PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 3, 1966

A NOTE ON TOPOLOGICAL TRANSFORMATION GROUPS WITH A FIXED END POINT

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Let (X, T, Π) be a topological transformation group, where X is a nontrivial Hausdorff continuum, and T is a topological group which leaves an endpoint e of X fixed. Wallace showed that if X is locally connected and T is cyclic, T has another fixed point. In a later paper, Wallace asked the following question: if X is a peano continuum and T is compact or abelian, does T have another fixed point?

In 1952, Wang showed that if X is arcwise connected and T is compact, T has another fixed point; Chu has recently extended this result by showing T has infinitely many fixed points. Gray has shown that in the abelian case, the answer to Wallace's question is "no" (in general). However, if T is a generative group, and if X is arcwise connected, T has another fixed point. In this paper we will generalize the last result. In fact, we show that if X is arcwise connected or locally connected, and T is a group of the form AH, where H is a connected subgroup, and A is an abelian group generated by a compact subset, and A lies in the center of T, then T has another fixed point. We will generalize several known theorems by studying ordered spaces similar to those introduced by Wallace in 1945; in particular, we will obtain a generalized solution of the compact group problem (Theorem 2).

2. In this section, X will denote a compact Hausdorff space consisting of more than two points on which a reflexive, transitive, and antisymmetric order $\leqq$ is defined; if $z \in X$, let

$$L(z) = \{x; z < x\}, M(z) = \{x; x \leqq z\}, N(z) = \{x; z \leqq x\} .$$

We assume that $\leqq$ satisfies the following conditions:

(a) The set $M(z)$ is closed.

(b) The set $L(z)$ is open, and $N(z)$ is closed.

(c) X has a least element e under $\leqq$.

(d) Each set $M(z)$ is a chain, i.e. $M(z)$ is simply ordered by $\leqq$.

(e) X is directed by $\leqq$ in the following sense: if $x, y \in X$ and $x \neq e, y \neq e$, then there exists $z \neq e$ such that $z \leqq x$ and $z \leqq y$.

Wallace, [6], has proved:

(f) Each nonvoid closed subset of X contains a maximal element under $\leqq$.

We show:

(g) If C is a closed nonvoid subset of X with $e \notin C$, we have

$z \in X, z \neq e$, for which $z \leqq c$ for every $c \in C$.

Proof. If for some $z \in C, z \leqq c$ for every $c \in C$, we are finished. Otherwise, if $x, y \in C$ and $x \neq y$, choose $z_{xy} \neq e$ satisfying (e): $z_{xy} \leqq x$ and $z_{xy} \leqq y$. We show that the collection $\{L(z_{xy}); x, y \in C, x \neq y\}$ is an open cover of C. If $x \in C$, we have $y \in C$ for which $x \nleqq y$; it follows that $z_{xy} < x$, and hence $x \in L(z_{xy})$. Since X is compact and C is closed in X, there is a finite subset $\{z_i, \cdots, z_n\}$ of the set $\{z_{xy}; x, y \in C, x \neq y\}$ for which $C \subset \cup \{L(z_j), 1 \leqq j \leqq n\}$; since $z_j \neq e$ for every j, by (e) we have $z \in X$ for which $z \neq e$ and $z \leqq z_j$ for $j = 1, \cdots, n$. z is the desired element of X.

By an order isomorphism: $X \rightarrow X$, we mean a homeomorphism which preserves $\leqq$. If (X, T, Π) is a transformation group, we will assume that for each $t \in T$ the t-transition of (X, T, Π) is an order isomorphism.

If $A \subset X$ and $B \subset X$, we write $A \leqq B$ $[A < B]$ if, given $a \in A$ and $b \in B$, we have $a \leqq b$ $[a < b]$.

LEMMA 2.1. *Let (X, T, Π) be a topological transformation group. If there is a closed nonempty T-invariant subset $A \subset X$ such that $e \notin A$, then T has a fixed point other than e.*

Proof. Let $A \neq \emptyset$ be *any* closed subset of X such that $e \notin A$. Define

$$M(A) = \cap \{M(a); a \in A\} .$$

$M(A)$ is a closed chain, and $M(A) = \{z; z \leqq A\}$. By (g), $M(A)$ does not consist of e alone. By (f), $M(A)$ contains a maximal element $\mathscr{M}(A)$. Since $M(A)$ is a chain, $\mathscr{M}(A)$ is the largest element of $M(A)$. If $t: X \rightarrow X$ is an order isomorphism, then $t\mathscr{M}(A) = \mathscr{M}(tA)$. Then if A is T-invariant, $\mathscr{M}(A)$ is fixed under T. It is clear that $\mathscr{M}(A) \neq e$, so that the proof is complete.

LEMMA 2.2. *Let (X, T, Π) be a transformation group. If there is a T-invariant chain $B \subset X$ which is not empty and does not consist of e alone, then T has a fixed point other then e.*

Proof. The collection of closed sets $\{N(b); b \in B\}$ has the finite intersection property since B is a chain. Hence the intersection, $N(B)$, of the $N(b)$ is not empty and is T-invariant since B is. Because $e \notin N(B)$, $N(B)$ satisfies the hypothesis of Lemma 2.1, and the proof is complete.

LEMMA 2.3. *Let $t_1, \cdots, t_n$ be commuting order isomorphisms: $X \to X$. Then the t_i have a fixed point other than e in common.*

Proof. Let z_0 be a maximal element of X. If $A = \{z_0, t_i^{-1}z_0; i = 1, \cdots, n\}$, then $e \notin A$. By (e) we have $z_1 \neq e, z_1 \leqq A$. For each i, $\{z_1, t_i z_1\} \subset M(z_0)$. We let T_i be the cyclic group generated by t_i and $T = T_1 T_2 \cdots T_n$. Then for each i, $T_i z_1$ is a chain.

(1) If $s \in T$ and $t \in T$ such that sz_1 and tz_1 both compare to z_1, then sz_1 and tz_1 compare.

For if $sz_1 \leqq z_1$ and $tz_1 \leqq z_1$, the result follows from (d). If $sz_1 \geqq z_1$ and $tz_1 \geqq z_1$, apply the last case to $s^{-1}z_1$ and $t^{-1}z_1$ and use the fact that T is abelian. The final case follows by transitivity of $\leqq$.

(2) Each element of Tz_1 compares to z_1.

Let $t_1^{K_1} \cdots t_n^{K_n} z_1 \in Tz_1$, where the K_i are integers. Then $t_1^{K_1} z_1$ compares to z_1. We proceed by induction. If $t_1^{K_1} \cdots t_j^{K_j} z_1$ compares to z_1, where $1 \leqq j \leqq n-1$, then since $t_{j+1}^{K_{j+1}} z_1$ compares to z_1 also, $t_1^{K_1} \cdots t_{j+1}^{K_{j+1}} z_1$ compares to z_1; the desired result follows.

From (1) and (2) it follows that Tz_1 is a chain. Now $e \notin Tz_1$ so that Lemma 2.2 applies. The proof is complete.

A group T is *generative* if T is abelian and is generated by a compact neighborhood of the identity of T.

THEOREM 1. *Let (X, T, Π) be a transformation group, where T acts as a generative group of order isomorphisms on X. Then T has a fixed point other than e.*

Proof. Since T is generative, it is known that T has the form $KZ^m R^n$ where Z and R denote the integers and reals, respectively, with the usual topology, and m and n are nonnegative integers. Thus T may be written in the form CA, where C is compact and A is a finitely generated abelian group. If x is a fixed point of X under A, with $x \neq e$, then $Tx = Cx$ is closed, T-invariant, and does not contain e. Hence Lemma 2.1 applies, and the proof is complete.

NOTE. Actually, in Theorem 1, we need only assume that the group T is abelian and is generated by a compact set. For if then C is a compact symmetric set which contains the identity of T and generates T, let x be a maximal element of X and let $z \leqq C^{-1}x$, where $e \neq z$. Then $Cz \subset M(x)$, hence Cz is a chain. Since T is abelian, we may argue as in the proof of Lemma 2.3 and show that $C^n z$ is a chain for each positive integer n. Thus the set $\cup \{C^n z; n = 1, 2, \cdots\}$ is a T-invariant chain not consisting of e alone, and T has a fixed point other than e. This proves

THEOREM 1′. *If (X, T, Π) is a transformation group, where T is abelian and is generated by a compact subset, and if T acts as a group of order isomorphisms on X, then T has a fixed point other than e.*

We now consider a strengthened form of axiom (e):

(e_s) X is strongly directed by $\leqq$ in the following sense: if $x, y \in X$ and $x \neq e$, $y \neq e$, then there is a $z \in X$ with $z \neq e$ for which $z < \{x, y\}$.

If X is a space which satisfies (a)—(e) but does not satisfy (e_s), then it is easy to see that there is an $x \in X$ with $x \neq e$ such that $tx = x$ for every order isomorphism $t: X \rightarrow X$. If X satisfies (e_s), then we have

(g_s) If C is a closed nonempty subset of X with $e \notin C$, there is a $z \in X$ with $z \neq e$ for which $z < C$.

THEOREM 2. *Let (X, T, Π) be a transformation group, where X has an order $\leqq$ which satisfies* (b)—(d) *and* (e_s), *and T acts as a compact group of order isomorphisms on X. Let $x \in X$ with $x \neq e$. Let*

$$M(Tx) = \{y; y \leqq Tx\} = \cap\{M(y); y \in Tx\} .$$

Then T leaves each point of $M(Tx)$ fixed. Furthermore $M(Tx)$ is an infinite set.

Proof. The set $M(Tx)$ is a T-invariant chain by axiom (d). Let $z \in M(Tx)$. Then Tz is a compact subchain of A, and since (f) holds for $\leqq$ without assuming (a), Tz contains a maximal element m. Since Tz is a chain, m is the largest element of Tz, hence is fixed under T. Thus the orbit of z contains a fixed point under T, so that T leaves z fixed. Now (g_s) also holds for $\leqq$, so that the set $M(Tx)$ is infinite, and the proof is complete.

In what follows, let X be a nontrivial Hausdorff continuum. If $e \in X$, then e is an *end point* of X if, given an open set U with $e \in U$, there exists $y \in U$ such that $y \neq e$ and

$$X - y = V \cup W, e \in V \subset U, (\bar{V} \cap W) \cup (V \cap \bar{W}) = \varnothing .$$

If $x \in X$, let $E(e, x) = \{e, x\} \cup \{z; z$ separates e and x in $X\}$ Given two points $x, y \in X$, define $x \leqq y$ if and only if $x \in E(e, y)$. Then $\leqq$ satisfies (b)—(e) and (e_s). Furthermore, a homeomorphism: $X \rightarrow X$ which leaves e fixed is an order isomorphism. If in addition X is locally connected, $\leqq$ satisfies (a), and the results of this section apply to such a space. Hence if (X, T, Π) is a transformation group, where X is locally connected and $Te = e$, and if there is a closed nonempty T-invariant subset $A \subset X$ such that $e \notin A$, then T has a fixed point other than e.

From Theorem 2 we obtain

COROLLARY 2.1. *Let* (X, T, Π) *be a transformation group, where* X *is a nontrivial Hausdorff continuum and* T *is a compact group which leaves an end point* e *of* X *fixed. If* $x \in X$ *and* $x \neq e$, *let*

$$E(e, Tx) = \{y; y \text{ separates } e \text{ and } Tx \text{ in } X\}.$$

Then T *leaves each point of* $E(e, Tx)$ *fixed.*

We will call a metric continuum a *dendrite* if each two distinct points of the continuum is separated by a third point of the continuum. It is known [10] that each point of a dendrite is either a cut point or an end point.

COROLLARY 2.2. *Let* X *be a dendrite with a finite number,* N, *of end points. Then the only compact groups which can act effectively on* X *are the subgroups of* S_N, *the permutation group on* N *symbols.*

Proof. Let E be the set of end points of X and T be a compact group which acts effectively on X. Then for each $t \in T$, the restriction, $t \mid E$, of t to E is in S_N, and the mapping $t \to t \mid E$ is a homomorphic mapping of T onto a subgroup of S_N.

Let P be the set of all elements of T which leave each point of E fixed. P is a closed subgroup of T, and since $X = \cup\{E(x, y); x, y \in E\}$, it follows from Corollary 2.1 that P leaves each element of X fixed, and because T is effective, P is the identity alone. Thus if $t \mid E = s \mid E$, then $s^{-1}t \in P$, hence $s = t$. Thus the mapping $t \to t \mid E$, all $t \in T$, is an isomorphism.

3. In this section, X will denote a nontrivial locally connected Hausdorff continuum, and T is a group which leaves an end point e of X fixed. We remark that all the results of this section hold when X is arcwise connected but not necessarily locally connected (we replace the remark immediately preceding Corollary 2.1 by Wang's Lemma, [9]).

LEMMA 3.1. *Let* (X, T, Π) *be a transformation group, where* T *is connected. Then* T *has a fixed point other than* e.

Proof. Since X contains at least two noncut points, [8], let $x \neq e$ be another noncut point, and

$$X - z = U \cup V, e \in U, x \in V, (\bar{U} \cap V) \cap (U \cup \bar{V}) = \varnothing ;$$

now Tx contains only noncut points, and so $z \notin Tx$ since z is a cut point. Since Tx is connected, it follows that $Tx \subset V$. Because

$V \cup \{z\}$ is closed, we have $\overline{Tx} \subset V \cup \{z\}$. We have found a nonempty closed T-invariant set not containing e, so that the remark preceding Corollary 2.1 applies.

THEOREM 3. *Let (X, T, Π) satisfy the hypothesis of Lemma 3.1. Either e is the only noncut point in one of its neighborhoods, or else T has infinitely many fixed points.*

Proof. We use the order and notation of § 2. Let x_0 be a noncut point of X with $x_0 \neq e$. From the proof of Lemma 2.1, we see that $\mathscr{M}(\overline{Tx_0})$ is a fixed point different from e. Let $A_1 = \mathscr{M}(\overline{Tx_0}) \cup \overline{Tx_0}$. Since e does not belong to the closed set A_1, we may find $z \in X$ for which

$$X - z = U \cup V, e \in U, A_1 \subset V, (\bar{U} \cap V) \cup (U \cap \bar{V}) = \varnothing .$$

Suppose every neighborhood of e contains a cut point other than e, and let $x_1 \in U$ be such a point. Since z is a cut point, $z \notin Tx_1$ so that $\overline{Tx_1} \subset U \cup \{z\}$. Furthermore, a separation argument shows that if $x \in \overline{Tx_1}$, then $M(x) \subset U \cup \{z\}$ so that $\mathscr{M}(\overline{Tx_1}) \subset U \cup \{z\}$. Since $\mathscr{M}(Tx_0) \in V$, we have $\mathscr{M}(Tx_1) \neq \mathscr{M}(Tx_0)$. Set

$$A_2 = \overline{Tx_0} \cup \overline{Tx_1} \cup \mathscr{M}(\overline{Tx_0}) \cup \mathscr{M}(\overline{Tx_1}) ,$$

and complete the proof by induction.

THEOREM 4. *Let (X, T, Π) be a transformation group, with $T \approx AH$, where A is an abelian group which is generated by a compact subset and lies in the center of T, and H is a connected subgroup. Then T has a fixed point other than e.*

Proof. Let X be a fixed point under A, where $x \neq e$. Then $\overline{Tx} = \overline{Hx}$ is connected. If $e \notin \overline{Hx}$, we are finished (in view of previous results). If $e \in \overline{Hx}$, since $\overline{Hx}$ is a nontrivial Hausdorff continuum, $\overline{Hx}$ contains a noncut point $y \neq e$. Then for some $z \in X$,

$$X - z = U \cup V, e \in U, y \in V, (\bar{U} \cap V) \cup (U \cap \bar{V}) = \varnothing .$$

Because $\overline{Hx}$ is connected, z is a cut point of Hx. Since Hy contains only noncut points of $\overline{Hx}$, $z \notin Hy$, and $\overline{Hy} \subset (V \cap \overline{Hx}) \cup \{z\}$, for the last set is closed in X. Now A lies in the center of T, hence every point of Hx is fixed under A, so that $\overline{Hy}$ is a T-invariant set not containing e. By the remark at the end of § 2, the proof is complete.

The author is indebted to Professor Hsin Chu for his encouragement during the preparation of this paper, and to the referee for pointing out two previously overlooked generalizations in the theorems.

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Received October 24, 1965, and in revised form February 5, 1966.

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