PACIFIC JOURNAL OF MATHEMATICS
Vol. 20, No. 1, 1967

EXISTENCE OF OPTIMAL CONTROLS

A. W. J. STODDART

Let $f = (f_1, f_2, \cdots, f_n)$ be a mapping to E_n from a set D in $E_1 \times E_n \times E_m$; and f_0 a real function on D. Consider a "control" function u from an interval $I = [t_0, t_1]$ in E_1 to E_m; and a "response" function x from I to E_n such that $(t, x(t), u(t)) \in D$ for almost every $t \in I, f_0(t, x(t), u(t))$ has an integral (finite or $+\infty$) on $I, f(t, x(t), u(t))$ is integrable on I, and

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), u(s))ds \qquad (1)$$

for all $t \in I$. In a class Γ of such control-response pairs (u, x), a pair (u^*, x^*) is called optimal (with respect to f_0) if the "cost" functional

$$C(u, x) = (I)\int f_0(t, x, u)dt$$

has a minimum at (u^*, x^*). Here we consider conditions sufficient for existence of such optimal pairs.

The problem of existence of optimal controls for various functions f, f_0 and classes Γ has been treated in [6], [11], [7], [5], [8], [9], [13], [10], [1], [2], and [3]. Gamkrelidze [6] assumed f_0 constant, f linear in (x, u), and u restricted to a cube U in E_m. Pontryagin [11] extended Gamkrelidze's work to the situation where U is any compact convex polyhedron. Lee and Markus [8] considered f and f_0 linear in u, and U any compact convex set. Simple integral restraints on u were treated by Krasovskii [7] and Neustadt [9].

The conditions on U and f for f_0 constant were relaxed remarkably by Filippov [5], who considered a variable compact restraint set $U(t, x)$ such that the set $f(t, x, U(t, x))$ is convex for each (t, x). Roxin [13], in effect, considered U a fixed compact set with $(f, f_0)(t, x, U)$ convex. By taking f and f_0 linear in x and U compact, Neustadt [10] avoided all convexity assumptions. Cesari [1] assumes $U(t, x)$ compact, $f(t, x, U(t, x))$ convex, and f_0 sufficiently convex in u compared with the curvature of f in u. In [3], Cesari extends considerations to restraint sets $U(t, x)$ which can be unbounded.

In this paper, we consider variations of the conditions above for the case in which f is linear in u, f_0 is convex in u, and the variable restraint set $U(t, x)$ is convex and closed but not necessarily bounded. In particular, integral restraints are taken into account, and used as an alternative source for the fundamental compactness condition. In a later section, we apply our results to classical existence problems of the calculus of variations.

2. Definitions. We shall call a real function $\phi(t, x, u)$ "linearly bounded below in u" if

$$\phi(t, x, u) \geqq p(t, x) + u \cdot q(t, x)$$

for some uniformly continuous and bounded functions p, q. The meaning of "linearly bounded in u" will be obvious.

Consider the following sets, functions, and numbers.

(2) The sets $J_0 = [T_0, T_0']$, $J_1 = [T_1, T_1']$ are compact intervals in E_1 with $T_0 \leqq T_1'$. Let $J = [T_0, T_1']$.

(3) The set B is a closed set in $J \times E_n$, and U is a closed convex set in E_m. Let $D = B \times U$.

(4) The real continuous functions $h_j(t, x, u)$ on D, at most countable in number, are convex and linearly bounded below in u. Let $U(t, x) = U \cap \{u: h_j(t, x, u) \leqq 0 \text{ for all } j\}$.

(5) The mapping $G_0(t)$, from J_0 to the class of compact sets in E_n, is continuous in the Hausdorff sense. The mapping $G_1(t)$, from J_1 to the class of closed sets in E_n, is also continuous in the Hausdorff sense.

(6) The real continuous functions $g_k(t, x, u)$ on D are convex and linearly bounded below in u; c_k are corresponding real numbers.

(7) The continuous mapping $f(t, x, u)$ from D to E_n is linear in u and with each component function f_i linearly bounded in u. Note that linear bounding of each f_i does not follow from linearity, even if the coefficients in f are bounded; for example, $f = u \sin x^2$ on $E_1 \times E_1$. However, if the coefficients in f are bounded and each component of u in U is bounded above or below (in particular, U bounded), then linearity implies linear bounding.

Define Γ to be the class of all control-response pairs (u, x) on intervals $I = [t_0, t_1]$, such that (1) holds, and

$$t_0 \in J_0 , \qquad t_1 \in J_1 ; \tag{8}$$

$$(t, x(t)) \in B \qquad \text{for every } t \in I ; \tag{9}$$

$$x(t_0) \in G_0(t_0) , \qquad x(t_1) \in G_1(t_1) ; \tag{10}$$

$$u(t) \in U(t, x(t)) \qquad \text{for almost every } t \in I ; \tag{11}$$

$$(I)\int g_k(t, x, u)dt \leqq c_k \qquad \text{for each } k . \tag{12}$$

We shall assume that

(13) $\int |u|$ is equi absolutely continuous on Γ;

that is, for any $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that $(M)\int |u| dt < \varepsilon$ for any $(u, x) \in \Gamma$ and measurable set $M \subseteqq I$ for which the Lebesgue measure $\mu(M) < \lambda(\varepsilon)$. (Conditions sufficient for this will be discussed in § 6.) Note that $(I)\int |u| dt$ is then bounded on Γ.

Note also that, without further restrictions on f, x is not necessarily determined through (1) by $x(t_0)$ and u.

Our general approach will be to prove that the class Γ is sequentially compact and closed in an appropriate convergence system. We then apply a general semicontinuity theorem of [14] to obtain the existence of a minimum for $C(u, x)$ on Γ.

3. A compactness theorem. We first prove a compactness theorem for Γ. It is essentially an abstraction of techniques of Tonelli [15] and Lee and Markus [8].

THEOREM 1. *Any infinite subclass of Γ contains a sequence (u^n, x^n) such that there exist a compact interval $I^* = [t_0^*, t_1^*]$, a continuous mapping x^* from I^* to E_n, and an integrable mapping u^* from I^* to E_m, for which*

(14) (a) $t_0^n \to t_0^*$, $t_1^n \to t_1^*$;

(b) $x^n(t_0^n) \to x^*(t_0^*)$, $x^n(t_1^n) \to x^*(t_1^*)$;

(c) $\sup\{|x^n(t) - x^*(t)| : t \in I^n \cap I^*\} \to 0$; *and*

(d) $(I^n \cap E)\int u^n dt \to (I^* \cap E)\int u^* dt$

for every measurable set $E \subseteqq E_1$.

Proof. The linear bounding of the component functions f_i gives $|f| \leqq a + b|u|$ for some constants a, b. Then

$$|x(t') - x(t)| \leqq a(t' - t) + b\int_t^{t'} |u| ds$$

from (1); thus x is equicontinuous on Γ.

All $G_0(t)$ and J_0 are compact; hence, by an elementary argument, $\bigcup G_0(t)$ is compact. In addition, x is equicontinuous and J is bounded; hence x is equibounded on Γ.

On J, define

$$\begin{aligned} x_+(t) &= x(t_0) \quad \text{on} \quad [T_0, t_0]\,, \\ &\; x(t) \quad \text{on} \quad [t_0, t_1]\,, \\ &\; x(t_1) \quad \text{on} \quad [t_1, T_1']\,; \\ u_+(t) &= 0 \quad \text{on} \quad [T_0, t_0) \text{ and } (t_1, T_1']\,, \\ &\; u(t) \quad \text{on} \quad [t_0, t_1]\,. \end{aligned}$$

Let Γ_+ be the corresponding class of pairs (u_+, x_+). On Γ_+, x_+ is equicontinuous and equibounded, and $\int |u_+|$ is equi absolutely continuous and $(J)\int |u_+|\,dt$ is bounded. Consequently, from any infinite subclass of Γ_+, we can extract in succession sequences $t_0^n \to t_0^*$, $t_1^n \to t_1^*$, $x_+^n \to x_+^*$ uniformly on J, and $u_+^n \to u_+^*$ weakly in $L_1(J)$ [4, p. 294] for some t_0^*, t_1^*, continuous x_+^*, and integrable u_+^*.

Define $x^* = x_+^* \mid [t_0^*, t_1^*]$, $u^* = u_+^* \mid [t_0^*, t_1^*]$. Then x^* is continuous and u^* is integrable. Since $x_+^n(t_0^*) \to x_+^*(t_0^*)$, x_+^n is equicontinuous, and $t_0^n \to t_0^*$, we have $x^n(t_0^n) \to x^*(t_0^*)$. Similarly, $x^n(t_1^n) \to x^*(t_1^*)$.

For any $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that $(E)\int |u_+|\,dt < \varepsilon$ for any set $E \subseteqq J$ with Lebesgue measure $\mu(E) < \lambda(\varepsilon)$. Now

$$(J)\int \phi u_+^n dt \to (J)\int \phi u_+^* dt$$

for every $\phi \in L_\infty(J)$. For any measurable set $E \subseteqq E_1$, take ϕ as the characteristic function of $I^* \cap E$. Then

$$\left|(I^* \cap E)\int u_+^n dt - (I^* \cap E)\int u_+^* dt\right| < \varepsilon$$

for n greater than some $N(\varepsilon, E)$. Now

$$\begin{aligned}&\mu(I^* \cap E - I^n \cap E) + \mu(I^n \cap E - I^* \cap E)\\ &\quad \leqq |t_0^n - t_0^*| + |t_1^n - t_1^*|\,,\end{aligned}$$

which is less than $\lambda(\varepsilon)$ for n greater than some $N(\varepsilon)$. Hence, for $n > N(\varepsilon)$ and $N(\varepsilon, E)$,

$$\left|(I^n \cap E)\int u^n dt - (I^* \cap E)\int u^* dt\right| < 2\varepsilon\,.$$

4: Continuity and semicontinuity. The following continuity theorem is required for the semicontinuity theorem.

THEOREM 2. *Let (u^n, x^n) be a sequence in Γ converging to (u^*, x^*) in the sense* (14). *Let the functions $p\colon B \to E_1$ and $q\colon B \to E_m$ be uniformly continuous and bounded. Then, for every measurable set $E \subseteqq E_1$,*

$$\begin{aligned}&(I^n \cap E)\int [p(t, x^n) + u^n \cdot q(t, x^n)]dt\\ &\qquad \to (I^* \cap E)\int [p(t, x^*) + u^* \cdot q(t, x^*)]dt\,.\end{aligned}$$

Proof. Note that, since B is closed, conditions (14) (a), (b), (c) ensure that $(t, x^*(t)) \in B$ for every $t \in I^*$.

We express our conditions in explicit form. For any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that, for $n > N(\varepsilon)$,

$$|t_0^n - t_0^*| < \varepsilon ,$$
$$|t_1^n - t_1^*| < \varepsilon ,$$
$$|x^n(t_0^n) - x^*(t_0^*)| < \varepsilon ,$$
$$|x^n(t_1^n) - x^*(t_1^*)| < \varepsilon ,$$

and

$$|x^n(t) - x^*(t)| < \varepsilon \qquad \text{if } t \in I^n \cap I^* ;$$

and, for any measurable set $E \subseteqq E_1$, there exists $N(\varepsilon, E)$ such that, for $n > N(\varepsilon, E)$,

$$\left|(I^n \cap E)\int u^n dt - (I^* \cap E)\int u^* dt\right| < \varepsilon .$$

Also, there exists $\lambda(\varepsilon) > 0$ such that $(M)\int |u^n| dt < \varepsilon$ for all n and for every measurable set $M \subseteqq I^n$ with $\mu(M) < \lambda(\varepsilon)$; and there exists α such that $(I^n)\int |u^n| dt < \alpha$ for all n. In addition, there exists $\delta(\varepsilon) > 0$ such that $|p(t, x) - p(t', x')|$ and $|q(t, x) - q(t', x')| < \varepsilon$ for $|t - t'|$ and $|x - x'| < \delta(\varepsilon)$; and there exists β such that $|p(t, x)|$ and $|q(t, x)| < \beta$ for all $(t, x) \in B$.

Now

$$\begin{aligned}
&\left|(I^n \cap E)\int p(t, x^n)dt - (I^* \cap E)\int p(t, x^*)dt\right| \\
&\quad \leqq \beta(|t_1^n - t_1^*| + |t_0^n - t_0^*|) \\
&\qquad + (I^n \cap E \cap I^*)\int |p(t, x^n) - p(t, x^*)| dt \\
&\quad < \beta\varepsilon + (T_1' - T_0)\varepsilon \qquad \text{if } n > N\left(\frac{1}{2}\varepsilon\right) \text{ and } N(\delta(\varepsilon)) .
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|(I^n \cap E)\int u^n \cdot q(t, x^n)dt - (I^n \cap E \cap I^*)\int u^n \cdot q(t, x^*)dt\right| \\
&\quad \leqq \beta(I^n - I^*)\int |u^n| dt \\
&\qquad + \left|(I^n \cap E \cap I^*)\int u^n \cdot [q(t, x^n) - q(t, x^*)] dt\right| \\
&\quad < \beta\varepsilon + \alpha\varepsilon \qquad \text{if } n > N\left(\frac{1}{2}\lambda(\varepsilon)\right) \text{ and } N(\delta(\varepsilon)) .
\end{aligned}$$

By uniform continuity of x^* on I^*, there exists $\gamma(\varepsilon) > 0$ such that $|x^*(t) - x^*(t')| < \varepsilon$ for $|t - t'| < \gamma(\varepsilon)$. Divide I^* into σ intervals I_s

with lengths less than $\delta(\varepsilon)$ and $\gamma(\delta(\varepsilon))$, and take $q_s = q(t, x^*(t))$ for some $t \in I_s$. Then

$$\begin{aligned}
&\left|(I^n \cap E \cap I^*)\int u^n \cdot q(t, x^*)dt - (I^* \cap E)\int u^* \cdot q(t, x^*)dt\right| \\
&\quad = \left|\sum (I^n \cap E \cap I_s)\int u^n \cdot q(t, x^*)dt - \sum (I_s \cap E)\int u^* \cdot q(t, x^*)dt\right| \\
&\quad < \left|\sum q_s \cdot \left[(I^n \cap E \cap I_s)\int u^n dt - (I_s \cap E)\int u^* dt\right]\right| \\
&\qquad + \varepsilon(I^*)\int |u^*| \, dt + \alpha\varepsilon \\
&\quad < \varepsilon + \varepsilon(I^*)\int |u^*| \, dt + \alpha\varepsilon \qquad \text{if } n > \max N(\varepsilon/\sigma \, |q_s|, I_s \cap E) .
\end{aligned}$$

We shall use repeatedly the following semicontinuity theorem.

THEOREM 3. *Let $\phi(t, x, u)$ be a real continuous function on D, convex and linearly bounded below in u. Consider a sequence $(u^n, x^n) \in \Gamma$ converging to (u^*, x^*) in the sense* (14). *(We shall prove in* § 5 *that $u^*(t) \in U$ for almost every $t \in I^*$; and $(t, x^*(t)) \in B$ for every $t \in I^*$.) Then, for every measurable set $E \subseteq E_1$,*

$$(I^* \cap E)\int \phi(t, x^*, u^*)dt \leq \liminf \, (I^n \cap E)\int \phi(t, x^n, u^n)dt .$$

Theorem 3 follows easily from Theorem 4 of [14]. Our convergence (14) satisfies condition (10) of [14]. The discussion of § 6 of [14] applies here, since the lower bound integral is continuous.

5. A closure theorem.

THEOREM 4. *Let (u^n, x^n) be a sequence in Γ converging to (u^*, x^*) in the sense* (14). *Then $(u^*, x^*) \in \Gamma$.*

Proof. By (14a), $t_0^* \in J_0$ and $t_1^* \in J_1$.

For $t_0^* < t < t_1^*$, $t \in I^n$ for all sufficiently large n, so $x^n(t) \to x^*(t)$ by (14c). Thus $(t, x^*(t)) \in B$. In addition, (14a) and (14b) give $(t_0^*, x^*(t_0^*))$ and $(t_1^*, x^*(t_1^*)) \in B$.

If $x^*(t_0^*)$ were not in $G_0(t_0^*)$, then it would not be in the closure N^c of some neighbourhood N of $G_0(t_0^*)$. But $G_0(t) \subseteq N$ for t sufficiently near t_0^*, so $x^n(t_0^n) \in N$ for all sufficiently large n, from which $x^*(t_0^*) \in N^c$! Similarly, $x^*(t_1^*) \in G_1(t_1^*)$.

The closed convex set U in E_m is the intersection of a countable number of half spaces $\{u: \beta + u \cdot b \leq 0\}$. Let

$$E = \{t: t \in I^*, \beta + u^*(t) \cdot b > 0\} .$$

Then

$$0 \geqq (I^n \cap E)\int(\beta + u^n \cdot b)dt \longrightarrow (E)\int(\beta + u^* \cdot b)dt \geqq 0 \ .$$

Thus $\mu(E) = 0$, and so $u^*(t) \in U$ for almost every $t \in I^*$.

Let $E_j = \{t: t \in I^*, h_j(t, x^*(t), u^*(t)) > 0\}$. Now

$$(I^n \cap E_j)\int h_j(t, x^n, u^n)dt \leqq 0 \ ,$$

so

$$(E_j)\int h_j(t, x^*, u^*)dt \leqq 0$$

by Theorem 3. Consequently, $\mu(E_j) = 0$. Thus (u^*, x^*) satisfies (11).

By Theorem 3 with $E = E_1$, the integral $(I)\int g_k(t, x, u)dt$ is lower semicontinuous in the convergence (14). Consequently

$$(I^*)\int g_k(t, x^*, u^*)dt \leqq c_k \ ,$$

that is, condition (12) is satisfied.

Consider t such that $t_0^* < t < t_1^*$. Theorem 3 with $E = \{s: s \leqq t\}$ shows that the integral

$$\int_{t_0}^{t} f(s, x(s), u(s))ds$$

is continuous in the convergence (14). Also, $x^n(t) \longrightarrow x^*(t)$ and $x^n(t_0^n) \longrightarrow x^*(t_0^*)$. Thus condition (1) on (u^n, x^n) carries over to (u^*, x^*). For $t = t_1^*$, a similar argument applies, but with $E = E_1$.

Thus (u^*, x^*) satisfies conditions (1) and (8) through (12).

6. The existence theorem.

THEOREM 5. *Let the real continuous function $f_0(t, x, u)$ on D be convex and linearly bounded below in u. Assume, as previously, that Γ satisfies condition* (13). *Then, if Γ is not empty, $C(u, x)$ has a minimum on Γ.*

Proof. Theorems 1 and 4 show that Γ is sequentially compact in itself with respect to the convergence (14).

Since f_0 is linearly bounded below in u and u is integrable, $f_0(t, x, u)$ has an integral, finite or $+\infty$. Theorem 3, with $E = E_1$, shows that $(I)\int f_0(t, x, u)dt$ is lower semicontinuous with respect to the convergence (14).

A lower semicontinuous funotional on a sequentially compact space has a minimum. Hence the result.

7. Equi absolute continuity of $\int |u|$. Condition (13) plays the key part in our compactness theorem. We now study conditions sufficient for equi absolute continuity of $\int |u|$ on Γ.

For example, if the set U and the functions h_j are such that

$$U(t, x) = U \cap \{u: h_j(t, x, u) \leqq 0 \text{ for all } j\}$$

is bounded uniformly on B, then condition (13) is obviously satisfied. This is the standard situation in problems of optimal control.

The following more general integral condition is quite standard in the calculus of variations.

THEOREM 6. *Let $\psi(u)$ be a real function on E_m, bounded below and such that $\psi(u)/|u| \to \infty$ as $|u| \to \infty$. If $(I)\int \psi(u(t))dt$ is bounded on Γ, then $\int |u|$ is equi absolutely continuous.*

Proof. Suppose that $(I)\int \psi(u)dt \leqq c$ on Γ; $\psi(u) \geqq b$; and, for any $\varepsilon > 0$, $(\psi(u) - b)/|u| > 1/\varepsilon$ for $|u| > m(\varepsilon)$. For any $(u, x) \in \Gamma$ and measurable set $M \subseteqq I$, define

$$M^+ = M \cap \{t: t \in I, |u(t)| > m(\varepsilon)\}, \qquad M^- = M - M^+ .$$

Then

$$\begin{aligned} (M)\int |u(t)|\, dt &= (M^+)\int |u|\, dt + (M^-)\int |u|\, dt \\ &\leqq \varepsilon(M^+)\int (\psi(u) + |b|)dt + m(\varepsilon)\mu(M^-) \\ &\leqq \varepsilon(I)\int (\psi(u) + |b|)dt + m(\varepsilon)\mu(M) \\ &< \varepsilon(c + |b|(T_1' - T_0) + 1) \end{aligned}$$

if $\mu(M) < \varepsilon/m(\varepsilon)$. Thus $\int |u|$ is equi absolutely continuous.

For example, a "growth condition" $g_k(t, x, u) \geqq \psi(u)$ on some g_k would be sufficient for the bounding of $(I)\int \psi(u)dt$ on Γ. Alternatively, the bounding of $(I)\int \psi(u)dt$, sufficiently for our purpose, would follow from a similar growth condition on f_0.

THEOREM 7. *Suppose that $f_0(t, x, u) \geqq \psi(u)$, where ψ has the properties stated in Theorem 6. Then our existence theorem, Theorem 5, holds without the direct assumption of condition* (13).

Proof. If $C(u, x) = \infty$ for all $(u, x) \in \Gamma$, then the result is trivial. Otherwise, there exists $(u_1, x_1) \in \Gamma$ with $C(u_1, x_1) < \infty$. In considering a minimum for $C(u, x)$ on Γ, we can restrict consideration to the class

$$\Gamma_1 = \Gamma \cap \{(u, x) : C(u, x) \leqq C(u_1, x_1)\} .$$

Then $(I)\int \psi(u)dt$ is bounded on Γ_1. Theorems 5 and 6 show that $C(u, x)$ has a minimum on Γ_1, which is obviously also a minimum on Γ.

8. Extension to unbounded intervals J_0, J_1. If J_1 has semi-infinite form, then our existence theorem still holds, provided f_0 is positively bounded below.

THEOREM 8. *Assume that $f_0(t, x, u) \geqq m$ for some positive constant m. Then Theorm 5 holds also for J_1 of the form $[T_1, \infty)$.*

Proof. If $C(u, x) = \infty$ for all $(u, x) \in \Gamma$, then the existence of a minimum for $C(u, x)$ is trivial. Otherwise, $C_1 = C(u_1, x_1) < \infty$ for some $(u_1, x_1) \in \Gamma$. We can restrict consideration to the class Γ_1 of those $(u, x) \in \Gamma$ for which $C(u, x) \leqq C_1$.

For $(u, x) \in \Gamma_1$, $C_1 \geqq C(u, x) \geqq m(t_1 - t_0)$, so

$$t_1 \leqq t_0 + C_1/m \leqq T_0' + C_1/m .$$

Thus the condition $t_1 \in [T_1, T_0' + C_1/m]$ does not further restrict Γ_1. Then Theorem 5 shows that $C(u, x)$ has a minimum on Γ_1, which is also a minimum on Γ.

Obviously, similar considerations apply when $J_0 = (-\infty, T_0']$; and, indeed, when J_0 and J_1 both have these semi-infinite forms.

9. Classical problems. If $U = E_m$ and the class of functions h_j is empty, then $U(t, x) = E_m$ for all (t, x), that is, there are no explicit restrictions (11). In this case, the fundamental condition (13) on u could come from a growth condition on f_0 or some g_k, as discussed in § 7.

If we take $f(t, x, u) = u$, so that $u = x'$ almost everywhere, then we have a minimum problem for $(I)\int f_0(t, x, x')dt$. The Tonelli theorem [16], on the existence of a minimum for nonparametric curve integrals, is just this problem with no explicit restrictions (11) and no integral restrictions (12); the condition (13) comes from a growth condition on f_0.

More generally, consider curves $y: I \rightarrow E_l$ with absolutely continuous derivatives $y^{(r-1)}$ of order $r - 1$. Take $x = (x_{(1)}, x_{(2)}, \cdots, x_{(r)})$

with $x_{(\alpha)}(t) = y^{(\alpha-1)}(t)$, and $u(t) = y^{(r)}(t)$. Then our work gives an existence theorem for the minimum of $(I)\int f_0(t, y, y', \cdots, y^{(r)})dt$. Here

$$f(t, x, u) = (x_{(2)}, x_{(3)}, \cdots, x_{(r)}, u) .$$

The linear bounding of the components of f is implied essentially by the bounding of $(I)\int |y^{(r)}| dt$.

Returning to first order problems, we can also consider parametric curve integrals $(I)\int f_0(x, x')dt$ with f_0 positively homogeneous of degree one in x'. In this case, a growth condition on f_0 of the form previously considered is impossible. However, if there are no explicit restrictions (11), the functions g_k are similarly independent of t and positively homogeneous of degree one in x', G_0 and G_1 are constant, and B is of the form $E_1 \times C$ for some closed set C in E_n, then we have a system invariant under Fréchet equivalence. We can reparametrize the curves of finite length $L \neq 0$ by their relative arc lengths s/L on the interval $I = [0, 1]$; here s is the arc length. In terms of the new parameter, $|x'| = L$ almost everywhere. If the curves in Γ have bounded lengths, then $\int |x'|$ is equi absolutely continuous. This is trivial for curves with $L = 0$. Thus condition (13) would be satisfied if the curves have bounded lengths.

We have really proved here part of Hilbert's theorem on compactness of a class of parametric curves. The bounding of the curve lengths L could come from the form of some g_k

$$\left(\text{for example, } (I)\int g_k(x, x')dt \rightarrow \infty \text{ as } L \rightarrow \infty\right);$$

or, effectively, from the form of f_0

$$\left(\text{for example, } (I)\int f_0(x, x')dt \rightarrow \infty \text{ as } L \rightarrow \infty\right).$$

REFERENCES

1. L. Cesari, *Un teorema di esistenza in problemi di controlli ottimi*, Ann. Scuola Norm. Sup. Pisa **19** (1965), 35-78.

2. ———, *An existence theorem in problems of optimal control*, J. SIAM Control (A) **3** (1965), 7-22.

3. ———, *Existence theorems for optimal solutions in Pontryagin and Lagrange problems*, J. SIAM Control (A) **3** (1966), 475-498.

4. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience Publishers, New York, 1958.

5. A. F. Filippov, *On certain questions in the theory of optimal control*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **2** (1959), 25-32 (Russian). English translation in J. SIAM Control (A) **1** (1962), 76-84.

6. R. V. Gamkrelidze, *The theory of time-optimal processes in linear systems*, Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 449-474 (Russian).

7. N. N. Krasovskii, *On the theory of optimal control,* Avtomat. i Telemeh. **18** (1957), 960–970 (Russian). English translation in Automation and Remote Control **18** (1957), 1005–1016.
8. E. B. Lee and L. Markus, *Optimal control for nonlinear processes,* Arch. Rational Mech. Anal. **8** (1961), 36–58.
9. L. W. Neustadt, *Time optimal control systems with position and integral limits,* J. Math. Anal. Appl. **3** (1961), 406–427.
10. ———, *The existence of optimal controls in the absence of convexity conditions,* J. Math. Anal. Appl. **7** (1963), 110–117.
11. L. S. Pontryagin, *Optimal control processes,* Uspehi Mat. Nauk **14** (1959), 3–20 (Russian). English translation in Automation Express **2** (1959), 26–30.
12. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (English translation), Interscience Publishers, New York, 1962.
13. E. Roxin, *The existence of optimal controls,* Michigan Math. J. **9** (1962), 109–119.
14. A. W. J. Stoddart, *Semicontinuity of integrals,* Trans. Amer. Math. Soc. **122** (1966), 120–135.
15. L. Tonelli, *Fondamenti di Calcolo delle Variazioni.* 2 vols. Zanichelli, Bologna, 1921–23.
16. ———, *Su gli integrali del calcolo delle variazioni in forma ordinaria,* Ann. Scuola Norm. Sup. Pisa **3** (1934), 401–450. Opere Scelte, Cremonese, Roma (1962) **3**, 192–254.

Received December 8, 1965.

THE UNIVERSITY OF OTAGO,
DUNEDIN, NEW ZEALAND