligned} \int|Ph|^p\,dm &\leqq \int_B \frac{|f|^p\cdot|E_{\mathfrak{B}}(h\cdot\bar{f}^{p-1})|^p}{\{E_{\mathfrak{B}}|f|^p\}^p}dm \\ &= \int_B E_{\mathfrak{B}}\left\{\left|\frac{E_{\mathfrak{B}}(h\cdot\bar{f}^{p-1})}{E_{\mathfrak{B}}|f|^p}\right|^p\cdot|f|^p\right\}dm \\ &= \int_B \left|\frac{E_{\mathfrak{B}}(h\cdot\bar{f}^{p-1})}{E_{\mathfrak{B}}|f|^p}\right|^p\cdot E_{\mathfrak{B}}|f|^p\,dm \\ &\leqq \int_B \frac{E_{\mathfrak{B}}|h|^p\cdot\{E_{\mathfrak{B}}|f|^p\}^{p-1}\cdot E_{\mathfrak{B}}|f|^p}{\{E_{\mathfrak{B}}|f|^p\}^p}dm \\ &= \int_B E_{\mathfrak{B}}|h|^p\,dm = \int_B|h|^p\,dm, \end{aligned}$$

thus P is a contraction in L_p. It is idempotent; in fact,

$$\begin{aligned} P^2h &= \frac{f}{E_{\mathfrak{B}}|f|^p}\cdot E_{\mathfrak{B}}\left\{\frac{E_{\mathfrak{B}}(h\cdot\bar{f}^{p-1})}{E_{\mathfrak{B}}|f|^p}\cdot|f|^p\right\} \\ &= \frac{f}{E_{\mathfrak{B}}|f|^p}\cdot\frac{E_{\mathfrak{B}}(h\cdot\bar{f}^{p-1})}{E_{\mathfrak{B}}|f|^p}\cdot E_{\mathfrak{B}}|f|^p = Ph \end{aligned}$$

by (2) and (5). This completes the proof.

COROLLARY 1. *A contractive projection P in L_p is isometrically equivalent to a conditional expectation (with respect to a measure), if $1<p<\infty$, or if $p=1$ and $PP_B=P$ where B is the maximum support of the range of P.*

Proof. When $1<p<\infty$, $p\neq 2$, with notations in the discussion preceding Theorem 2, consider the measure m' on $\mathfrak{A}$, defined by

$$m'(A)=m_p(A\cap B)+m(A\cap B^c),$$

then the operator T is identified with a conditional expectation with respect to m' and on account of $V=0$

$$P=UTU^{-1}$$

where U is the isometry which assigns to $h\in L_p(m')$ a function $h(f+\chi_{B^c})$ it L_p. When $p=2$, the assertion follows from the fact that unitary equivalence is determined only by the dimension of the range. When $p=1$, $PP_B=P$ implies $V=0$ so that the same arguments as in the first case can be applied.

COROLLARY 2. *P is a positive contractive projection in L_p $(1 \leqq p < \infty)$ if and only if in Theorem 2 f can be chosen as a nonnegative function and V is positive, when $p = 1$.*

Proof. If P has a representation with nonnegative f and positive V, it is obviously positive, because the conditional expectation is positive. The converse statement follows from the construction in the proof of Theorem 2 combined with Corollary of Lemma 3.

COROLLARY 3. *P is a contractive projection both in L_p and L_q $(1 < p < 2, p^{-1} + q^{-1} = 1)$ if and only if in Theorem 2 f can be chosen as*

$$|f(\omega)|^2 = |f(\omega)| .$$

Proof. If P has the representation with such f,

$$|Ph| = |f \cdot E_{\mathfrak{B}}(h \cdot \bar{f})| \leqq E_{\mathfrak{B}}|f|$$

by (4), and the assertion (for all p with $1 < p \leqq 2$) follows from the fact that the conditional expectation is contractive in every L_p. Conversely if P is contractive both in L_p and L_q for some p, it is also contractive in L_2 by Riesz's convexity theorem ([3], VI, 10), hence is self-adjoint in L_2. Take a function g with maximum support in the range $\mathfrak{R}_P$, then the self-adjointness implies, by Lemma 1, $g^{r^n} \in \mathfrak{R}_P$ $n = 1, 2, \cdots$ where $r = p - 1$. Since g^{r^n} converges to $\operatorname{sgn}(g)$ so that $f = \operatorname{sgn}(g)$ meets the requiremet.

When $1 > p > 0$, the duality method is no longer available, but the concavity of the function $|\xi|^p$ is a tool.

THEOREM 3. *P is a contractive projection in L_p $(0 < p < 1)$, if and only if there are a set B and a contraction V such that $P_B V = V$, $VP_B = 0$, and P is represented in the form*

$$P = P_B + V .$$

Proof. If P admits the representation, by the property of V

$$\begin{aligned}\int |Ph|^p \, dm &\leqq \int \{|P_B h| + |Vh|\}^p dm \\ &\leqq \int |P_B h|^p \, dm + \int |Vh|^p \, dm \\ &\leqq \int_B |h|^p \, dm + \int_{B^c} |h|^p \, dm\end{aligned}$$

so P is a contraction. It is idempotent; in fact,

$$\begin{aligned} P^2 &= P_B(P_B + V) + V(P_B + P_B V) \\ &= P_B + V = P . \end{aligned}$$

According to the discussion preceding Theorem 2, for the converse assertion, it suffices to prove that a contractive projection P in L_p which makes constant functions invariant is necessarily the identity operator. P is positive and $P\chi_A$ is a characteristic function, as in the proof of Lemma 2. The correspondence which assigns to A the support of $P\chi_A$ preserves inclusion relation by the positivity of P. Since P is linear, it preserves disjointness. The idempotency of the correspondence follows from the idempotency of P. Then the correspondence must be the identity, which means that P makes all characteristic functions invariant, so that it is the identity operator. This completes the proof.

4. **Geometric description.** A Banach space will be called of L_p *type* if it is isometrically isomorphic to an L_p space on a measure space. When $1 < p < \infty$, a Banach space is of L_p type if and only if its conjugate space is of L_q type with $p^{-1} + q^{-1} = 1$. Given an L_p space, the simplest subspace of L_p type is the collection of functions which vanish outside a fixed set B; such a subspace will be called an L_p *section.* A closed linear subspace is an section if and only if it is the range of an operator P_B. The collection of all functions measurable with respect to a Borel subring is clearly of L_p type; such a subspace will be called an L_p *subspace.* A closed linear subspace is an L_p subspace if and only if it is the range of a conditional expectation. Inspection of the proof of Lemma 3 shows that a closed linear subspace $\mathfrak{M}$ is an L_p subspace if it contains constant functions and if $f \in \mathfrak{M}$ implies $\{\mathrm{Re}(f)\}^+ \in \mathfrak{M}$. The representation of a contractive projection in terms of a conditional expectation will answer the question of when a closed linear subspace can be the range of a contractive projection.

THEOREM 4. *A closed linear subspace can be the range of a contractive projection, if and only if it is of L_p type or and L_p section according as $1 \leqq p < \infty$ or $0 < p < 1$.*

Proof. The assertion is well known in case $p = 2$ and is an immediate consequence of Theorem 3 in case $0 < p < 1$. Douglas [2] treated the case $p = 1$. If P is a contractive projection in L_p $(1 < p < \infty, p \neq 2)$, by Corollary 1 of Theorem 2 it is isometrically equivalent to a conditional expectation, so that its range is isometrically isomorphic to that of a conditional expectation, hence it is of

L_p type. Conversely if $\mathfrak{M}$ is a closed linear subspace of L_p type, there exists by definition a measure space $(\Omega', \mathfrak{A}', m')$ and an isometry W from $L_p(\Omega', \mathfrak{A}', m')$ onto $\mathfrak{M}$. Let $f = W1$ where 1 denotes the constant function with value 1 on Ω', and consider the isometry V from $L_p(m'')$ onto L_p which assigns $h(f + \chi_{B^c})$ to h where B is the support of f and

$$dm'' = \{|f|^p + \chi_{B^c}\}dm .$$

Then the image of $\mathfrak{M}$ under V^{-1} is just the image of $L_p(m')$ under the isometry $U = V^{-1}W$. On account of Lamperti's result [5] U preserves disjointness in the sense that $g_1 \cdot g_2 = 0$ implies $Ug_1 \cdot Ug_2 = 0$. Since U maps the constant function 1 on Ω' to the characteristic function χ_B, it results that the image of a characteristic function is also a characteristic function, and U is positive in the sense that it preserves nonnegativity. A discussion similar to the proof of Lemma 4 shows that the image of U is a L_p subspace, hence is the range of a conditional expectation, so that $\mathfrak{M}$ itself is the range of a contractive projection, isometrically equivalent to the conditional expectation. This completes the proof.

Given a closed linear subspace $\mathfrak{M}$ of L_p $(1 < p < \infty)$, consider the operator $P_{\mathfrak{M}}$ which assigns to f the function in $\mathfrak{M}$ with minimum distance from it, that is, $P_m f \in \mathfrak{M}$ and $\|f - P_{\mathfrak{M}} f\|_p \leqq \|f - g\|_p$ for all $g \in \mathfrak{M}$; the operator is well defined because of the weak compactness of the unit ball and of the strict convexity of the L_p norm. $P_{\mathfrak{M}}$ will be called the *prediction* relative to $\mathfrak{M}$. It is idempotent, but not linear in general.

Suppose that the prediction relative to $\mathfrak{M}$ is linear, then the operator $P = I - P_{\mathfrak{M}}$ is a contractive projection which annihilates exactly functions in $\mathfrak{M}$, so that its adjoint P^* is a contractive projection in L_q, with $p^{-1} + q^{-1} = 1$, having the annihilator $\mathfrak{M}^\perp$ as its range. Then by Theorem 4 $\mathfrak{M}^\perp$ is of L_q type. Conversely if the annihilator $\mathfrak{M}^\perp$ is of L_q type, it is the range of a contractive projection Q in L_q, and $I - Q^*$ is readily shown to coincide with the prediction $P_{\mathfrak{M}}$. On the other hand, by duality theorem, the annihilator $\mathfrak{M}^\perp$ is isometrically isomorphic to the conjugate space of the quotient space $L_p/\mathfrak{M}$. These observations lead to the assertion.

THEOREM 5. *The prediction $P_{\mathfrak{M}}$ relative to a closed linear subspace $\mathfrak{M}$ of L_p $(1 < p < \infty)$ is linear, if and only if the quotient space $L_p/\mathfrak{M}$ is of L_p type.*

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