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ON REAL NUMBERS HAVING NORMALITY OF ORDER k

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This paper contains three theorems concerning real numbers having normality of order k. The first theorem gives a simple construction of a periodic decimal having normality of order k to base r. After introducing the notion of c-uniform distribution modulo one, we prove in the second theorem that α has normality of order k to base r if and only if the function αr^x is r^k-uniformly distributed modulo one. In the third theorem we show that α has normality of order k to base r if and only if, for every integer b and every positive integer $t \leqq k$,

$$\lim \frac{N(b, n)}{r} = r^{-t}$$

where $N(b, n)$ is the number of integers x with $1 \leqq x \leqq n$ for which

$$[\alpha r^x] \equiv b (\mathrm{mod}\ r^t)\ .$$

Let α be a real number, $0 < \alpha < 1$. Let r be a positive integer greater than one and construct the "decimal" representation of α to base r. Suppose that a certain sequence of digits occurs $N(n)$ times among the first n digits in the representation of α. If $N(n)/n$ tends to a limit f as n tends to infinity, then f is called the *relative frequency* with which the sequence occurs in α. If the sequence has k digits and appears in α with relative frequency r^{-k}, then it is said to occur with *normal frequency.* If every sequence of k digits appears in α with normal frequency, then α is said to have *normality of order* k. If α has normality of order k for every integer $k \geqq 1$ then it was proved by Niven and Zuckerman [7] and later by Cassels [2] that α is a normal number as defined by Borel [1]. Borel proved that almost all real numbers are normal. We also note that α has normality of order one if and only if it is *simply normal* to base r. This notion is also due to Borel.

The expression "normality of order k" is due to I. J. Good who gave a method [5] for constructing decimals of period r^k having normality of order k for any $k \geqq 1$. The problem was also studied by Rees [8], de Bruijn [4] and Korobov [6] who gave a variety of methods of constructing such decimals. In Section 2 of this paper we give yet another construction for a periodic decimal having normality of order k. While the method does not yield a decimal of minimum period, it has the advantage of being extremely simple.

In addition to the problem of constructing numbers having normality of order k, it is of interest to ask what characteristic properties such numbers possess. For example, D. D. Wall [9] proved that a real number α is normal to base r if and only if the function αr^x is uniformly distributed modulo one. Wall also showed that α is normal to base r if and only if, for every positive integer c and every integer b, $[\alpha r^x] \equiv b (\mathrm{mod}\ c)$ with relative frequency $1/c$ where $[\alpha r^x]$ denotes the largest integer less than or equal to αr^x. In Section 3 we introduce the notion of c-uniform distribution modulo one and show that a real number α has normality of order k if and only if αr^x is r^k-uniformly distributed modulo one. We also show that α has normality of order k if and only if for every integer b and every integer t with $0 < t \leqq k$,

$$[\alpha r^k] \equiv b (\mathrm{mod}\ r^t)$$

with relative frequency r^{-t}.

2. Construction of a number having normality of order k.

Perhaps the simplest example of a normal number was given by D. G. Champernowne [3] who showed that the decimal

$$\alpha = .12345678910111213 \cdots$$

is normal to base 10 where α is formed by writing the decimal representations of the natural numbers in order after the decimal point. Analogously, we prove the following theorem.

THEOREM 1. *Let r and k be integers with $r \geqq 2$ and $k \geqq 1$. Working to base r^k form the periodic decimal*

$$\alpha = .\dot{0}12 \cdots (r^k \dot{-} 1) .$$

Written to base r, α has period kr^k and normality of order k.

Proof. Let Y_n denote the block $a_1a_2 \cdots a_n$ of the first n digits of the representation of α to base r and let $B_k = b_1b_2 \cdots b_k$ denote an arbitrary sequence of k digits to base r. Let C_i denote the ith digit in the representation of α to base r^k. We will also use C_i to denote the block of k digits in the representation of α to base r which corresponds to the digit C_i in the representation of α to base r^k. Thus, we use C_1 to denote 0 and also to denote the block of k zeros with which the representation of α to base r begins. In any given instance the intended meaning will be clear from the context.

Since the representation of α is periodic, it clearly suffices to show that every B_k appears precisely k times starting in Y_{kr^k}. We note that B_k appears precisely once starting in Y_{kr^k} as one of the C_i; i.e., starting in Y_{kr^k} in a position congruent to one modulo k. The problem is to determine how many times B_k appears starting in Y_{kr^k} in a position congruent to $k - j + 1$ for each $j = 1, 2, \cdots, k - 1$. This is equivalent to asking how many times B_k appears with the mid-point of two adjacent C_i's coming between the jth and $(j + 1)$st digits of B_k for each j. And this occurs when and only when, for some i,

$$C_i = c_1c_2 \cdots c_{k-j}b_1b_2 \cdots b_j$$

and

$$C_{i+1} = b_{j+1}b_{j+2} \cdots b_kd_1d_2 \cdots d_j \ .$$

Case 1. Suppose that at least one of $b_1, b_2, \cdots, b_j$ is different from $r - 1$. Then, for some i,

$$C_i = b_{j+1}b_{j+2} \cdots b_kb_1b_2 \cdots b_j$$

and

$$C_{i+1} = b_{j+1}b_{j+2} \cdots b_kd_1d_2 \cdots d_j$$

where $d_1d_2 \cdots d_j$ is the successor to $b_1b_2 \cdots b_j$ in the sequence of j-tuples

$$(1) \qquad 00 \cdots 0, 00 \cdots 01, \cdots, (r - 1) \cdots (r - 1) \ .$$

Thus, in this case, B_k does appear starting in Y_{kr^k} in a position congruent to $k - j + 1$ and this is the only way it can appear in this position.

Case 2. Suppose that $b_1 = b_2 = \cdots = b_j = r - 1$ and that at least one of $b_{j+1}, b_{j+2}, \cdots, b_k$ is different from zero. If $d_{j+1}d_{j+2} \cdots d_k$ is the predecessor of $b_{j+1}b_{j+2} \cdots b_k$ in the sequence of $(k - j)$-tuples

$$(2) \qquad 00 \cdots 0, 00 \cdots 01, \cdots, (r - 1) \cdots (r - 1) \ ,$$

then, for some i,

$$C_i = d_{j+1}d_{j+2} \cdots d_kb_1b_2 \cdots b_j$$

and

$$C_{i+1} = b_{j+1}b_{j+2} \cdots b_k\, 00 \cdots 0 \ .$$

Thus, in this case, B_k again appears starting in Y_{kr^k} in a position congruent to $k - j + 1$ modulo k and this is the only way it can appear in this position.

Case 3. Finally, suppose that $b_1 = b_2 = \cdots = b_j = r - 1$ and that $b_{j+1} = b_{j+2} = \cdots = b_k = 0$. The only way such a B_k can appear starting in Y_{kr^k} in a position congruent to $k - j + 1$ is for

$$b_{j+1}b_{j+2} \cdots b_k = 00 \cdots 0$$

to have a predecessor in the sequence (2). Thus, in this case, B_k cannot appear in the desired position entirely contained in Y_{kr^k}. However, it clearly does appear starting in a position congruent to $k - j + 1$ modulo k in Y_{kr^k} and overlapping the mid-point between Y_{kr^k} and the next sequence of kr^k digits in the representation of α to base r.

Therefore, for each $j = 1, 2, \cdots, k$, B_k occurs in the representation of α to base r starting in Y_{kr^k} in a position congruent to $k - j + 1$ modulo k precisely once. Since B_k was arbitrary, it follows that each sequence of k digits to base r appears in the representation of α to base r equally often. Thus, α has normality of order k as claimed.

Since the α of the preceding theorem is simply normal to base r^k, it is natural to ask if normality of order k to base r is implied by simple normality to base r^k. However, since $\beta = .\dot{1}02\dot{3}$ is simply normal to base 4 but does not have normality of order 2 to base 2, this is clearly not the case.

3. Properties of numbers having normality of order k.

Let $(\alpha) = \alpha - [\alpha]$ denote the fractional part of the real number α. A real valued function $f(x)$ is said to be uniformly distributed modulo one if, for every real λ with $0 \leqq \lambda \leqq 1$, $\lim n_\lambda / n = \lambda$ where n_λ denotes the number of values of $x = 1, 2, \cdots, n$ for which $(f(x)) < \lambda$. Analogously, for any integer $c > 1$, we say that $f(x)$ is *c-uniformly distributed modulo one* if the preceding definition holds for all λ's which are positive rational fractions with denominator c. It then follows that $f(x)$ is uniformly distributed modulo one if and only if it is c-uniformly distributed modulo one for every integer $c > 1$. We also have the following result concerning numbers having normality of order k.

THEOREM 2. *The real number α has normality of order k to base r if and only if the function αr^x is r^k-uniformly distributed modulo one.*

Proof. Let αr^x be r^k-uniformly distributed modulo one. Let $b_1 b_2 \cdots b_k$ denote an arbitrary sequence of digits to base r and let

$$\varepsilon = b_1 r^{-1} + b_2 r^{-2} + \cdots + b_k r^{-k} .$$

It then follows that $\varepsilon \leqq (\alpha r^x) < \varepsilon + r^{-k}$ with relative frequency r^{-k}.

But this simply says that the sequence $b_1b_2 \cdots b_k$ appears in the representation of α to base r with normal frequency so that α has normality of order k.

Conversely, suppose that α has normality of order k to base r. Let $\lambda = br^{-k}$ where b is an integer and $0 < b < r^k$. Then λ can be written in the form

$$\lambda = b_1r^{-1} + b_2r^{-2} + \cdots + b_kr^{-k},\ 0 \leqq b_i < r$$

and $(\alpha r^x) < \lambda$ if and only if

$$a_{1+x}r^{-1} + a_{2+x}r^{-2} + \cdots + a_{k+x}r^{-k} < b_1r^{-1} + b_2r^{-2} + \cdots + b_kr^{-k} .$$

This inequality is equivalent to

$$a = a_{1+x}r^{k-1} + \cdots + a_{k+x} < b_1r^{k-1} + \cdots + b_k = b$$

and it follows that $(\alpha r^x) < \lambda$ if and only if $a < b$. Clearly there are just b nonnegative integers a having this property and, by hypothesis, each k-tuple corresponding to such an a appears in the representation of α to base r with frequency r^{-k}. Therefore, $(\alpha r^x) < \lambda$ with frequency $br^{-k} = \lambda$ and α is r^k-uniformly distributed modulo one.

As noted above the following theorem is also analogous to a result of Wall.

THEOREM 3. *The real number α has normality of order k to base r if and only if, for every positive integer $t \leqq k$ and every integer b, we have $[\alpha r^x] \equiv b (mod\ r^t)$ with relative frequency r^{-t}.*

Proof. There is no loss in generality in assuming that $0 \leqq b < r^t$.

Suppose first that α has normality of order k to base r. Then αr^{-t} also has normality of order k. Therefore, by Theorem 2, αr^{x-t} is r^k-uniformly distributed modulo one and it follows that

$$br^{-t} \leqq (\alpha r^{x-t}) < (b + 1)r^{-t}$$

with relative frequency $r^{-t} = r^{k-t}r^{-k}$. Thus, there exist positive integers n_x with relative frequency r^{-t} such that

$$n_x + br^{-t} \leqq \alpha r^{x-t} < n_x + (b + 1)r^{-t}$$

or, equivalently, such that

$$n_xr^t + b \leqq \alpha r^x < n_xr^t + b + 1 .$$

But this says that

$$[\alpha r^x] \equiv b (\mathrm{mod}\ r^{-t})$$

with relative frequency r^{-t}.

To prove the converse, we simply reverse the preceding argument reading k for t at each step.

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