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MANY-ONE DEGREES OF THE PREDICATES $H_a(x)$

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Spector proved in his Ph. D. Thesis that if $|a| = |b|$ $(a, b \in O)$, then $H_a(x)$ and $H_b(x)$ have the same degree of unsolvability; Davis had already shown that if $|a| = |b| < \omega^2$, then $H_a(x)$ and $H_b(x)$ are in fact recursively isomorphic, i.e.,

$$(1) \qquad H_a(x) \equiv H_b(f(x)) ,$$

where $f(x)$ is a recursive permutation.

In this note we prove that if $|a| = |b| = \xi$, then $H_a(x)$ and $H_b(x)$ need not have the same many-one degree, unless $\xi = 0$ or is of the form $\eta + 1$ or $\eta + \omega$; if $\xi \neq 0$ is not of the form $\eta + 1$ or $\eta + \omega$, then the partial ordering of the many-one degrees of the predicates $H_a(x)$ with $|a| = \xi$ contains well-ordered chains of length ω_1 as well as incomparable elements. The proof rests on a combinatorial result which relates the many-one degree of $H_{a'}(x)$ $(a' = 3.5^a \in O)$ to the rate with which the sequence of ordinals $|a_n|$ approaches $|a'|$.

Summary of results. We denote the relations of many-one and one-one reducibility by $\leqq_m$ and $\leqq_1$. By a result of Myhill [5], if $P(x) \leqq_1 Q(x)$ and $Q(x) \leqq_1 P(x)$, then $P(x)$ and $Q(x)$ are recursively isomorphic.

Let $a' = 3.5^a$ and $b' = 3.5^b$ be names in O of the same limit ordinal $|a| = |b'| = \xi$. We say that a' is *recursively majorized* by b' and write $a' \prec b'$, if there is a recursive function $f(n)$ such that for all n,

$$(2) \qquad |a_n| \leqq |b_{f(n)}| .$$

(Here $a_n \simeq \{a\}(n_O)$; in dealing with constructive ordinals and hyperarithmetic predicates we use without apologies and sometimes without reference the notations of Kleene's [2] and [3].) If $a' \prec b'$ and $b' \prec a'$, a' and b' are *equivalent*, $a' \sim b'$; if neither $a' \prec b'$, nor $b' \prec a'$, a' and b' are incomparable, $a' | b'$. Notations such as $a' \precsim\!\!\!\!/\; b'$ are self-explanatory.

THEOREM 1. *Let $a' = 3.5^a \in O$, $b' = 3.5^b \in O$, $|a'| = |b'| = \xi$. Then $H_{a'}(x) \leqq_m H_{b'}(x)$ if and only if $H_{a'}(x) \leqq_1 H_{b'}(x)$ if and only if $a' \prec b'$.*

THEOREM 2. *If ξ is of the form $\eta + 1$ or $\eta + \omega$ and $|a| = |b| = \xi$, then $H_a(x)$ and $H_b(x)$ are recursively isomorphic.*

For each constructive ordinal ξ, let $\mathscr{L}(\xi)$ be the partial ordering of the many-one degrees of the predicates $H_{a'}(x)$ with $|a'| = \xi$.

THEOREM 3. *If $\xi \neq 0$ is not of the form $\eta + 1$ or $\eta + \omega$, then $\mathscr{L}(\xi)$ contains well-ordered chains of length ω_1.*

THEOREM 4. *If $\xi \neq 0$ is not of the form $\eta + 1$ or $\eta + \omega$, then $\mathscr{L}(\xi)$ contains incomparable elements.*

2. Proof of Theorem 1.

LEMMA 1. (*Kleene's Lemma* 3 *in* [2]). *There is a partial recursive function $\sigma_1(a, b, x)$, such that*

$$\text{(3)} \qquad \textit{if} \quad a \leqq_O b\,, \quad \textit{then} \quad H_a(x) \equiv H_b(\sigma_1(a, b, x))\,.$$

Let $P'(x)$ denote the *jump* of the predicate $P(x)$,

$$\text{(4)} \qquad P'(x) \equiv (Ey)T_1^P(x, x, y)\,.$$

LEMMA 2. (a) *There is a primitive recursive $\sigma_2(e, x)$ such that if $Q(x)$ is recursive in $P(x)$ with Gödel number e, then*

$$\text{(5)} \qquad Q(x) \equiv P'(\sigma_2(e, x))\,.$$

(b) *There is a primitive recursive $\sigma_3(e)$ such that*

$$\text{(6)} \qquad \begin{aligned} &\textit{if} \quad t = \sigma_3(e) \quad \textit{and} \quad \{e\}(t) \quad \textit{is defined}\,, \\ &\textit{then} \quad P'(t) \not\equiv P(\{e\}(t))\,. \end{aligned}$$

(Both of these facts are implicit in Section 1.4 of [4] and the references given there to [1] and [6].)

LEMMA 3. *There is a partial recursive $\sigma_4(a, b, c, x)$ such that for a, b, c in O,*

$$\text{(7)} \quad \textit{if} \quad |a| \leqq |b| \quad \textit{and} \quad b <_O c\,, \quad \textit{then} \quad H_a(x) \equiv H_c(\sigma_4(a, b, c, x))\,.$$

Proof. By Spector's Uniqueness Theorem in [7], if $|a| \leqq |b|$, then $H_a(x)$ is recursive in $H_b(x)$ with Gödel number $\tau(a, b)$ (τ recursive). Since $b <_O c$ implies $2^b \leqq_O c$, Lemma 1 together with Lemma 2(a) imply that

$$H_a(x) \equiv H_{2^b}(\sigma_2(\tau(a, b), x)) \equiv H_c(\sigma_1(2^b, c, \sigma_2(\tau(a, b), x)))$$

and we can define σ_4 as the argument of H_c in this equivalence.

LEMMA 4. *There is a partial recursive $\sigma(a, b, e)$, such that if $a <_O b$, then $\sigma(a, b, e)$ is defined and*

(8) $$\textit{if} \quad t = \sigma(a, b, e) \quad \textit{and} \quad \{e\}(t) \quad \textit{is defined,} \quad \textit{then} \quad H_b(t) \not\equiv H_a(\{e\}(t)) .$$

Proof is by induction on $b \in O$ for fixed $a \in O$ and the recursion theorem, utilizing Lemma 2 (b).

Case 1. $b = 2^a$. Set $\sigma(a, b, e) = \sigma_3(e)$.

Case 2. $b = 2^c$ and $c \neq a$. In this case, if $a <_O b$ we must have $a <_O c$ and the Ind. Hyp. applies to a and c. Put

(9) $$y \simeq \sigma(a, c, \Lambda x\{e\}(\sigma_1(c, b, x))) ,$$

and

(10) $$\sigma(a, b, e) \simeq \sigma_1(c, b, y) .$$

(For a partial recursive $f(x_1, \cdots, x_n, y)$, $\Lambda y f(x_1, \cdots, x_n, y)$ is a primitive recursive function of $x_1, \cdots, x_n$ and a Gödel number of f such that

$$\{\Lambda y f(x_1, \cdots, x_n, y)\}(y) \simeq f(x_1, \cdots, x_n, y) ;$$

see [1], Section 65.)

Since $c <_O b$, $\sigma_1(c, b, x)$ is totally defined; since $a <_O c$, the induction hypothesis implies that y is defined, hence $\sigma(a, b, e)$ is defined. We now derive a contradiction from the assumption

(11) $$\text{for} \quad t = \sigma(a, b, e), \{e\}(t) \quad \text{is defined and} \quad H_b(t) \equiv H_a(\{e\}(t)) .$$

Since

$$H_c(y) \equiv H_b(\sigma_1(c, b, y)) \equiv H_b(t) ,$$

we have

$$H_c(y) \equiv H_a(\{e\}(t)) ;$$

but

$$\{e\}(t) \simeq \{e\}(\sigma_1(c, b, y)) \simeq \{\Lambda x\{e\}(\sigma_1(c, b, x))\}(y) ,$$

hence

$$H_c(y) \equiv H_a(\{\Lambda x\{e\}(\sigma_1(c, b, x))\}(y))$$

which by induction hypothesis is false if y is given by (9).

Case 3. $b = 3.5^z$. In this case $a <_O b$ implies $a \leqq_O z_{\iota(a,z)}$, where $\iota(a, z)$ is partial recursive ([2], Lemma 2). Now the definition and proof of Case 2 apply if we substitute $\iota(a, z)$ for c throughout.

The proof is completed by securing via the recursion theorem a partial recursive function $\sigma(a, b, e)$ such that

$$\sigma(a, b, e) \simeq \begin{cases} \sigma_3(e) \quad \text{if} \quad b = 2^a , \\ \sigma_1((b)_0, b, \sigma(a, (b)_0, \Lambda x\{e\}(\sigma_1((b)_0, b, x)))) \\ \qquad\qquad \text{if} \quad b = 2^{(b)_0}, (b)_0 \neq a , \\ \sigma_1(\iota(a, (b)_2), b, \sigma(a, \iota(a, (b)_2), \Lambda x\{e\}(\sigma_1(\iota(a, (b)_2), b, x)))) \\ \qquad\qquad \text{if} \quad b = 3.5^{(b)_2} , \\ 0 \quad \text{otherwise} . \end{cases}$$

LEMMA 5. *Let* $a' = 3.5^a$, $b' = 3.5^b \in O$, $|a'| = |b'|$. *If* $H_{a'}(x) \leqq_m H_{b'}(x)$, *then* $H_{a'}(x) \leqq_1 H_{b'}(x)$.

Proof. Suppose that $H_{a'}(x) \equiv H_{b'}(f(x))$, with $f(x)$ general recursive, possibly many-one. Put

$$g(x) = 2^u 3^v ,$$

where

$$u = 2^x 3^{(f(x))_0} , \quad v = \sigma_1(b_{(f(x))_0}, b_u, (f(x))_1)$$

and σ_1 is the partial recursive function of Lemma 1. It is clear that $g(x)$ is general recursive and one-one. To complete the proof we compute:

$$\begin{aligned} H_{b'}(g(x)) &\equiv H_{b_u}(v) \equiv H_{b_u}(\sigma_1(b_{(f(x))_0}, b_u, (f(x))_1)) \\ &\equiv H_{b_{(f(x))_0}}((f(x))_1) \equiv H_{b'}(f(x)) \equiv H_{a'}(x) . \end{aligned}$$

Proof of Theorem 1. First assume that $a' \prec b'$, i.e., for some general recursive $f(n)$ we have $|a_n| \leqq |b_{f(n)}|$, all n. Since, for each n, $b_{f(n)} <_O b_{f(n)+1}$, Lemma 3 yields

$$H_{a_n}(x) \equiv H_{b_{f(n)+1}}(\sigma_4(a_n, b_{f(n)}, b_{f(n)+1}, x)) .$$

Hence

$$H_{a'}(x) \equiv H_{b'}(2^{u(x)} \cdot 3^{v(x)}) ,$$

with

$$\begin{aligned} u(x) &= f((x_0)) + 1 , \\ v(x) &= \sigma_4(a_{(x)_0}, b_{f((x)_0)}, b_{f((x)_0)+1}, (x)_1) , \end{aligned}$$

which implies $H_{a'}(x) \leqq_m H_{b'}(x)$; by Lemma 5 this is equivalent to $H_{a'}(x) \leqq_1 H_{b'}(x)$.

To prove the converse assume that for all x

$$H_{a'}(x) \equiv H_{b'}(\{e\}(x)) , \tag{12}$$

with $\{e\}(x)$ general recursive. For fixed n we compute:

$$H_{a_{n+2}}(x) \equiv H_{a'}(2^{n+2}\cdot 3^x) \equiv H_{b'}(\{e\}(2^{n+2}\cdot 3^x)) \equiv H_{b_{x_0}}(x_1) \ , \tag{13}$$

where

$$x_0 = (\{e\}(2^{n+2}\cdot 3^x))_0 \ , \tag{14}$$

$$x_1 = (\{e\}(2^{n+2}\cdot 3^x))_1 \ . \tag{15}$$

Now assume that for a fixed x

$$| b_{x_0} | \leqq | a_n | \ ; \tag{16}$$

this implies that for each y

$$H_{b_{x_0}}(y) \equiv H_{a_{n+1}}(\sigma_4(b_{x_0}, a_n, a_{n+1}, y)) \ , \tag{17}$$

which for $y = x_1$ yields

$$H_{a_{n+2}}(x) \equiv H_{a_{n+1}}(\sigma_4(b_{x_0}, a_n, a_{n+1}, x_1)) \ . \tag{18}$$

Equivalence (18) however is impossible if

$$x = \sigma(a_{n+1}, a_{n+2}, \Lambda x \sigma_4(b_{x_0}, a_n, a_{n+1}, x_1)) \tag{19}$$

by Lemma 4, hence for this x the negation of (16) must be true. Thus to prove $a' \prec b'$ it is enough to set

$$f(n) = x_0 \ , \tag{20}$$

where x is given by (19) and x_0 by (14).

3. **Proof of Theorem 2.** It is implicit in [4], Section 1.4, that if $P(x)$ is recursive in $Q(x)$, then $P'(x) \leqq_1 Q'(x)$. Thus if $|a| = |b| = \eta + 1$, Spector's Uniqueness Theorem implies that $H_a(x)$ and $H_b(x)$ are one-one reducible to each other and hence recursively isomorphic. The case $|a'| = |b'| = \eta + \omega$ is settled by the following Lemma in view of Theorem 1.

LEMMA 6. *If $|a'| = |b'| = \eta + \omega$, then $a' \prec b'$.*

Proof. It is easy to define primitive recursive functions $L(x)$ and $N(x)$ so that for $x \in O$,

$$x = L(x) +_O N(x) \ , \tag{21}$$

where $L(x) = 1$ or $|L(x)|$ is a limit ordinal and $|N(x)| < \omega$ (with these requirements $L(x)$ and $N(x)$ are uniquely determined on members of O).

Let a^0 and b^0 be the uniquely determined elements of O such that

$$a^0 <_O a' \ , \qquad | a^0 | = \eta \ ; \qquad b^0 <_O b' \ , \qquad | b^0 | = \eta \ . \tag{22}$$

Set

$$f(n) = \mu y[b^0 +_O N(a_n) \leqq_O b_y] . \tag{23}$$

That $f(n)$ is totally defined follows from the fact that if z is any ordinal notation for an integer (in particular if $z = N(a_n)$), then $b^0 +_O z <_O b'$ and hence there is a y so that $b' +_O z \leqq_O b_y$. That $f(n)$ is recursive follows from the fact that $\leqq_O$ is recursive on the $<_O$-predecesors of b' (see [3], Section 21.).

If $|a_n| \leqq \eta$, then $|a_n| \leqq |b_{f(n)}|$, since for each n, $|b_{f(n)}| \geqq \eta$. If $|a_n| > \eta$, then $L(a_n) = a^0$, hence $|a_n| = |a^0 +_O N(a_n)| = |a^0| + |N(a_n)| = |b^0| + |N(a_n)| = |b^0 +_O N(a_n)| \leqq |b_{f(n)}|$, which completes the proof.

4. Proof of Theorem 3 for special ordinals. Call an ordinal ξ *special* if $\xi > \omega$ and whenever $\eta, \eta' < \xi$, then $\eta + \eta' < \xi$.

LEMMA 7. *There is a primitive recursive $\rho_1(a')$ such that if $a' \in O$ and $|a'|$ is special, then $\rho_1(a') \in O$, $|\rho_1(a')| = |a'|$ and $a' \precnsim \rho_1(a')$.*

Proof. Define $f(n, t)$ by the recursion

$$\begin{aligned} f(n, 0) &\simeq a_n \\ f(n, t+1) &\simeq \begin{cases} 2 & \text{if} \quad \bar{T}_1(n, n, t+1) \\ a_{\{n\}(n)} & \text{otherwise} . \end{cases} \end{aligned} \tag{24}$$

It is clear that if $a' = 3.5^a \in O$, then $f(n, t)$ is general recursive and its range is a subset of O. Moreover:

$$\sum_{t=0}^{\infty} |f(n, t)| = \begin{cases} |a_n| + \omega & \text{if} \quad \{n\}(n) \quad \text{is not defined} , \\ |a_n| + |a_{\{n\}(n)}| + \omega & \text{if} \quad \{n\}(n) \quad \text{is defined} . \end{cases} \tag{25}$$

Put

$$\begin{aligned} \xi_0 &= \sum_{t=0}^{\infty} |f(0, t)| , \\ \xi_{n+1} &= \xi_n + \sum_{t=0}^{\infty} |f(n+1, t)| . \end{aligned} \tag{26}$$

Since ξ is special, for each n, $\xi_n < \xi$; since for each n, $|a_n| < \xi_n$, $\{\xi_n\}$ is a fundamental sequence converging to ξ.

By an elementary construction one can define a primitive recursive $\rho(a')$ such that if $a' = 3.5^a \in O$, then $\rho(a') = b' = 3.5^b \in O$ and for each n, $|b_n| = \xi_n$.

Since, for each n, $|a_n| < \Sigma_t |f(n, t)| < \xi_n$, it is trivial that $a' \prec b'$. To show that the converse is impossible assume that for all n $|b_n| = \xi_n \leqq |a_{\{m\}(n)}|$; this is absurd for $n = m$, since

$$\xi_m = \xi_{m-1} + \Sigma_t \,|\, f(m, t) \,| = \xi_{m-1} + |\, a_m \,| + |\, a_{\{m\}(m)} \,| + \omega > |\, a_{\{m\}(m)} \,| \,.$$

This lemma already shows that for each a' with $|\, a' \,| = \omega^2$ there is a b', $|\, b' \,| = \omega^2$ such that the many-one degree of $H_{b'}(x)$ is strictly greater than the many-one degree of $H_{a'}(x)$.

LEMMA 8. *Let $a' = 3.5^a \in O$, $|\, a' \,|$ be special. There is a primitive recursive $\rho_2(e)$ such that if for each t, $\{e\}(t) \in O$ and $|\, \{e\}(t) \,| = |\, a' \,|$, then $\rho_2(e) \in O$, $|\, \rho_2(e) \,| = |\, a' \,|$ and for each t, $\{e\}(t) \prec \rho_2(e)$.*

Proof. If e satisfies the hypothesis, then for each t, $\{e\}(t) = 3.5^{m(t)}$ and $|\, m(t)_0 \,|, |\, m(t)_1 \,|, \cdots$, is a fundamental sequence converging to $|\, a' \,|$. Put

$$\begin{aligned} f(0) &= m(0)_0 \\ f(t+1) &= f(t) +_O m(0)_{t+1} +_O m(1)_{t+1} +_O \cdots +_C m(t)_{t+1} \\ &\qquad +_O m(t+1)_0 +_O m(t+1)_1 +_O \cdots +_O m(t+1)_{t+1} \,, \end{aligned}$$

where the association is to the left; since by [3], XVII if $x \in O$ and $y >_O 1$, then $x <_O x +_O y$, we have for each t,

$$f(t) <_O f(t+1) \,.$$

Since $|\, a' \,|$ is special, for each t, $|\, f(t) \,| < |\, a' \,|$; since for each t $|\, m(0)_t \,| \leqq |\, f(t) \,|$, the sequence $|\, f(0) \,|, |\, f(1) \,|, \cdots$, is fundamental and converges to $|\, a' \,|$.

It is easy to construct a primitive recursive $\rho_2(e)$ such that if the hypotheses are fulfilled then $\rho_2(e) = 3.5^b$ and for each t, $b_t = f(t)$. Now $\rho_2(e) \in O$, $|\, \rho_2(e) \,| = |\, a' \,|$ and for each t, n

$$|\, m(t)_n \,| \leqq |\, m(t)_{n+t} \,| \leqq |\, f(n+t) \,| = |\, b_{n+t} \,| \,,$$

which proves that $\{e\}(t) \prec 3.5^b$.

LEMMA 9. *Let $a' = 3.5^a \in O$, $|\, a' \,|$ be special. There is a primitive recursive $\rho(x)$ such that*

(i) *$\rho(1) = a'$*

(ii) *if $x \in O$, then $\rho(x) \in O$ and $|\, \rho(x) \,| = |\, a' \,|$,*

(iii) *if $x <_O y$, then $\rho(x) \precsim\!\!\!\!/\; \rho(y)$.*

Proof. Using the recursion theorem we obtain a $\rho(x)$ satisfying:

$$\begin{aligned} \rho(1) &= a' \,, \\ \rho(2^x) &= \rho_1(\rho(x)) \,, \\ \rho(3.5^z) &= \rho_2(\Lambda t \rho(z_t)) \,. \end{aligned}$$

Proof that $\rho(x)$ is the required function is by induction on $x \in O$. To treat the case $x = 3.5^z$—here the induction hypothesis is that for each t, $\rho(z_t) \in O$, $|\rho(z_t)| = |a'|$ and $\rho(z_t) \precsim \rho(z_{t+1})$. Lemma 8 assures us that for each t $\rho(z_t) \prec \rho(3.5^z)$; if for some t $\rho(3.5^z) \prec \rho(z_t)$, the transitivity of $\prec$ would imply that $\rho(z_{t+1}) \prec \rho(z_t)$, violating the induction hypothesis.

Theorem 3 for special ordinals follows from Lemma 9 by letting A be a subset of O, linearly ordered under $<_O$ and containing a notation for each constructive ordinal and considering $\rho(A)$.

5. **Proof of Theorem 4 for special ordinals.** Let $\xi = |3.5^a|$ be a special ordinal. In the proof of Lemma 6 we constructed a notation $b' = 3.5^b$ of ξ determined by a fundamental sequence $\{\xi_n\}$ which was in turn defined from a double sequence $f(n, t)$ by equations (26). Here we will define two such double sequences, $f(n, t)$ and $g(n, t)$, such that the notations $b' = 3.5^b$ and $c' = 3.5^c$ for sequences $\{\xi_n\}$ and $\{\zeta_n\}$ determined as in equations (26) from $f(n, t)$ and $g(n, t)$ respectively will be incomparable.

We define the functions $f(n, t)$ and $g(n, t)$ in stages; at stage $2s$ we will define $f(n, t)$ for $n, t \leqq s$ and at stage $2s + 1$ we will define $g(n, t)$ for $n, t \leqq s$. At each stage s we will also define finite sets F_s and G_s of pairs $\langle m, k\rangle$ of integers which will determine partial functions —i.e., if $\langle m, k\rangle \in F_s$ and $\langle m, k'\rangle \in F_s$, then $k = k'$, and similarly for G_s. We give the definitions informally, but it is a routine matter to derive Herbrand-Gödel-Kleene equations for f and g from our instructions.

Basis 0. $s = 0$. Put $f(0, 0) = a_0$; $F_0 = \{\langle 0, 0\rangle\}$; $G_0 = \{\langle 0, 0\rangle\}$.

Basis 1. $s = 1$. Put $g(0, 0) = a_0$; $F_1 = F_0 \cup \{\langle 1, 1\rangle\}$; $G_1 = G_0 \cup \{\langle 1, 1\rangle\}$.

Even Induction Step $2s + 2$.

Case 1. For every pair $\langle m, k\rangle \in F_{2s+1}$ and for every $y \leqq 2s + 1$, $\bar{T}_1(m, k, y)$. In this case set:

$$(27) \qquad \begin{cases} f(n, s+1) = 2 & (n \leqq s)\,, \\ f(s+1, 0) = a_{s+1}\,, & \\ f(s+1, t) = 2 & (1 \leqq t \leqq s+1)\,. \end{cases}$$

Put $F_{2s+2} = F_{2s+1} \cup \{\langle 2s + 2, k'\rangle\}$ where k' is the smallest integer larger than all the second members of the pairs in F_{2s+1}; put $G_{2s+2} = G_{2s+1} \cup \{\langle 2s + 2, k'\rangle\}$ where k' is the smallest integer larger than all the second members of the pairs in G_{2s+1}.

Case 2. Otherwise. Let m be the smallest integer such that some k, $\langle m, k\rangle \in F_{2s+1}$ and for some $y \leqq 2s + 1$, $T_1(m, k, y)$; let k and y be the corresponding (unique) k and y.

Subcase 2a. $U(y) = z \leqq s$.

For any stage (in particular $2s+1$) and any $x \leqq s$ (in particular z) consider the array of values of $g(u, v)$ with $u \leqq x$ and $v \leqq s$. Put

$$(28) \quad J_g(x, s) = \begin{cases} g(0, 0) +_O g(0, 1) +_O \cdots +_O g(0, s) +_O \omega_O \\ +_O g(1, 0) +_O g(1, 1) +_O \cdots +_O g(1, s) +_O \omega_O \\ +_O \cdots \\ \qquad\qquad \cdots \\ +_O g(x, 0) +_O g(x, 1) +_O \cdots +_O g(x, s) +_O \omega_0 , \end{cases}$$

where ω_O is some fixed ordinal notation of ω and the association in the sum is to the left. It is clear that if all the values of $g(u, v)$ for $u \leqq x$, $v \leqq x$ are elements of O, then so is $J_g(x, s)$. Put

$$(29) \quad \begin{cases} f(n, s+1) = 2 & (n \leqq s, n \neq k) , \\ f(k, s+1) = J_g(z, s) , \\ f(s+1, 0) = a_{s+1} , \\ f(s+1, t) = 2 & (1 \leqq t \leqq s+1) . \end{cases}$$

Put $F_{2s+2} = F_{2s+1} - \{\langle m, k\rangle\} \cup \{\langle 2s+2, k'\rangle\}$, where k' is the smallest integer larger than all the second members of the pairs in F_{2s+1}.

To define G_{2s+2}, first remove from G_{2s+1} all pairs $\langle m', k'\rangle$ with $m' \geqq m$; then introduce one pair $\langle m', k'\rangle$ for each m', $m \leqq m' \leqq 2s+2$ in some systematic way, so that if $m' \neq m''$, then $k' \neq k''$, and all the second members of the new pairs are larger than all the second members of the pairs in G_{2s+1} and also larger than z.

Subcase 2b. $U(y) = z > s$. Give exactly the same definitions as in Subcase 2a, except for the second equation of (29) for which we substitute

$$(30) \quad f(k, s+1) = J_g(s, s) +_O a_{s+1} +_O \omega_O +_O a_{s+2} +_O \omega_O +_O \cdots +_O a_z +_O \omega_O .$$

(Remark: the last conditions on the definition of G_{2s+2}, that all new second members be larger than z, will be utilized for this subcase.)

Odd Induction Step $2s+3$. The definitions are symmetric to those in the Even Ind. Step, except for the following differences:

(i) In Subcase 2a we put $J_g(z, s+1)$ where complete symmetry would suggest $J_f(z, s)$.

(ii) In Subcase 2b we put $g(k, s+1) = J_f(s+1, s+1) +_O \omega_O +_O \cdots +_O a_z +_O \omega_O$.

(iii) In Case 2 we define F_{2s+3} by removing from and reintroducing in F_{2s+2} all pairs with first members $m' > m$ (rather than $m' \geqq m$).

It is easy to prove by induction on s that for all n, t $f(n, t)$, $g(n, t) \in O$ and $|f(n, t)| < \xi$, $|g(n, t)| < \xi$. Put

$$(31)\qquad \begin{aligned} \xi_0 &= \sum_{t=0}^{\infty} |f(0, t)| , & \zeta_0 &= \sum_{t=0}^{\infty} |g(0, t)| , \\ \xi_{n+1} &= \xi_n + \sum_{t=0}^{\infty} |f(n+1, t)| , & \zeta_{n+1} &= \zeta_n + \sum_{t=0}^{\infty} |g(n+1, t)| . \end{aligned}$$

By a routine construction numbers $b' = 3.5^b$ and $c' = 3.5^c$ can be defined such that $b' \in O$, $c' \in O$ and for all n,

$$|b_n| = \xi_n , \qquad |c_n| = \zeta_n .$$

We will prove that $|b'| = |c'| = \xi$ and that b' and c' are incomparable.

Say that m *F-joins* k at stage s if $\langle m, k\rangle \in F_s$ but $\langle m, k\rangle \notin F_{s-1}$; m *F-leaves* k at stage s if $\langle m, k\rangle \notin F_s$ but $\langle m, k\rangle \in F_{s-1}$. (Similarly with G in place of F throughout.)

Clearly at each stage s, some m F-joins some k. Using this we can show by an induction on s that if m F-joins k at stage s, then k is larger than all the second members of all the pairs in F_t, with $t < s$. This in turn implies that for a fixed k and in the course of the whole computation there is at most one stage s at which some m F-joins k, and consequently at most one stage s at which some m F-leaves k. Hence for each k there is a t_0 such that for $t \geqq t_0$, $f(k, t) = 2$, since only if $t = 0$ or some m F-leaves k at stage t is $f(k, t) \neq 2$, and we have

$$(32)\qquad \sum_{t=0}^{\infty} |f(k, t)| = |f(k, t_0)| + \omega < \xi ,$$

since ξ is special. Now a simple induction on n shows that for each n, $\xi_n < \xi$, and since clearly $|a_n| < \xi_n$, we have proved that $\lim \xi_n = |b'| = \xi$.

(Exactly the same considerations for g prove that $|c'| = \xi$.)

We prove by induction the following proposition depending on m: m F-joins only finitely many k's, and G-joins only finitely many k's.

If $m = 0$ this is trivial since $\{0\}(x)$ is the totally undefined function.

If m F-joins k at stage s either $m = s$ or there is an $m' < m$ such that m' G-leaves some k' at stage s; by ind. hyp. each $m' < m$ G-joins some k' only for finitely many x's, hence each $m' < m$ G-leaves some k' only for finitely many s's, which completes the proof of half the induction step.

If m G-joins k at stage s, either $m = s$ or there is an $m' \leqq m$ such that m' F-leaves some k' at stage s; we now use the ind. hyp. and the first half of the ind. step which has been already proved to see that this can only happen finitely often.

For a fixed m, let k be the largest integer such that m F-joins k and assume that $\{m\}(k) \simeq z$ is defined. An easy induction on m shows that there must be some stage $2s + 2$ where Case 2 applies with this

m and k, and $z = U(y)$. We prove that $\xi_k > \zeta_z$.

Subcase 2a. Since $f(k, s+1) = J_g(z, s)$, $\xi_k > |J_g(z, s)|$. We assert that if $u \leqq z$, $v > s$, then $g(u, v) = 2$. Because if $g(u, v) \neq 2$, then some m' G-leaves u at stage $2v + 1 > 2s + 2$; since at stage $2s + 2$ each $m'' \geqq m$ G-joins some $k'' > z$, we must have $m' < m$; but this implies that m F-joins some $k' > k$, contrary to hyp. that k is the largest integer that m F-joins.

Now the above implies that $\zeta_z = |J_g(z, s)| < \xi_k$.

Subcase 2b. Now we can prove that if $u \leqq s$ and $v > s$ or $s < u \leqq z$ and $v > 0$, then $g(u, v) = 2$, by exactly the same argument. Hence $\zeta_z = |f(k, s+1)| < \xi_k$.

For a fixed m let k be the largest integer such that m G-joins k and assume that $\{m\}(k) \simeq z$ is defined. As before there must be some stage $2s + 3$ where case 2 applies for this m and this k. We give one of the cases of the proof that $\zeta_k > \xi_z$.

Subcase 2a. We assert that if $u \leqq z$, $v > s + 1$, then $f(u, v) = 2$. Because if $f(u, v) \neq z$, then some m' F-leaves u at stage $2v > 2s + 3$; since at stage $2s + 3$ each $m'' > m$ F-joins some $k'' > z$, we must have $m' \leqq m$; but this implies that m G-joins some $k' > k$, contrary to hyp. that k is the largest integer that m G-joins.

The above remarks complete the proof that b' and c' are incomparable. Because if $b' \prec c'$, then there is an m such that for each k, $|b_k| \leqq |c_{\{m\}(k)}|$, i.e., $\xi_k < \zeta_{\{m\}(k)}$, which we showed to be false if k is the largest integer that m F-joins, and similarly for $c' \prec b'$.

6. Reduction of the general to the special case. In this section we prove that if $\xi = \eta + \zeta$ ($\zeta \neq 0$), then $\mathscr{L}(\xi)$ and $\mathscr{L}(\zeta)$ are similar and that if ξ is $\neq 0$ and not of the form $\eta + 1$ or $\eta + \omega$, then there is a unique special ordinal ζ such that for some η, $\xi = \eta + \zeta$.

LEMMA 10. *There is a primitive recursive $\delta(a, b)$ such that if $a \leqq_O b$, then $\delta(a, b) \in O$ and*

$$|a| + |\delta(a, b)| = |b| . \tag{33}$$

Proof. We obtain via the recursion theorem a primitive recursive $\delta(a, b)$ satisfying the following conditions:

$$\begin{aligned} \delta(a, a) &= 1 , \\ \delta(a, 2^b) &= 2^{(a,b)} , \\ \delta(a, 3.5^z) &= 3.5^y , \quad \text{where for each } t,\ y_t \simeq \delta(a, z_{\iota(a,z)+t}) , \\ \delta(a, x) &= 0 \qquad \text{otherwise} \end{aligned}$$

(recall that $\iota(a, z)$ is partial recursive and such that if $a <_O 3.5^z$, then $a \leqq_O z_{\iota(a,z)}$).

We prove by induction on $b \in O$ the following statement: if $a \leqq_O b$, then $\delta(a, b) \in O$ and for each x, if $a \leqq_O x <_O b$, then $\delta(a, x) <_O \delta(a, b)$. The following cases arise: (1) $b = a$, (2) $b = 2^a$, (3) $b = 2^c$ and $a <_O c$ and (4) $b = 3.5^z$ and for some t, $a \leqq_O z_t$.

Case 3. By Ind. Hyp. $\delta(a, c) \in O$, hence $\delta(a, b) = 2^{\delta(a,c)} \in O$. If $x <_O b$, either $x = c$ or $x <_O c$; in the first case it is clear that $\delta(a, c) <_O \delta(a, b)$, while in the second case the Ind. Hyp. implies that $\delta(a, x) <_O \delta(a, c)$, hence $\delta(a, x) <_O \delta(a, b)$.

Case 4. Since $a <_O 3.5^z$, $\iota(a, z)$ is defined and for each t, $a <_O z_{\iota(a,z)+t}$. Thus the Ind. Hyp. implies that for each t, y_t is defined, $y_t \in O$ and $y_t <_O y_{t+2}$, hence $\delta(a, b) \in O$. If $x <_O 3.5^z$, then for some t, $x <_O z_{\iota(a,z)+t}$, hence by Ind. Hyp. $\delta(a, x) <_O \delta(a, z_{\iota(a,z)+t}) = y_t <_O \delta(a, b)$.

Equation (33) is proved easily by induction on $|b|$, using the continuity of ordinal addition, e.g.,

$$\begin{aligned} |a| + |\delta(a, 3.5^z)| &= |a| + \lim_t |\delta(a, z_{\iota(a,z)+t})| \\ &= \lim_t (|a| + |\delta(a, z_{\iota(a,z)+t})|) \\ &= \lim_t |z_{\iota(a,z)+t}| \\ &= |3.5^z| . \end{aligned}$$

This lemma allows us to represent a constructive limit ordinal as an infinite sum of smaller ordinals,

$$|3.5^z| = |z_0| + |\delta(z_0, z_1)| + |\delta(z_1, z_2)| + \cdots .$$

LEMMA 11. *Assume that $\xi = \eta + \zeta$, where ζ is a limit ordinal. Then $\mathscr{L}(\xi)$ and $\mathscr{L}(\zeta)$ are similar.*

Proof. Let u be a fixed notation in O for η. For each $a' = 3.5^a \in O$ we define by induction

$$\begin{aligned} g(0) &= u +_O a_0 \\ g(n+1) &= g(n) +_O \delta(a_n, a_{n+1}) . \end{aligned}$$

A routine construction yields a primitive recursive $\tau(a')$ such that if $a' = 3.5^a \in O$, then $\tau(a') = 3.5^x \in O$ and for each n, $x_n = g(n)$. Notice that by the definition of δ,

$$|x_n| = \eta + |a_n| . \tag{34}$$

It is clear that if $|a'| = \zeta$, then $|x'| = \lim_n |x_n| = \eta + \zeta = \xi$.

Assume that $|b'| = \zeta$ and $a' \prec b'$, i.e., there is a general recursive

$f(n)$ such that for each n, $|a_n| \leqq |b_{f(n)}|$. Now if $\tau(b') = 3.5^y$,

$$|x_n| = \eta + |a_n| \leqq \eta + |b_{f(n)}| = |y_{f(n)}| ,$$

hence $\tau(a') \prec \tau(b')$.

Assume that $\tau(a') \prec \tau(b')$, i.e., there is a general recursive $f(n)$ such that for each n, $|x_n| \leqq |y_{f(n)}|$. Then $\eta + |a_n| \leqq \eta + |b_{f(n)}|$, i.e., $|a_n| \leqq |b_{f(n)}|$ which proves that $a' \prec b'$.

We have shown that $\tau(a')$ induces a mapping from $\mathscr{L}(\zeta)$ into $\mathscr{L}(\xi)$ which is a similarity imbedding. To complete the proof we must show that this mapping is onto, i.e., that given y', $|y'| = \xi$, there is an a', $|a'| = \zeta$, such that if $\tau(a') = x'$, then $x' \sim y'$.

If $|y'| = \xi$, there is a unique $v <_O y'$ such that $|v| = \eta$, and some t such that $v <_O y_t$. Put

$$h(0) = \delta(v, y_t) ,$$
$$h(n+1) = h(n) +_O \delta(y_{t+n}, y_{t+n+1})$$

and choose $a' = 3.5^a$ so that for each n, $a_n = h(n)$. Surely $a' \in O$ and since for each n, $\eta + |a_n| = |y_{t+n}|$, we have $|a'| = \lim_n |a_n| = \zeta$. If $x' = \tau(a')$, then for each n we have

$$|x_n| = \eta + |a_n| = |y_{t+n}|$$

which implies $x' \sim y'$, which completes the proof.

LEMMA 12. *Let $\xi > 0$ be given and assume that ξ is not of the form $\eta + 1$ or $\eta + \omega$. Then there is a unique special ordinal ζ such that for some η, $\xi = \eta + \zeta$.*

Proof. Let ζ be the smallest nonzero ordinal for which there is an η such that $\xi = \eta + \zeta$. Our assumptions imply that $\zeta > \omega$. If ζ is not special, there exist $\zeta_1, \zeta_2 < \zeta$ such that $\zeta_1 + \zeta_2 \geqq \zeta$. The continuity of ordinal addition implies that there exist $\zeta_1, \zeta_2 < \zeta$ such that $\zeta_1 + \zeta_2 = \zeta$ (hence $\zeta_2 \neq 0$); but this is turn implies that $\xi = \eta + \zeta_1 + \zeta_2$ with $0 < \zeta_2 < \zeta$, which violates the defining condition of ζ.

To prove that ζ is unique assume that $\xi = \eta_1 + \zeta_1 = \eta_2 + \zeta_2$ and without loss of generality further assume $\eta_1 \leqq \eta_2$. Then there is a θ such that $\eta_1 + \theta = \eta_2$ which implies $\eta_1 + \zeta_1 = \eta_1 + \theta + \zeta_2$, i.e., $\zeta_1 = \theta + \zeta_2$. Now if ζ_1 is special we must have $\zeta_1 = \zeta_2$, which completes the proof.

7. Open problems. We do not have answers for the following questions:

1. Is $\mathscr{L}(\xi)$ for special ξ an upper semi-lattice, a lower semi-lattice or a lattice?

2. Does $\mathscr{L}(\xi)$ have a minimum for each special ξ? It is easy to show that $\mathscr{L}(\omega^2)$ has a minimum; we conjecture that $\mathscr{L}(\omega^3)$ does not.

3. If ξ and ζ are special and $\xi \neq \zeta$, is it possible that $\mathscr{L}(\xi)$ and $\mathscr{L}(\zeta)$ are similar? We conjecture that it is not.

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