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SECOND ORDER DISSIPATIVE OPERATORS

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A theory of dissipative operators has been developed and successfully applied by R. S. Phillips to the Cauchy problem for hyperbolic and parabolic systems of linear partial differential equations with time invariant coefficients. Our purpose is to show that the Cauchy problem for another system of equations can be brought within the scope of this theory. For this system of equations, we shall parallel the early work of Phillips on dissipative hyperbolic systems. This system of equations is general enough to include, as special cases, such equations as the one dimensional Schrödinger equation and the fourth order equation describing the damped vibrations of a rod.

Several of the results necessary to accomplish this task provide generalizations of the work of A. R. Sims on secondary conditions for nonselfadjoint second order ordinary differential operators.

The system of linear partial differential equations which we consider is of the following form

$$y_t = (Ay_x)_x + (By)_x + Cy \tag{1.1}$$

where $t \geqq 0$ and $x \in J = (a, b)$, an open subinterval (possibly improper) of the real line. The coefficients are assumed to be time invariant, that is to say, they depend only on the variable x. Several other conditions are imposed on the coefficients. Specifically, for each $x \in J$, $A(x)$ is nonsingular and skew-hermitian, $B(x)$ is hermitian and

$$D(x) \equiv [B_x + C + C^*](x) \tag{1.2}$$

is nonpositive under the inner product $(y, z) = \sum y^i \bar{z}^i$ in E^m (complex Euclidean m-space). This last condition is referred to as a dissipative condition and in associated physical models reflects a supposition of no energy sources in the interior of the system.

Besides these conditions the elements of A and B are required to be absolutely continuous on compact subintervals of J, and their derivatives and the elements of C are required to be square integrable on such subintervals. Attention is called to the fact that no conditions are imposed on the coefficients at the ends of the interval J.

Before proceeding it will be convenient to announce some conventions with regard to notation. Let $F(x) = I - D(x)$. The symbol H_1 will be used to denote the Hilbert Space $L^2(a, b, F)$ with weight F. Since $F(x) \geqq I$ we may introduce the Hilbert space $H_2 = L^2(a, b, F^{-1})$.

Finally, $H_0 = L^2(a, b, I)$. Since $F^{-1}(x) \leqq I \leqq F(x)$, it follows that $H_1 \subset H_0 \subset H_2$ and if $y \in H_1$ and $z \in H_2$ then $|\langle y, z\rangle_0| \leqq \|y\|_1 \|z\|_2$. Frequently it will be necessary to form a vector in E^{2m} from two vectors in E^m. For instance, if y_1 and y_2 are vectors in E^m, then the vector $\eta = [y_1, y_2]$, whose first m components are those of y_1 and whose second m components are those of y_2, is a vector in E^{2m}.

Suppose now that y is a sufficiently smooth solution of equation (1.1) and let L^0 denote the ordinary differential operator on the right hand side of equation (1.1). A purely formal integration by parts leads to the equations

$$\begin{aligned}\frac{d}{dt}\langle y, y\rangle_0 &= \langle L^0 y, y\rangle_0 + \langle y, L^0 y\rangle_0 \\ &= -(\mathfrak{A}\eta, \eta)^a + (\mathfrak{A}\eta, \eta)^b + \langle Dy, y\rangle_0\end{aligned} \tag{1.3}$$

where

$$\mathfrak{A}(x) = \begin{bmatrix} B(x) & A(x) \\ A^*(x) & \theta \end{bmatrix}$$

and $\eta = [y, y_x]$. It will be shown by example later that it is reasonable to require that $d/dt\langle y, y\rangle_0 \leqq 0$ for all $t \geqq 0$. Keeping in mind the dissipative condition (1.2), the right hand side of the second equation in (1.3) will be nonpositive provided the solution satisfies conditions at the boundary of the form

$$-(\mathfrak{A}\eta, \eta)^a + (\mathfrak{A}\eta, \eta)^b \leqq 0 , \qquad t \geqq 0 . \tag{1.4}$$

The principal result of this paper is that corresponding to each member of a class of boundary conditions which possess property (1.4), an operator L may be defined which is of the form L^0 and which is the infinitesimal generator of a strongly continuous semi-group $[S(t); t \geqq 0]$ of contraction operators; that is, a one parameter family of linear bounded operators on H_0 to itself such that $S(t_1 + t_2) = S(t_1)S(t_2)$, $\|S(t)\|_0 \leqq 1$ and $\lim S(t)f = f$ as $t \to 0$ for each $f \in H_0$. In addition, $d/dt\, S(t)f = LS(t)f$, $f \in D(L)$. The initial values are assumed in the mean of order two and the derivative is taken in the strong sense. This is the sense in which equation (1.1) is to be satisfied.

Notice that under these circumstances the operator L will satisfy the condition

$$\langle Ly, y\rangle + \langle y, Ly\rangle \leqq 0$$

for each $y \in D(L)$. Operators satisfying a one-sided condition of this type are called dissipative and have been treated from an operator-theoretic point of view by R. S. Phillips [4, 5].

In the present development, the principal tool employed is the Hille-Yosida theorem [2, Theorem 12.3.1 and corollary]. Thus, in Section 2, a fixed $\lambda > 0$ is chosen and a number of linear bounded operators $R(\lambda)$ are constructed. These operators are defined on H_0 and take values in H_1. In addition, for each $f \in H_0$, $(\lambda I - L^0)R(\lambda)f = f$. Selecting a particular $R(\lambda)$, the operator L is introduced whose domain $D(L)$ is the range of $R(\lambda)$ and whose values are those of L^0 restricted to $D(L)$. It is then shown that $D(L)$ is dense in H_0, the resolvent $R(\lambda, L)$ of L exists at λ, and $R(\lambda, L) = R(\lambda)$. Moreover, $\lambda \| R(\lambda, L) \|_0 \leqq 1$.

The operators $R(\lambda)$ are constructed from solutions of

$$L^0 y = (Ay_x)_x + (By)_x + Cy = \lambda y \tag{1.5}$$

and the formal dual equation

$$M^0 z = -(Az_x)_x - (Bz)_x + (B_x + C^*)z = \lambda z \,. \tag{1.6}$$

These solutions depend on λ so that, ostensibly, the $D(L)$ depends on λ. In Section 3, $D(L)$ is shown to be independent of λ and at the same time its members are characterized entirely by their behavior at the ends of the interval J. In addition, L is found to satisfy the requirements of the Hille-Yosida theorem and is therefore an infinitesimal generator. It is also found that these operators constitute all infinitesimal generators within a certain class.

As previously mentioned, this paper parallels the theory presented by R. S. Phillips [3] for the hyperbolic case $A \equiv \theta$. By an appropriate interpretation, most of the theorems and their proofs in that case apply directly to the present case. The exceptions are those results which we have called Theorem 2.1 and Lemma 2.3 [3, Theorem 3.3 and Lemma 4.2]. The proof of Lemma 2.3 requires that the explicit form of the resolvents be known, and hence, their construction is given in some, but not complete, detail in Section 2. The principal content of Section 3 is a statement of the main results of the theory. The statement of the main results without proofs is possible because of the aforementioned circumstance.

Two equations occurring in mathematical physics may be considered as examples. The first example is the one dimensional Schrödinger equation

$$y_t = \frac{ih}{m} y_{xx} - ih^{-1} V(x) y$$

which is directly of the form (1.1). The second example, the fourth order equation describing the motion of a damped rod,

$$y_{tt} + y_{xxxx} + ky_t = 0 \,, \qquad k \geqq 0$$

is not directly of the form (1.1). However, setting $u^1 = y_t$ and $u^2 = y_{xx}$, the vector $\mu = [u^1, u^2]$ satisfies the system

$$\mu_t = A\mu_{xx} + C\mu$$

were

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -k & 0 \\ 0 & 0 \end{bmatrix}.$$

For this system we have

$$\langle \mu, \mu \rangle_0 = \int_a^b (y_t^2 + y_{xx}^2)dx .$$

Notice that this is essentially the total energy of the system. Thus the dissipative requirement is that the total energy of the system should be nonincreasing in time.

Finally, it should be noted that, from the point of view of ordinary differential operators, Theorem 2.1, and to some extent Theorem 2.2, provide generalizations of the results of A. R. Sims [6, Theorem 1 and Theorem 6] to systems of equations.

II. Construction of the resolvents.

Before constructing the operators $R(\lambda)$ it is necessary to be assured of sufficiently smooth solutions to equations (1.5) and (1.6). It is necessary to discuss only equation (1.5) since both equations are of the same form and corresponding coefficients possess the same degree of smoothness.

LEMMA 2.1. *Let $c \in J$ and let η^1 and η^2 be two vectors in E^m. There exists a unique vector-valued function y satisfying equation* (1.5) *almost everywhere on J with $y(c) = \eta^1$ and $y_x(c) = \eta^2$. In addition, on compact subintervals of J, y and Ay_x are absolutely continuous with square integrable derivatives. If $\eta_k = [\eta_k^1, \eta_k^2]$, $k = 1, \cdots, 2m$, are linearly independent vectors in E^{2m}, then the solutions y^k with $y^k(c) = \eta_k^1$ and $y_x^k(c) = \eta_k^2$ span the solution space of equation* (1.5).

The proof of this need not be given in detail. Suffice it to say that equation (1.5) can be reduced to an equivalent first order system for the function $u = [u^1, u^2]$, where $u^1 = y$ and $u^2 = Ay_x + By$. An appeal to standard existence theorems [1, p. 97, problem 1], guarantees the existence and uniqueness of a solution to the reduced system with the required smoothness. The remaining statements follow from uniqueness and linearity.

The following lemma, whose proof may be obtained as a simple application of Rouche's theorem, will be needed later.

LEMMA 2.2. *Let M be a matrix-valued function whose elements are continuous on J. If M is hermitian and nonsingular at each point, then the number of positive and negative eigenvalues of M is the same at each point of the interval.*

COROLLARY. *If P and Q are matrices of order m such that P is nonsingular and Q hermitian, then the matrix*

$$G = \begin{bmatrix} Q & P \\ P^* & \theta \end{bmatrix}$$

has m positive and m negative eigenvalues.

Proof. Consider the one parameter family of matrices obtained from G by replacing Q by xQ in the definition of G. This matrix-valued function satisfies the hypothesis of Lemma 2.2. $G(0)$ has m positive and m negative eigenvalues these being the square roots, both positive and negative, of the m positive eigenvalues of the matrix PP^*. Thus $G(1) = G$ has the same number of positive and negative eigenvalues as $G(0)$.

Turning now to the construction of the $R(\lambda)$, the basic theorem from which the construction follows is reminiscent, for the system (1.5) or (1.6), of the familiar Weyl limit point-limit circle classification for singular self-adjoint second order equations [1, pp. 225–231]. More closely related is the presentation by A. R. Sims [6, pp. 254–257]. The presentation followed here is to be found in [3, Theorem 3.3].

Let y and z be solutions, in the sense of Lemma 2.1, of equations (1.5) and (1.6) respectively, for some fixed $\lambda > 0$. The identities

$$d/dx(\mathfrak{A}\eta, \eta) = 2\lambda(y, y) - (Dy, y) \tag{2.1}$$

$$d/dx(\mathfrak{A}\zeta, \zeta) = -2\lambda(z, z) + (Dz, z) \tag{2.2}$$

$$d/dx(\mathfrak{A}\eta, \zeta) = 0 \tag{2.3}$$

where $\eta = [y, y_x]$ and $\zeta = [z, z_x]$, hold for almost all $x \in J$.

It follows from these identities that $(\mathfrak{A}\eta, \eta)$ is an increasing function, $(\mathfrak{A}\zeta, \zeta)$ is a decreasing function and $(\mathfrak{A}\eta, \zeta)$ is a constant function on J. Moreover, if $c \in J$ then $y \in L^2(c, b, F)$ $[z \in L^2(c, b, F)]$ if, and only if, $(\mathfrak{A}\eta, \eta)^b < +\infty$ $[(\mathfrak{A}\zeta, \zeta)^b > -\infty]$. (REMARK. An expression such as $(\mathfrak{A}\eta, \eta)^u$ denotes the limit of $(\mathfrak{A}\eta, \eta)$ as x tend to u.) The proof of the last statement follows from the simple inequality

$$\min(1, 2\lambda)(Fy, y) \leqq 2\lambda(y, y) - (Dy, y) \\ \leqq \max(1, 2\lambda)(Fy, y) .$$

The dual statement is proved in the same way and a similar pair of statements hold at $x = a$.

The following theorem is basic.

THEOREM 2.1. *If $F_b[F_a]$ denotes the collection of all solutions of equation* (1.5) *such that $(\mathfrak{A}\eta, \eta)^b < +\infty$ $[(\mathfrak{A}\eta, \eta)^a > -\infty]$ then $F_b[F_a]$ is a linear subspace of the solution space of equation* (1.5) *of dimension $l_b[l_a] \geqq m$. If $C_b^1[C_a^1]$ denotes the subset of $F_b[F_a]$ such that $(\mathfrak{A}\eta, \eta)^b \leqq 0$ $[(\mathfrak{A}\eta, \eta)^a \geqq 0]$ then $C_b^1[C_a^1]$ contains at least one m-dimensional subspace. The same statements apply to the solution space of equation* (1.6) *where $G_b[G_a]$ denotes the set of solutions such that $(\mathfrak{A}\zeta, \zeta)^b > -\infty$ $[(\mathfrak{A}\zeta, \zeta)^a < +\infty]$.*

Proof. By the remarks preceding the statement of the theorem $F_b \subset L^2(c, b, F)$. In fact, it is a linear subspace since equation (1.5) is linear. To complete the proof we need only show that C_b^1 contains at least one m-dimensional subspace. Let y^i, $i = 1, \cdots, 2m$, form a basis for the solution space of equation (1.5) and let $\eta^i = [y^i, y_x^i]$. If y is an arbitrary solution of equation (1.5) and $\eta = [y, y_x]$, then $(\mathfrak{A}\eta, \eta)^x = (Y^*\mathfrak{A}Y\alpha, \alpha)^x$ where Y is the matrix whose i^{th} column is η^i and $\alpha \in E^{2m}$. For some $c \in J$ we can suppose that $Y(c) = I$. Now $Y^*\mathfrak{A}Y$ is hermitian nonsingular and continuous on J, and by the corollary to Lemma 2.2 has m positive and m negative eigenvalues at c, so that by Lemma 2.2 it has m positive and m negative eigenvalues for each $x \in J$. Thus, the set

$$C_x = [y; (\mathfrak{A}\eta, \eta)^x \leqq 0]$$

contains at least one m-dimensional subspace of the solution space. Since $(\mathfrak{A}\eta, \eta)^x$ is increasing, $C_s \supset C_t$ if $s \leqq t$. It follows that C_b^1 is the intersection of all C_x for $x < b$ and must therefore contain at least one m-dimensional subspace. The same proof is valid at the left end of the interval J.

The proof of this theorem for the dual equation is the same. The dimension of $G_b[G_a]$ will be denoted by $m_b[m_a]$ and the subset of solutions of equation (1.6) such that $(\mathfrak{A}\zeta, \zeta)^b \geqq 0$ $[(\mathfrak{A}\zeta, \zeta)^a \leqq 0]$ by $C_b^2[C_a^2]$.

Let $S_1(\lambda)$ and $S_2(\lambda)$ denote, respectively, the Cartesian products of the solution spaces of equation (1.5) and (1.6) with themselves. Clearly, $\dim S_i(\lambda) = 4m$. Suppose $N_{ab} \subset S_1(\lambda)$ such that $\dim N_{ab} = 2m$ and for each $\{y_a, y_b\} \in N_{ab}$

$$-(\mathfrak{A}\eta_a, \eta_a)^a + (\mathfrak{A}\eta_b, \eta_b)^b \leqq 0 \,, \tag{2.4}$$

where, for instance, $\eta_a = [y_a, y_{ax}]$. Theorem 2.1 guarantees the existence of such subsets. Let $P_{ab} \subset S_2(\lambda)$ such that whenever $\{z_a, z_b\} \in P_{ab}$

$$-(\mathfrak{A}\eta_a, \zeta_a)^a + (\mathfrak{A}\eta_b, \zeta_b)^b = 0 \tag{2.5}$$

for all $\{y_a, y_b\} \in N_{ab}$. P_{ab} is called the $\mathfrak{A}_{ab}$-orthogonal complement of N_{ab}. It follows by an argument similar to one employed in [3, Theorem 3.4] and an appeal to Lemma 2 and its corollary that $P_{ab} \subset G_a \times G_b$, $\dim P_{ab} = 2m$ and

$$-(\mathfrak{A}\zeta_a, \zeta_a)^a + (\mathfrak{A}\zeta_b, \zeta_b)^b \geqq 0 \tag{2.6}$$

for all $\{z_a, z_b\} \in P_{ab}$. Moreover, N_{ab} is $\mathfrak{A}_{ab}$-orthogonal to P_{ab}. Dually, of course, we can start with P_{ab} and obtain an N_{ab} with the appropriate properties.

The procedure now is to construct solutions to the equation $(\lambda I - L^0)y = f$ where $f \in H_0$ and vanishes outside some compact subinterval of J and L^0 has for its domain those functions which possess the basic smoothness of the solutions to equation (1.5) guaranteed by Lemma 2.1. Choose arbitrary bases $\{y_a^i, y_b^i\}$ and $\{z_a^i, z_b^i\}$ for N_{ab} and P_{ab}, respectively, and define the $4m \times 4m$ matrices

$$Y(x) = \begin{bmatrix} \eta_a^1, \cdots, \eta_a^{2m}, \eta_b^1, \cdots, \eta_b^{2m} \\ \eta_b^1, \cdots, \eta_b^{2m}, \eta_b^1, \cdots, \eta_a^{2m} \end{bmatrix} \tag{2.7}$$

$$Z(x) = \begin{bmatrix} \zeta_b^1, \cdots, \zeta_b^{2m}, -\zeta_a^1, \cdots, -\zeta_a^{2m} \\ \zeta_a^1, \cdots, \zeta_a^{2m}, -\zeta_b^1, \cdots, -\zeta_b^{2m} \end{bmatrix}. \tag{2.8}$$

Reasoning similar to that used in establishing the duality of N_{ab} and P_{ab} shows that $Y(x)$ and $Z(x)$ are nonsingular for each $x \in J$. Defining $Q = Z^*(x)\mathfrak{A}(x, x)Y(x)$ with

$$\mathfrak{A}(x, x) = \begin{bmatrix} -A(x) & \theta \\ \theta & A(x) \end{bmatrix},$$

then Q may be partitioned into four $2m \times 2m$ submatrices Q_{ij}, $i, j = 1, 2$, such that $Q_{ij} = \theta$; $i \neq j$ and Q_{ii}, $i = 1, 2$, are independent of x. Letting $V(x) = Z(x)[Q^*]^{-1}$, it is clear that $V(x)$ is nonsingular and of the same form as $Z(x)$. Now call the solution pairs from which $V(x)$ is constituted $\{z_a^i, z_b^i\}$ and use these as a basis for P_{ab}. For this choice of basis

$$I = V^*(x)\mathfrak{A}(x, x)Y(x) = \mathfrak{A}(x, x)Y(x)V^*(x)\,. \tag{2.9}$$

Let

$$(2.10)\quad \begin{aligned} y_1(x) = R_1(\lambda)f = &-Y_b(x)\int_a^x [Z_a^*(s) + Z_b^*(s)]f(s)ds \\ &- Y_a(x)\int_x^b [Z_a^*(s) + Z_b^*(s)]f(s)ds \\ y_2(x) = R_2(\lambda)f = &-[Y_a(x) + Y_b(x)]\int_a^x Z_a^*(s)f(s)ds \\ &- [Y_a(x) + Y_b(x)]\int_x^b Z_b^*(s)f(s)ds . \end{aligned}$$

Here, for instance, $Y_a(x)$ is the $m \times 2m$ matrix whose columns are the $2m$ solutions y_a^i chosen in the basis for N_{ab}. Provided the relation (2.9) is kept in mind, it is easily verified that $R_i(\lambda)f$ solves $(\lambda I - L^0)y = f$.

The previous construction was carried out without any reference to special properties possessed by N_{ab}. If $N_{ab} = N_a \times N_b$ where N_a and N_b are m-dimensional subspaces of C_a^1 and C_b^1, respectively, then it is possible to form m-dimensional subspaces $P_a \subset C_a^2$ and $P_b \subset C_b^2$ such that $P_{ab} = P_a \times P_b$ is the $\mathfrak{A}_{ab}$-orthogonal complement of N_{ab} in $S_2(\lambda)$. In this case it is possible to choose bases for N_a, P_a, N_b, and P_b, and then adjust the bases for N_b and P_b so that (2.9) is satisfied. In this case $R_1(\lambda)$ and $R_2(\lambda)$ are both of the same form:

$$(2.11)\quad \begin{aligned} y(x) = R(\lambda)f = &-Y_b(x)\int_a^x Z_a^*(s)f(s)ds \\ &- Y_a(x)\int_x^b Z_b^*(s)f(s)ds , \end{aligned}$$

where, for example, $Y_a(x)$ is the $m \times m$ matrix whose columns are the solutions to equation (1.5) chosen as a basis for N_a. Again, it is easily verified that $y = R(\lambda)f$ satisfies $(\lambda I - L^0)y = f$ for almost all $x \in J$. It should be observed that the same construction can be performed for the dual equation $(\lambda I - M^0)z = f$.

From this point on the development may proceed exactly as the theory presented in [3] with the exception of one lemma [3, Lemma 4.2] which requires an extended argument. Let L_∞ and M_∞ be formally given by L^0 and M^0. Choose $D(L_\infty)$ and $D(M_\infty)$ such that y and Ay_x are absolutely continuous on compact subintervals of J.

LEMMA 2.3. *Let $\{y_n\} \subset D(L_\infty)$ and suppose there is a $y_0 \in D(L_\infty)$ such that $\{y_n, L_\infty y_n\} \to \{y_0, L_\infty y_0\}$ in $H_0 \times H_0$ [or $H_1 \times H_2$]. Let $\beta \in C^2(a, b)$ with $0 \leqq \beta(x) \leqq 1$ such that $\beta(x) \equiv 0$ for $a < x \leqq a'$ and $\beta(x) \equiv 1$ for $b' \leqq x < b$. Setting $u_n = \beta y_n$, then $\{u_n, L_\infty u_n\} \to \{u_0, L_\infty u_0\}$ in $H_0 \times H_0$ [or $H_1 \times H_2$].*

Proof. Since $Au_{nx} = \beta Ay_{nx} + \beta_x Ay_n$, it is clear that $u_n \in D(L_\infty)$ for all n. Now $\| u_n - u_0 \|_{0,1} \leqq \| y_n - y_0 \|_{0,1}$.

REMARK. The subscripts 0, 1, 2 refer to the norms in H_0, H_1 or H_2 for which an inequality holds with the proviso that only one distinct subscript may be used in any inequality unless it occurs as a single subscript.

The last inequality means that $u_n \to u_0$ in H_0 [or H_1]. It is easily verified that

$$L_\infty u_n = \beta L_\infty y_n + (\beta_{xx} A + \beta_x A_x + \beta_x B) y_n + 2\beta_x A y_{nx}$$

so that

$$\begin{aligned} \| L_\infty u_n - L_\infty u_0 \|_{0,2} \leqq & \| L_\infty y_n - L_\infty y_0 \|_{0,2} \\ & + 2 \| \beta_x A(y_{nx} - y_{0x}) \|_{0,2} \\ & + \| (\beta_{xx} A + \beta_x A_x + \beta_x B) y_n \|_{0,2} . \end{aligned}$$

By hypothesis, the first term on the right of this inequality tends to zero as n tends to infinity. It remains to show that this is true for the second and third terms. To this end, suppose $f_n = \lambda y_n - L_\infty y_n$. Now $f_n \in H_0$ [or H_2], since y_n and $L_\infty y_n$ belong to H_0 [or $L_\infty y_n \in H_2$ and $y_n \in H_1 \subset H_2$], and $f_n \to f_0$ in H_0 [or H_2]. It is possible to represent y_n as follows:

$$y_n = R(\lambda) f_n + \sum_{i=1}^{r} \gamma^i(y_n) y^i(x) . \tag{2.12}$$

The first expression on the right represents a particular solution of the equation $\lambda y_n - L_\infty y_n = f_n$ of type (2.11). The integrals appearing in the definition of $R(\lambda)$ are well defined in either of the two cases under consideration since, by construction, the columns of Z_a belong to $L^2(a, c, F) \subset L^2(a, c, I)$; and the columns of Z_b belong to $L^2(c, b, F) \subset L^2(c, b, I)$. The last sum merely represents the most general solution of $(\lambda I - L^0) y = 0$ which lies in H_0 [or H_1]. Now

$$\| V(y_n - y_0) \|_{0,2} \leqq \| V(y_n - y_0) \|_0'$$

where $V \equiv \beta_{xx} A + \beta_x A_x + \beta_x B$ and the prime indicates that the interval of integration implied by the norm sign may be taken as $[a', b']$ due to the nature of β. Since the elements of $V = [v_{ij}]$ are square integrable on compact subintervals of J, we obtain the inequality

$$\| V(y_n - y_0) \|_0' \leqq \mathrm{k} \operatorname*{Max}_j \left[\operatorname*{Sup}_x | y_n^j(x) - y_0^j(x) | \right] ,$$

where k is some positive constant depending on $\| v_{ij} \|_0'$. The representation (2.12) chosen for y_n shows that $y_n \to y_0$ uniformly in each component on compact subintervals of J. Thus, the right side of the above inequality may be made arbitrarily small for sufficiently large n. Turning to the second term, we may differentiate (2.12) with respect to x and obtain the inequality

$$\|\beta_x A(y_{nx} - y_{0x})\|_{0,2} \leqq k_0 \|f_n - f_0\|_{0,2} + k_1 \operatorname*{Max}_i |\gamma^i(y_n) - \gamma^i(y_0)|$$

where k_1 depends only on the numbers $\|Ay^i_x\|'_0$ and

$$k_0 = \operatorname*{Max}_j [\|z^j_b\|'_{0,1} \|Ay^j_{ax}\|'_0, \|z^j_a\|'_{0,1} \|Ay^j_{bx}\|'_0] .$$

Both terms on the right of the above inequality tend to zero as $n \to \infty$, completing the proof.

III. Statement of results. We introduce the operators L_1 and M_1 which are restrictions of L_∞ and M_∞ with domains

$$D(L_1) \text{ (or } D(M_1)) = [y; y \in H_1, L_\infty y \text{ (or } M_\infty y) \in H_0] .$$

It is then possible to prove [3, Theorem 4.1]:

THEOREM 2.2. *For fixed* $\lambda > 0$, *let* $P_{ab} \subset S_2(\lambda)$ *with* $\dim P_{ab} = 2m$ *and*

$$-(\mathfrak{A}\zeta_a, \zeta_a)^a + (\mathfrak{A}\zeta_b, \zeta_b)^b \geqq 0$$

for all $\{z_a, z_b\} \in P_{ab}$. *Let* L *be a restriction of* L_1 *whose domain is the* $\mathfrak{A}_{ab}$*-orthogonal complement of* P_{ab} *in* $D(L_1)$. *Then* L *has a dense domain and for each* $y \in D(L)$,

$$-(\mathfrak{A}\eta, \eta)^a + (\mathfrak{A}\eta, \eta)^b \leqq 0 .$$

The resolvent of L *at* λ *exists and is given by either of the expressions* $R_1(\lambda)$ *or* $R_2(\lambda)$. *The solution pairs* $\{z^i_a, z^i_b\}$ *and* $\{y^i_a, y^i_b\}$ *are bases chosen for* P_{ab} *and* N_{ab}, *adjusted in such a way that* (2.9) *is satisfied, and* N_{ab} *is the* $\mathfrak{A}_{ab}$*-orthogonal complement of* P_{ab} *in* $S_1(\lambda)$. *Moreover, if* $R(\lambda, L)$ *represents this resolvent, then* $\lambda \|R(\lambda, L)\| \leqq 1$.

The principle result of the theory may now be stated. Introduce the sets

$$D(L_a) = [y; y \in D(L_1), (\mathfrak{A}\eta, \zeta)^a = 0 \text{ for all } z \in D(M_1)]$$

and

$$D(M_a) = [z; z \in D(M_1), (\mathfrak{A}\zeta, \eta)^a = 0 \text{ for all } y \in D(L_1)]$$

with similar definitions for the $D(L_b)$ and $D(M_b)$.

It can be shown that the quotient spaces

$$\mathfrak{L}_a = D(L_1)/D(L_a) , \qquad \mathfrak{L}_b = D(L_1)/D(L_b)$$
$$\mathfrak{M}_a = D(M_1)/D(M_a) , \qquad \mathfrak{M}_b = D(M_1)/D(M_b)$$

are finite dimensional and

$$\dim \mathfrak{L}_a = \dim \mathfrak{M}_a = d_a = l_a + m_a - 2m ,$$
$$\dim \mathfrak{L}_b = \dim \mathfrak{M}_b = d_b = l_b + m_b - 2m .$$

Introducing the $d_a + d_b = d$ dimensional product spaces

$$\mathfrak{L} = \mathfrak{L}_a \times \mathfrak{L}_b$$

and

$$\mathfrak{M} = \mathfrak{M}_a \times \mathfrak{M}_b$$

it is possible to construct a hermitian nonsingular operator Φ on $\mathfrak{L}$ with $l_a + l_b - 2m$ positive and $m_a + m_b - 2m$ negative eigenvalues. In addition, there is a connective inner product $\{Y, Z\}$, $Y \in \mathfrak{L}$ and $Z \in \mathfrak{M}$, between these two spaces. The main result is as follows: Let $\mathfrak{N}$ be a $m_a + m_b - 2m$ dimensional subspace of $\mathfrak{L}$ such that $(\Phi Y, Y) \leqq 0$ for all $Y \in \mathfrak{N}$. Let

$$\mathfrak{P} = [Z: Z \in \mathfrak{M}, \{Y, Z\} = 0 \text{ for all } Y \in \mathfrak{N}] ;$$

then $\mathfrak{P}$ is $l_a + l_b - 2m$ dimensional and if L is the restriction of L_1 with

$$D(L) = [y; y \in D(L_1), \{y, y\} \rightarrow \mathfrak{N}] ,$$

and M the restriction of M_1 with

$$D(M) = [z; z \in D(M_1); \{z, z\} \rightarrow \mathfrak{P}]$$

then L and M generate contraction semi-groups and $L^* = M$ and $M^* = L$. Moreover, these constitute all of the dissipative restrictions of the operators L_1 and M_1 which generate contraction semi-groups and are at the same time extensions of the restrictions L_0 and M_0 with domins $D(L_0) = D(L_a) \cap D(L_b)$ and $D(M_0) = D(M_a) \cap D(M_b)$. A word of explanation is due here. The symbol $\{y, y\} \rightarrow \mathfrak{N}$ means that y behaves like a function in one of the cosets of $\mathfrak{L}_a$ near $x = a$ and like a function in one of the cosets of $\mathfrak{L}_b$ near $x = b$; this pair of cosets belonging, of course, to $\mathfrak{N}$. For $\lambda > 0$ fixed, the set P_{ab} of solution pairs $\{z_a, z_b\}$ in $S_2(\lambda)$ such that $\{z_a, z_b\} \rightarrow \mathfrak{P}$ is $2m$-dimensional and is $\mathfrak{A}_{ab}$-orthogonal to the set N_{ab} of solution pairs $\{y_a, y_b\}$ in $S_1(\lambda)$ such that $\{y_a, y_b\} \rightarrow \mathfrak{N}$. N_{ab} is also $2m$-dimensional and is a negative subspace of $\mathfrak{A}_{ab}$. Thus, P_{ab} and N_{ab} define a restriction of L_1 by the process described in Theorem 2.2. The content of the principle result is that this restriction is an infinitesimal generator.

We include here another result of some interest. Let $L_{00} \subset L_1$ with $D(L_{00})$ taken as all functions in $D(L_1)$ which vanish near the ends of the interval J. Now L_1 is closed [3, Lemma 5.2] so that the closure of L_{00}, which we denote by L_0', is again a restriction of L_1. Clearly, $D(L_{00}) \subset D(L_a) \cap D(L_b) = D(L_0)$. It follows as a corollary to Lemma 2.3 that $D(L_0') \subset D(L_0)$. Under certain conditions $L_0' = L_0$.

LEMMA 3.1. *Suppose the largest subspace of the solution space of equation* (1.6) *which lies in* $L^2(c, b; I)$ *coincides with* G_b. *Then, for each* $y \in D(L_b)$, *there exists a sequence of* $\{y_n\} \subset D(L_1)$ *such that* $\{y_n, L_1 y_n\} \to \{y, L_1 y\}$ *in* $H_0 \times H_0$ *and each* y_n *vanishes near* $x = b$. *A similar statement holds at the* a *end with* G_a *replacing* G_b.

Proof. We may assume without loss of generality that y vanishes near $x = a$. To see this, observe that $u = \beta y$ and $w = (1 - \beta)y$, where β is defined as in Lemma 2.3, vanish near a and b, respectively. If u can be approximated by a sequence of the required variety, then the sequence $\{y_n = u_n + w\}$ will be a sequence of the required kind for y.

Let L be a restriction of L_1 whose resolvent is given by an expression of the form (2.11). Recall that in this case $N_{ab} = N_a \times N_b$ and $P_{ab} = P_a \times P_b$ and that the bases for P_b and N_b are adjusted according to what bases are chosen for N_a and P_a in such a way that (2.9) is satisfied. Since P_a and P_b together span the solution space of equation (1.6), whatever basis is chosen for P_a, $m_b - m$ members of this basis together with the basis for P_b span G_b. Since y vanishes near $x = a$ and belongs to $D(L_b)$ it is contained in the domain of every restriction of L_1 of the kind described in Theorem 2.2. In particular then, setting $f = \lambda y - Ly$ we have $y = R(\lambda, L)f$. In general, if $y_0 \in D(L_1)$ and $z_a \in P_a$

$$\int_a^x (\lambda y_0 - L_1 y_0, z_a) ds = -(\mathfrak{A}\eta_0, \zeta_a)^x + (\mathfrak{A}\eta_0, \zeta_a)^a .$$

When $y_0 = y$ this reduces to

$$\int_a^x (f, z_a) ds = -(\mathfrak{A}\eta, \zeta_a)^x .$$

In particular, when z_a is one of the $m_b - m$ columns of Z_a (the columns of Z_a constitute the basis chosen for P_a) described above, $z_a \in D(M_1)$ and

$$\int_a^b (f, z_a) ds = 0$$

since $y \in D(L_b)$. The hypothesis imposed on G_b implies that the largest subspace of P_a which lies in H_0 is the subspace spanned by the $m_b - m$

columns of Z_a which together with the basis for P_b span G_b. By a theorem due essentially to Rellich [3, Lemma 5.1] the set S of all bounded, measurable, vector-valued functions which vanish near a and b and belong to the orthogonal complement J of P_a in H_0 is contained in and dense in the orthogonal complement of the $m_b - m$ dimensional subspace of P_a which lies in H_0. Thus, since $f \in J$, there is a sequence $\{f_n\} \subset H_0$ such that $f_n \to f$ in H_0 and each f_n vanishes near a and b. Let $y_n = R(\lambda, L)f_n$, then $y_n \to y = R(\lambda, L)f$ in H_0 and since for x sufficiently near b

$$y_n(x) = -Y_b(x)\int_a^b Z_a^*(s)f(s)ds$$

each y_n vanishes near b.

Finally, $Ly_n = LR(\lambda, L)f_n = \lambda y_n - f_n \to \lambda y - f = Ly$ in H_0. This completes the proof.

THEOREM 3.1. *If the largest subspaces of the solution space of equation* (1.6) *which lie in* $L^2(a, c; I)$ *and* $L^2(c, b; I)$ *coincide with* G_a *and* G_b, *respectively, then* $L_0' = L_0$.

Proof. Let $y \in D(L_0)$. According to Lemma 3.1, there exist sequences $\{y_a^n\}$ and $\{y_b^n\}$ in $D(L_1)$ which vanish near a and b, respectively, and $\{y_a^n, L_1 y_a^n\} \to \{y, L_1 y\}$ and $\{y_b^n, L_1 y_b^n\} \to \{y, L_1 y\}$ in $H_0 \times H_0$. Let

$$y^n = \alpha y_a^n + \beta y_b^n ,$$

where β is as before and $\alpha = 1 - \beta$. By Lemma 2.3 the sequence $\{y^n\}$ has the same convergence properties as the original sequences and vanishes near a and b. This proves the theorem.

COROLLARY. *If* $H_1 = H_0$, *then* $L_0' = L_0$.

The content of the above theorem is not empty. Examples constructed by Sims [6, pp. 256–257] provide cases in point for which G_b (or G_a) does not contain all solutions of equations of type (1.6) which lie in $L^2(c, b; I)$ or $L^2(a, c; I)$.

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