PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 3, 1966

ON DUAL SERIES RELATIONS INVOLVING LAGUERRE POLYNOMIALS

K. N. SRIVASTAVA

In this paper, we shall consider the problem determining the sequence $\{An\}$, such that

$$\sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1)\} L_n^{\alpha}(x) = f_1(x)\,, \qquad 0 \leqq x < y\,,$$

$$\sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1/2\} L_n^{\alpha}(x) = f_2(x)\,,\ y < x \leqq \infty, \alpha > -1/2\,,$$

where $L_n^{\alpha}(x)$ is a Laguerre polynomial, the functions $f_1(x)$ and $f_2(x)$ being prescribed. By expressing the sequence $\{An\}$ in terms of a sequence of integrals involving an unknown function $g(u)$ the problem is reduced to that of solving an Abel integral equation for $g(u)$.

In recent years, dual series relations involving Fourier-Bessel, Dini series, trigonometric series and series of Jacobi polynomials have been investigated by various workers [1, 2, 5 to 12]. Here we shall apply the method developed by Sneddon and Srivastav for obtaining a solution of the dual series relations involving Laguerre polynomials.

As pointed out by Sneddon and Srivastav [6], with a view to simplify the calculations, we split the problem posed by the pair of dual equations given above into two parts: Problem (a). Determine the constants $\{An\}$ satisfying the dual series relations

$$\sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1)\} L_n^{\alpha}(x) = f_1(x)\,, \qquad 0 \leqq x < y\,, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1/2)\} L_n^{\alpha}(x) = 0\,, \qquad y < x \leqq \infty,\ \alpha > -1/2\,. \tag{1.2}$$

Problem (b). Determine the constants $\{An\}$ satisfying the dual series relations

$$\sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1)\} L_n^{\alpha}(x) = 0\,, \qquad 0 \leqq x < y\,, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1/2)\} L_n^{\alpha}(x) = f_2(x)\,, \quad y < x \leqq \infty, \alpha > -1/2. \tag{1.4}$$

The solution of the general problem is obviously obtained merely by adding the solutions of problem (a) and (b). We suppose that functions $f_1(x)$ and $f_2(x)$ satisfy the following conditions:

(i) $F_1(x) = x^{\alpha} f_1(x)$ is finite and continuously differentiable for $0 \leqq x < y$,

(ii) $F_2(x) = \int_x^\infty e^{-x} f_2(x) dx$ is finite and continuously differentiable for $y < x \leqq \infty$.

As we shall presently see the classes of functions $f_1(x)$ and $f_2(x)$ for which the problem under discussion is solvable, must satisfy the above conditions,

2. In this section we list some results for ready reference. By combining the results [3, p. 292 (2), (3)], we have

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = (n+1)_\alpha \cdot \delta_{mn} , \tag{2.1}$$

where δ_{mn} is a Kronecker delta. From [4, p. 193 (27), (28)] we have

$$\frac{d}{dn}\{x^\alpha L_n^\alpha(x)\} = (n+\alpha)\, x^{\alpha-1} L_n^{\alpha-1}(x) , \tag{2.2}$$

$$\int_x^\infty e^{-y} L_n^\alpha(y) dy = e^{-x} L_n^{\alpha-1}(x) . \tag{2.3}$$

We shall also require the following results which are easily derived from the more general results given in [3, p. 293 (5), p. 405 (20)]. For $\alpha > -1/2$

$$\int_x^\infty (y-x)^{-1/2} e^{-y} L_n^\alpha(y) dy = \Gamma(1/2) e^{-x} L_n^{\alpha-1/2}(x) , \tag{2.4}$$

$$\int_0^x (x-y)^{-1/2} y^\alpha L_n^\alpha(y) dy = \frac{\Gamma(n+\alpha+1)\,\Gamma(1/2)}{\Gamma(n+\alpha+3/2)}\, x^{\alpha+1/2} L_n^{\alpha+1/2}(x) . \tag{2.5}$$

We also note that if $f(x)$ is continuously differentiable then Abel integral equation

$$f(x) = \int_0^x \frac{\phi(y)}{(x-y)^{1/2}} dy \tag{2.6}$$

has a continuous solution given by the equation

$$\phi(y) = \frac{1}{\Pi} \frac{d}{dy} \int_0^y \frac{f(x)}{(y-x)^{1/2}} dx . \tag{2.7}$$

Furthermore, if $f(x)$ is continuously differentiable then the integral equation

$$f(x) = \int_x^\infty \frac{\phi(y)}{(y-x)^{1/2}} dy \tag{2.8}$$

has a continuous solution

$$\phi(y) = -\frac{1}{\Pi}\frac{d}{dy}\int_y^\infty \frac{f(x)}{(x-y)^{1/2}}\,dx\,. \tag{2.9}$$

This can be easily established by simple methods given in [13, p. 229]. The analysis given here is purely formal and no attempt is made to justify the interchange of various limiting processes.

3. Solution of the problem (a). Let us suppose that for $0 \leqq x < y$

$$\sum_{n=0}^\infty \{An/\Gamma(n+\alpha+1/2)\}\, L_n^\alpha(x) = -e^x \frac{d}{dx}\int_x^y \frac{g_1(u)}{(u-x)^{1/2}}\,du\,. \tag{3.1}$$

Using the orthogonal property (2.1), it can be shown that

$$An = -\frac{\Gamma(n+\alpha+1/2)\,\Gamma(1/2)}{\Gamma(n+\alpha+1)}\int_0^y x^\alpha L_n^\alpha(x)\Big(\frac{d}{dx}\int_x^y \frac{g_1(u)}{(u-x)^{1/2}}\,du\Big)dx. \tag{3.2}$$

Since

$$-\frac{d}{dx}\int_x^y \frac{g_1(u)}{(u-x)^{1/2}}\,du = \frac{g_1(y)}{(y-x)^{1/2}} - \int_x^y \frac{\frac{d}{du}\{g_1(u)\}}{(u-x)^{1/2}}\,du \tag{3.3}$$

we obtain with the help of (2.5), the equation

$$An = \Gamma(n+1)\,\Gamma(1/2)\int_0^y g_1(u)u^{\alpha-1/2}L_n^{\alpha-1/2}(u)du\,, \qquad n = 0, 1, 2, \cdots \tag{3.4}$$

If in the equation (1.1), we substitute for the coefficients An from (3.4), on interchanging the order of summation and integration, we get

$$f_1(x) = \int_0^y g_1(u)u^{\alpha-1/2}K_1(u, x)du\,, \qquad 0 \leqq x < y\,, \tag{3.5}$$

where

$$K_1(u, x) = \sum_{n=0}^\infty \frac{\Gamma(n+1)\,\Gamma(1/2)}{\Gamma(n+\alpha+1)}\, L_n^{\alpha-1/2}(u)L_n^\alpha(x) \tag{3.6}$$

with the help of equations (2.1) and (2.4) it can be shown that

$$K_1(u, x) = e^u x^{-\alpha}(x-u)^{-1/2}\, H(x-u) \tag{3.7}$$

where $H(t)$ is Heaviside's unit function. (2.7) is easily proved. Let

$$K_1(u, x) = \sum_{n=0}^\infty a_n L_n^\alpha(x)$$

where the coefficients a_n are given by

$$\begin{aligned} a_n &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^\infty K_1(u, x) x^\alpha e^{-n} L_n^\alpha(x) \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} e^u \int_u^\infty e^{-x}(x-u)^{-1/2} L_n^\alpha(x) dx \\ &= \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+\alpha+1)} L_n^{\alpha-1/2}(u) . \end{aligned}$$

Thus the equation (3.5) is equivalent to

$$F_1(x) = x^\alpha f_1(x) = \int_0^x \frac{g_1(u) u^{\alpha-1/2} e^u}{(x-u)^{1/2}} du , \qquad 0 \leqq x < y . \tag{3.8}$$

This is Abel integral equation, since $F_1(x)$ is finite and continuously differentiable, its solution is given by

$$u^{\alpha-1/2} e^u g_1(u) = \frac{1}{\Pi} \frac{d}{du} \int_0^u \frac{x^\alpha f_1(x)}{(u-x)^{1/2}} dx . \tag{3.9}$$

The coefficients An may now be calculated with the help of the relations (3.4) and (3.9).

4. Solution of the problem (b). We start with the assumption that for $y < x \leqq \infty$

$$\sum_{n=0}^\infty \{An/\Gamma(n+\alpha+1)\} L_n^\alpha(x) = x^{-\alpha} \int_y^x \frac{g_2(u)}{(x-u)^{1/2}} du . \tag{4.1}$$

This is equivalent to assuming that

$$An = \Gamma(n+1)\Gamma(1/2) \int_y^\infty g_2(u) e^{-u} L_n^{\alpha-1/2}(u) du, \ n = 0, 1, 2, \cdots \tag{4.2}$$

If we multiply both sides of the equation (1.4) by $\exp(-x)$ and integrate with respect to x from x to ∞, $y < x \leqq \infty$, we obtain

$$F_2(x) = \int_x^\infty e^{-x} f_2(x) dx = \sum_{n=0}^\infty \{An/\Gamma(n+\alpha+1)\} e^{-x} L_n^{\alpha-1}(x) . \tag{4.3}$$

Substituting the values of the coefficients from (4.2) in the equation (4.3) we find on interchanging the order of summation and integration that

$$e^x F_2(x) = \int_y^\infty g_2(u) e^{-u} K_2(u, x) du, \ y < x \leqq \infty , \tag{4.4}$$

where

$$K_2(u, x) = \sum_{n=0}^\infty \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+\alpha+1/2)} L_n^{\alpha-1/2}(u) L_n^{\alpha-1}(x) . \tag{4.5}$$

From the relations (2.1) and (2.5) it easily follows that

$$K_2(u, x) - e^x u^{-\alpha+1/2}(u - x)^{-1/2} H(u - x) \, . \tag{4.6}$$

Consequently the equation (4.4) reduces to the integral equation

$$F_2(x) = \int_y^\infty \frac{g_2(u) u^{1/2-\alpha} e^{-u}}{(u - x)^{1/2}} du \, , \qquad y < x \leqq \infty \, . \tag{4.7}$$

Since $F_2(x)$ is finite and continuously differentiable, the solution of the above equation is given by

$$g_2(u) = -\frac{e^u u^{\alpha-1/2}}{\Pi} \frac{d}{du} \int_0^u \frac{F_2(x)}{(x - u)^{1/2}} dx \, . \tag{4.8}$$

The coefficients A_n are given by the relations (4.2) and (4.8).

REFERENCES

1. W. D. Collins, *On some dual series equations and their applications to electrostatic problem of spheroidal caps*, Proc. Camb. Phil. Soc. **57** (1961), 367-384.
2. J. C. Cooke and C. J. Tranter, *Dual Fourier-Bessel series*, Quart. J. Mech. **12** (1959), 379-385.
3. A. Erdelyi (Editor), *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, 1954.
4. ———, *Higher Transcendental Functions*, Vol 2 McGraw-Hill, 1953.
5. B. Noble, *Some dual series equations involving Jacobi polynomials*, Proc. Camb. Phil. Soc. **59** (1963), 363-372.
6. I. N. Sneddon and R. P. Srivastav, *Dual series relations*-I. *Dual relations involving Fourier-Bessel series*, Proc. Roy. Soc. Edin. A **66** (1964), 150-160.
7. R. P. Srivastav, *Dual series relations*-II. *Dual relations involving Dini series*, Proc. Roy. Soc. Edin. A **66** (1964), 161-172.
8. ———, *Dual series relations*-III. *Dual relations involving trigonometric series*, Proc. Roy. Soc. Edin. A **66** (1964) 173-184.
9. ———, *Dual series relations*-IV. *Dual relations involving series of Jacobi polynomials*, Proc. Roy. Soc. Edin. A **66** (1964), 185-191.
10. C. J. Tranter, *Dual trigonometric series*, Proc. Glasgow Math. Assoc. **4** (1959), 49-57.
11. ———, *A further note on dual trigonometric series*, Proc. Glasgow Math. Assoc. **4** (1960), 198-200.
12. ———, *An improved method for dual trigonometrical series*, Proc. Glasgow Math. Assoc. **6** (1964), 136-140.
13. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, 1920.

Received June 29, 1965.

M. A. COLLEGE OF TECHNOLOGY, BHOPAL
INDIA