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JESSEN'S THEOREM ON RIEMANN SUMS FOR LOCALLY COMPACT GROUPS

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Throughout this paper G denotes a locally compact group and $\{H_n\}$ denotes an increasing sequence of closed subgroups of G whose union H is dense in G. For each n, Δ_n denotes the modular function on H_n and Δ denotes the modular function on G. Then $\lim_n \Delta_n(x) = \Delta(x)$ for each $x \in H$. For each n, λ_n denotes a left Haar measure on H_n and λ denotes a left Haar measure on G. For a function f on G and an x in G, ${}_xf$ denotes the function ${}_xf(y) = f(xy)$. The main theorem states that if Δ_n is the restriction of Δ to H_n for all sufficiently large n, then there is a "normalizing" sequence $\{\alpha_n\}$ of positive numbers such that for every f in $\mathfrak{L}_1(G,\lambda)$

$$\text{(1)} \qquad \mathrm{m}_n\, \alpha_n \int_{H_n} {}_xf \, d\lambda_n = \int_G f \, d\lambda$$

for λ-locally almost all x in G. The hypotheses regarding the Δ_n's and Δ hold in all cases known to the authors. In particular, they hold if the H_n's are unimodular (hence if they are Abelian, compact, or discrete) or if the H_n's are open subgroups or normal subgroups. If G is the compact group $[0,1[$ with addition modulo 1, if the H_n's are the finite groups $\{k2^{-n}\colon 0 \leqq k \leqq 2^n - 1\}$ with counting measure λ_n, and if $\alpha_n = 2^{-n}$, then the left side of (1) is a Riemann sum and (1) becomes Jessen's theorem.

Jessen's theorem [10] states that if f is a function on the real line that has period 1 and is Lebesgue summable on $[0, 1]$, then

$$\text{(2)} \qquad \lim_m 2^{-n} \sum_{k=0}^{2^n-1} f\left(x + \frac{k}{2^n}\right) = \int_0^1 f(y)\, dy$$

for almost all x in $[0, 1]$. Jessen observed that in proving (2) he actually proved that

$$\text{(3)} \qquad \lim_n \frac{1}{m_n} \sum_{k=0}^{m_n-1} f\left(x + \frac{k}{m_n}\right) = \int_0^1 f(y)\, dy \qquad \text{a.e.}$$

for any sequence $\{m_n\}$ of positive integers where $m_n \mid m_{n+1}$ for all n. Since such sequences $\{m_n\}$ correspond to all possible increasing sequences of closed subgroups of $[0, 1[$, the generalization stated in (1) gives no new information about the case $G = [0, 1[$.

Relation (3) fails for some functions in $\mathfrak{L}_1([0, 1[)$ in the case that $m_n = n$. This was shown by Marcinkiewicz and Zygmund [12] and by Ursell [15]. Rudin [13] showed that there are many sequences $\{m_n\}$ and *bounded* functions f in $\mathfrak{L}_1([0,1[)$ for which (3) fails and his strong negative theorem emphasizes that the divisibility properties of the m_n's are crucial in Jessen's theorem. These results show that (1) cannot be proved for an arbitrary sequence $\{H_n\}$ whose union is dense. Salem [14] gives a generalization on $[0,1[$ of Jessen's theorem. Another generalization is given by Civin [3].

Notation and terminology not explicitly defined here can be found in [8] or [9]. The first theorem contains a number of equivalent natural conditions any of which could serve as the definition of a "normalizing sequence". We make a formal definition after the theorem.

THEOREM 1. *For a sequence $\{\alpha_n\}$ of positive numbers, the following conditions are equivalent*:

(i) $\lim_n \alpha_n \int_{H_n} f_0 d\lambda_n = \int_G f_0 d\lambda$

for some nonzero f_0 in $\mathfrak{C}_{00}^+(G)$;

(ii) $\lim_n \alpha_n \int_{H_n} f d\lambda_n = \int_G f d\lambda$

for all $f \in \mathfrak{C}_{00}(G)$;

(iii) $\lim_n \alpha_n \lambda_n(U_0 \cap H_n) = \lambda(U_0)$

for some nonvoid open set U_0 in G such that U_0^- is compact and $\lambda(\text{bdry}\, U_0) = 0$;

(iv) $\lim_n \alpha_n \lambda_n(U \cap H_n) = \lambda(U)$

for all open sets U such that U^- is compact and $\lambda(\text{bdry}\, U) = 0$.[1]

Proof. (i) $\Rightarrow$ (ii). There is an $h \in H$ such that $f_0(h) \neq 0$. Then ${}_h(f_0)(e) \neq 0$ and (i) is satisfied by ${}_h(f_0)$. We select a sequence $\{\beta_n\}$ such that

$$\beta_n \int_{H_n} {}_h(f_0) d\lambda_n = \int_G {}_h(f_0) d\lambda \tag{1}$$

for all n; clearly $\lim_n \beta_n \alpha_n^{-1} = 1$. We now use the fact that if $\{H_\gamma\}$ is a net of closed subgroups converging to a closed subgroup H_0 in the sense of Hausdorff and if the Haar measures λ_γ on H_γ are normalized so that $\int_{H_\gamma} g d\lambda_\gamma = \int_{H_0} g d\lambda_0$ for some $g \in \mathfrak{C}_{00}^+$ where $g(e) \neq 0$, then

$$\lim_\gamma \int_{H_\gamma} f d\lambda_\gamma = \int_{H_0} f d\lambda_0 \tag{2}$$

for all $f \in \mathfrak{C}_{00}$. This is due to J. M. G. Fell; see the appendix to [7]; the proof uses an earlier result of Fell [5]. This fact is also proved by Bourbaki [1] (in § 5) and by Flachsmeyer and Zieschang [6]

[1] Such sets U are sometimes called "continuity sets for λ".

(Satz 1).[2] Clearly G is the limit of the sequence $\{H_n\}$ in the sense of Hausdorff and relation (1) is a special case of (2). Therefore

$$\lim_n \beta_n \int_{H_n} f d\lambda_n = \int_G f d\lambda$$

for all $f \in \mathfrak{C}_{00}$. Assertion (ii) follows from this since $\lim_n \beta_n \alpha_n^{-1} = 1$.

(ii) $\Rightarrow$ (iv). Let U be an open set such that U^- is compact and $\lambda(\text{bdry}\, U) = 0$. If $f \in \mathfrak{C}_{00}$ and $f \geqq \xi_{U^-}$, then

$$\begin{aligned}\limsup_n \alpha_n \lambda_n(U \cap H_n) &\leqq \lim_n \alpha_n \int_{H_n} f d\lambda_n \\ &= \int_G f d\lambda \,.\end{aligned}$$

Since

$$\lambda(U) = \lambda(U^-) = \inf\left\{\int_G f d\lambda : f \in \mathfrak{C}_{00},\, f \geqq \xi_{U^-}\right\},$$

we obtain

$$\limsup_n \alpha_n \lambda_n(U \cap H_n) \leqq \lambda(U) \,.$$

A similar argument using

$$\lambda(U) = \sup\left\{\int_G f d\lambda : f \in \mathfrak{C}_{00},\, f \leqq \xi_U\right\}$$

shows that

$$\lambda(U) \leqq \liminf_n \alpha_n \lambda_n(U \cap H_n) \,.$$

(iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let f_0 be any nonzero function in $\mathfrak{C}_{00}^+$. Clearly there is a sequence $\{\beta_n\}$ of positive numbers for which

$$\lim_n \beta_n \int_{H_n} f_0 d\lambda_n = \int_G f_0 d\lambda \,. \tag{3}$$

The already proved implication (i) $\Rightarrow$ (iv) applied to $\{\beta_n\}$ yields

$$\lim_n \beta_n \lambda_n(U_0 \cap H_n) = \lambda(U_0) \,.$$

By supposition

[2] Yet another proof, which uses the Hahn-Banach theorem, can be given for this result. There are at least two other proofs for the compact case. One uses the fact that the semigroup of probability measures on G is compact in the weak-* topology and the other uses the fact that $\lim_n \alpha_n \hat{\lambda}_n = \hat{\lambda}$ pointwise where $\hat{\lambda}_n$ and $\hat{\lambda}$ are the Fourier-Stieltjes transforms on the space of equivalence classes of irreducible unitary representations of G.

$$\lim_n \alpha_n \lambda_n(U_0 \cap H_n) = \lambda(U_0)$$

and it follows that $\lim_n \alpha_n \beta_n^{-1} = 1$. This equality and (3) imply that (i) holds for $\{\alpha_n\}$ and f_0.

As noted in the proof of Theorem 1, sequences $\{\alpha_n\}$ satisfying (i) always exist trivially. If G is compact and if $\lambda_n(H_n) = \lambda(G)$ for all n, then all the conditions of Theorem 1 hold for the sequence $\alpha_n = 1$.

DEFINITION. A sequence $\{\alpha_n\}$ satisfying the equivalent conditions of Theorem 1 is called a *normalizing sequence* for the family $\{\lambda, \lambda_1, \lambda_2, \cdots\}$ of left Haar measures.

It is easy to prove that if $\{\alpha_n\}$ is a normalizing sequence, then another sequence $\{\beta_n\}$ of positive numbers is a normalizing sequence if and only if $\lim_n \alpha_n \beta_n^{-1} = 1$.

The next two theorems tell us more about normalizing sequences.

LEMMA 1. *If $\{\alpha_n\}$ is a normalizing sequence and if F is a compact subset of G, then there is a finite constant c_F, depending only upon F, such that*

(i) $\alpha_n \lambda_n(xF \cap H_n) \leqq c_F$

for all $x \in G$ and all n. The constant c_F can be taken to be $\sup_n \alpha_n \lambda_n(F^{-1}F \cap H_n)$.[3]

Proof. Choose g in $\mathfrak{C}_{00}^+$ such that $g \geqq \xi_{F^{-1}F}$; then

$$c_F = \sup_n \alpha_n \lambda_n(F^{-1}F \cap H_n) \leqq \sup_n \alpha_n \int_{H_n} g d\lambda_n < \infty .$$

Consider any x and n. If $xF \cap H_n = \varnothing$, then (i) is plain. Otherwise $xa = h$ for some $a \in F$ and $h \in H_n$. Then

$$\begin{aligned} \alpha_n \lambda_n(xF \cap H_n) &= \alpha_n \int_{H_n} \xi_{xF} d\lambda_n \\ &= \alpha_n \int_{H_n} {}_{h^{-1}}(\xi_{a^{-1}F}) d\lambda_n \\ &= \alpha_n \int_{H_n} \xi_{a^{-1}F} d\lambda_n \leqq \alpha_n \int_{H_n} \xi_{F^{-1}F} d\lambda_n \leqq c_F . \end{aligned}$$

NOTATION. For the remainder of the paper, whenever F is a compact subset of G, c_F will denote the constant in Lemma 1. If G is compact and $\lambda_n(H_n) = \lambda(G) = 1$ for all n, then we take $\alpha_n = 1$ for all n and $c_F = 1$ for all F.

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[3] The existence of c_F can also be deduced from the proof of Fell's theorem [**7**] or from Hilfssatz 1 of [**6**].

THEOREM 2. *Let* $\{\alpha_n\}$ *be a normalizing sequence. If* $f \in \mathfrak{C}_{00}(G)$, *then*

$$\text{(i)} \quad \lim_n \alpha_n \int_{H_n} {}_x f d\lambda_n = \int_G f d\lambda$$

and

$$\text{(ii)} \quad \lim_n \alpha_n \int_{H_n} f_x d\lambda_n = \Delta(x^{-1}) \int_G f d\lambda$$

uniformly on compact subsets of G.

Proof.[4] The pointwise convergence of (i) and (ii) follows from Theorem 1. Let F be any compact subset of G. An Ascoli theorem (Theorem 15, page 232 of [11]) states that pointwise convergence implies uniform convergence on compact sets provided that the functions involved belong to an equicontinuous family of functions. Thus it suffices to prove that the family of functions consisting of all

$$\phi_n(x) = \alpha_n \int_{H_n} {}_x f d\lambda_n \quad \text{and all} \quad \psi_n(x) = \alpha_n \int_{H_n} f_x d\lambda_n$$

is equicontinuous on F. Let E be a compact set containing the support of f. Let $c = c_{E \cup EF^{-1}}$; by Lemma 1,

$$\alpha_n \lambda_n(x(E \cup EF^{-1}) \cap H_n) \leqq c$$

for all x and n. Given $\varepsilon > 0$, select a neighborhood V of the identity e such that $xy^{-1} \in V$ implies $\| {}_x f - {}_y f \|_u < \varepsilon/2c$ and $\| f_x - f_y \|_u < \varepsilon/c$. Then $xy^{-1} \in V$ implies

$$\begin{aligned} | \phi_n(x) - \phi_n(y) | &\leqq \alpha_n \int_{H_n} | {}_x f - {}_y f | d\lambda_n \\ &= \alpha_n \int_{H_n} | {}_x f - {}_y f | \xi_{x^{-1}E \cup y^{-1}E} d\lambda_n \\ &\leqq \frac{\varepsilon}{2c} \alpha_n \lambda_n(x^{-1}E \cap H_n) + \frac{\varepsilon}{2c} \alpha_n \lambda_n(y^{-1}E \cap H_n) \leqq \varepsilon ; \end{aligned}$$

if, in addition, x and y are in F, then

$$\begin{aligned} | \psi_n(x) - \psi_n(y) | &\leqq \alpha_n \int_{H_n} | f_x - f_y | \xi_{EF^{-1}} d\lambda_n \\ &\leqq \frac{\varepsilon}{c} \alpha_n \lambda_n(EF^{-1} \cap H_n) \leqq \varepsilon . \end{aligned}$$

THEOREM 3. *Let* $\{\alpha_n\}$ *be a normalizing sequence. Then* G/H_n *is compact for some* n *if and only if*

[4] The proof for compact G was kindly given us by Thomas Paine.

(i) $\lim_n \alpha_n \int_{H_n} {}_x f d\lambda_n = \int_G f d\lambda$
uniformly on G *for all* $f \in \mathfrak{C}_{00}(G)$.

Proof. Let $\phi_n(x) = \alpha_n \int_{H_n} {}_x f d\lambda_n$.

Suppose that G/H_{n_0} is compact for some n_0. Then there is a compact set F in G such that $FH_n = G$ for all $n \geqq n_0$; see 5.24.b of [8]. By Theorem 2 there is an $n_1 \geqq n_0$ such that $\left|\phi_n(y) - \int_G f d\lambda\right| < \varepsilon$ for all $y \in F$ and $n \geqq n_1$. For any x in G and $n \geqq n_1$, $x = yh$ for some $y \in F$ and $h \in H_n$ and hence

$$\begin{aligned}\phi_n(x) &= \alpha_n \int_{H_n} {}_x f d\lambda_n = \alpha_n \int_{H_n} {}_h({}_y f) d\lambda_n \\ &= \alpha_n \int_{H_n} {}_y f d\lambda_n = \phi_n(y) .\end{aligned}$$

It follows that $\left|\phi_n(x) - \int_G f d\lambda\right| < \varepsilon$ for all $x \in G$ and $n \geqq n_1$.

Suppose now that (i) holds. Let f be a nonzero function in $\mathfrak{C}_{00}^+$ and let F be a compact set containing its support. Since $\int_G f d\lambda > 0$, there is an n such that $\alpha_n \int_{H_n} {}_x f d\lambda_n > 0$ for all $x \in G$. Then $x \in G$ implies that ${}_x f(h) \neq 0$ for some $h \in H_n$, hence $xh \in F$ and $x \in FH_n^{-1}$. Therefore $G = FH_n$ and G/H_n is compact.

The next theorem relates the modular function on G to the modular functions on the H_n's.

THEOREM 4. *If* F *is a compact subset of some* H_m, *then* $\lim_n \Delta_n(x) = \Delta(x)$ *uniformly on* F. *In particular,* $\lim_n \Delta_n(x) = \Delta(x)$ *for all* $x \in H$.

Proof. Let $\{\alpha_n\}$ be a normalizing sequence and let f be a nonzero function in $\mathfrak{C}_{00}^+$. By Theorem 2, we have

$$\begin{aligned}\lim_n \Delta_n(x)\alpha_n \int_{H_n} f d\lambda_n &= \lim_n \alpha_n \int_{H_n} f_{x^{-1}} d\lambda_n \\ &= \Delta(x) \int_G f d\lambda\end{aligned}$$

uniformly on F. Since

$$\lim_n \alpha_n \int_{H_n} f d\lambda_n = \int_G f d\lambda \neq 0 ,$$

we infer that $\lim_n \Delta_n(x) = \Delta(x)$ uniformly on F.

Note that $\Delta_n = \Delta | H_n$ whenever H_n is a normal subgroup of G; see 15.23 of [8]. If H_n is not normal, then the identity $\Delta_n = \Delta | H_n$ may fail to hold. It seems unlikely that $\Delta_n = \Delta | H_n$ must hold for sufficiently large n, but the authors unfortunately have not been able to produce an example to settle this question. Further comments about this question follow Theorem 5.

We next prove two lemmas that are needed in order to prove in Theorem 5 our main result, namely, our generalization of Jessen's theorem. The first lemma is a consequence of a result of Edwards and Hewitt [4].

LEMMA 2. *Suppose that* $\{\mu_n\}$ *is a sequence of nonnegative Borel measures on* G. *Then, for every* $f \in \mathfrak{L}_1(G, \lambda)$, ${}_xf$ *is* μ_n*-measurable for* λ*-locally almost all* $x \in G$. *Suppose also that*

(i) $\lim_n \int_G f d\mu_n = \int_G f d\lambda$

for all $f \in \mathfrak{C}_{00}(G)$ *and that*

(ii) $\sup_n \int_G {}_xf d\mu_n < \infty$ λ*-locally a.e.*

for all $f \in \mathfrak{L}_1^+(G, \lambda)$. *Then*

(iii) $\lim_n \int_G {}_xf d\mu_n = \int_G f d\lambda$ λ*-locally a.e.*

for all $f \in \mathfrak{L}_1(G, \lambda)$.

Proof. In their Theorem 1.6, Edwards and Hewitt [4] prove the following. Suppose E is a real semimetrizable topological vector space of the second category and that $(S, \mathscr{M}, \mu)$ is a measure space. Let $\mathfrak{F}$ be the family of all $\mathscr{M}$-measurable functions from S into $[0, \infty]$, where any two functions in $\mathfrak{F}$ that are equal μ-locally almost everywhere are identified. Suppose $\{P_\alpha\}$ is a countable net of sublinear operators from E into $\mathfrak{F}$ satisfying

(1) for each α, $\lim_n f_n = f$ in E implies that $\lim_k P_\alpha f_{n_k} = P_\alpha f$ μ-locally a.e. for some subsequence $\{f_{n_k}\}$ of $\{f_n\}$,

and

(2) $Pf(s) = \sup_\alpha P_\alpha f(s)$ is finite μ-locally a.e. for every $f \in E$.

Let E_0 be the set of f in E for which $\lim_\alpha P_\alpha f(s) = 0$ μ-locally a.e. Then E_0 is a closed vector subspace of E.

It suffices to prove (iii) for f in $\mathfrak{L}_1^r(G)$. Let $E = \mathfrak{L}_1^r$ and, for each positive integer m and $f \in \mathfrak{L}_1^r$, let

$$P_m f(x) = \left| \int_G {}_xf d\mu_m - \int_G f d\lambda \right| \qquad \text{for } x \in G .$$

Suppose that G is σ-compact. If h is a λ-null function on G, then $|h|$ is dominated by a Borel measurable λ-null function k. Then $(x, y) \rightarrow k(xy)$ is Borel measurable on $G \times G$ and an application of Fubini's theorem (13.9 of [8]) shows that ${}_xk$ is μ_m-measurable for λ-almost all x and that $\int_G {}_xk\, d\mu_m$ is λ-null. The same remarks thus apply to h. A similar application of Fubini's theorem shows that $\int_G {}_xf\, d\mu_m$ is λ-measurable for any $f \in \mathfrak{L}_1(G, \lambda)$. Therefore $P_m f$ is λ-measurable and is well-defined in the sense that if $f = g$ λ-a.e., then $P_m f = P_m g$ in $\mathfrak{F}$. If G is not σ-compact, the same statement can be proved by making a similar argument on its open σ-compact subgroups. It is easy to see that each P_m is sublinear:

$$P_m(\alpha f) = |\alpha| P_m(f) \quad \text{and} \quad P_m(f + g) \leqq P_m f + P_m g$$

λ-locally a.e., where α is real.

To prove (1), fix m and suppose that $\lim_n \|f_n - f\|_1 = 0$ where f and each f_n belong to $\mathfrak{L}_1^r$. Since $\lim_n \int_G f_n d\lambda = \int_G f d\lambda$, it suffices to prove that the sequence

$$s_n(x) = \left| \int_G {}_x(f_n) d\mu_m - \int_G {}_xf\, d\mu_m \right|$$

of functions has a subsequence that converges λ-locally a.e. to 0. For positive integers k, choose n_k so that $\|f_{n_k} - f\|_1 < 4^{-k}$ and let $g = \sum_{k=1}^{\infty} 2^k |f_{n_k} - f|$. Then g belongs to $\mathfrak{L}_1$ and $\int_G {}_xg\, d\mu_m$ exists and is finite λ-locally a.e. For any x such that $\int_G {}_xg\, d\mu_m < \infty$, we have

$$s_{n_k}(x) = \left| \int_G {}_x(f_{n_k} - f) d\mu_m \right| \leqq 2^{-k} \int_G {}_xg\, d\mu_m$$

and hence $\lim_k s_{n_k}(x) = 0$. This proves (1), and (2) follows immediately from (ii).

Let E_0 consist of all f in $\mathfrak{L}_1^r$ such that $\lim_m P_m f(x) = 0$ λ-locally a.e. Equivalently E_0 consists of the functions in $\mathfrak{L}_1^r$ for which (iii) holds and so we need only prove that $E_0 = \mathfrak{L}_1^r$. For any $f \in \mathfrak{C}_{00}^r$ and $x \in G$, (i) applied to ${}_xf$ shows that $\lim_m P_m f(x) = 0$. Therefore $\mathfrak{C}_{00}^r \subset E_0$; the theorem of Edwards and Hewitt asserts that E_0 is closed in $\mathfrak{L}^r$ and hence $E_0 = \mathfrak{L}_1^r$.

LEMMA 3. *Suppose that* $\{\alpha_n\}$ *is a normalizing sequence and that* $\Delta_n = \Delta | H_n$ *for all* n. *Let* f *be a nonnegative Borel measurable function on* G. *Let*

$$f^*(x) = \sup_n \alpha_n \int_{H_n} {}_xf\, d\lambda_n$$

for $x \in G$ *and for* $t \geqq 0$, *let* $B_t^* = \{x \in G : f^*(x) > t\}$. *Then for* $t \geqq 0$ *and every compact subset* F *of* G, *we have*

(i) $t\lambda(B_t^* \cap F) \leqq c_F \int_{B_t^*} f d\lambda$.

If f *is also in* $\mathfrak{L}_1^+(G, \lambda)$, *then*

(ii) $f^*(x) < \infty$

for λ-*locally almost all* x *in* G.

Proof.[5] Let $\phi_n(x) = \alpha_n \int_{H_n} {}_x f d\lambda_n$. Let N be a fixed positive integer and let $D_N = \{x \in G : \sup_{1 \leqq n \leqq N} \phi_n(x) > t\}$. For $n = 1, 2, \cdots, N$, let $E_n = \{x \in G : \phi_n(x) > t\}$ and let $A_n = E_n \cap (\bigcup_{k=n+1}^N E_k)'$; note that $A_N = E_N$. Note also that $E_k H_n = E_k$ for $n \leqq k$ and hence $A_n H_n = A_n$ for all $n \leqq N$. Recall that $\alpha_n \lambda_n(xF \cap H_n) \leqq c_F$ for all x and n by Lemma 1. For all n, we have

$$\begin{aligned}
t\lambda(A_n \cap F) &\leqq \int_{A_n \cap F} \phi_n d\lambda \\
&= \int_{A_n \cap F} \alpha_n \int_{H_n} f(xy) d\lambda_n(y) d\lambda(x) \\
&= \alpha_n \int_{H_n} \int_G \xi_{A_n \cap F}(x) f(xy) d\lambda(x) d\lambda_n(y) \\
&= \alpha_n \int_{H_n} \Delta(y^{-1}) \int_G \xi_{(A_n \cap F)y^{-1}}(x) f(x) d\lambda(x) d\lambda_n(y) \\
&= \int_G f(x) \alpha_n \int_{H_n} \Delta_n(y^{-1}) \xi_{(A_n \cap F)^{-1}x}(y) d\lambda_n(y) d\lambda(x) \\
&= \int_G f(x) \alpha_n \int_{H_n} \xi_{x^{-1}(A_n \cap F)}(y) d\lambda_n(y) d\lambda(x) \\
&= \int_G f(x) \alpha_n \lambda_n(x^{-1}(A_n \cap F) \cap H_n) d\lambda(x) \\
&= \int_{A_n} f(x) \alpha_n \lambda_n(x^{-1}(A_n \cap F) \cap H_n) d\lambda(x) \\
&\leqq c_F \int_{A_n} f d\lambda \,.
\end{aligned}$$

The last equality follows from the fact that $A_n H_n = A_n$. Since $D_n = \bigcup_{n=1}^N A_n$ and the union is disjoint, we infer that $t\lambda(D_N \cap F) \leqq c_F \int_{D_N} f d\lambda$. Inequality (i) now follows from the fact that B_t^* is the union of the increasing sequence $\{D_N\}$ of sets.

To prove (ii) we need to show that $B = \bigcap_{t=1}^\infty B_t^*$ is locally null. For a compact set F, (i) shows that $t\lambda(B_t^* \cap F) \leqq c_F \|f\|_1$ and hence $\lim_{t \to \infty} \lambda(B_t^* \cap F) = 0$. Therefore $\lambda(B \cap F) = 0$ and B is locally null.

[5] This proof of (i) is a modification of the proof of one of Jessen's lemmas [**10**].

An example showing the necessity of the hypothesis regarding Δ and the Δ_n's will be given after Theorem 5. In Theorem 6, we will obtain sharper results about the function f^* for the case that G is compact.

THEOREM 5. *Suppose that* $\{\alpha_n\}$ *is a normalizing sequence and that* $\Delta_n = \Delta | H_n$ *for all sufficiently large* n. *If* f *is in* $\mathfrak{L}_1(G, \lambda)$, *then*

(i) $\lim_n \alpha_n \int_{H_n} {}_x f d\lambda_n = \int_G f d\lambda$

for λ*-locally almost all* x *in* G.

Proof. Choose n_0 so that $\Delta_n = \Delta | H_n$ for $n \geqq n_0$. We apply Lemma 2 to the sequence $\{\alpha_n \lambda_n : n \geqq n_0\}$ of measures; these measures may, of course, be regarded as defined on G. Hypothesis (i) follows from Theorem 1. To prove (ii), we replace f by a Borel measurable function that is equal to it λ-a.e. and then apply (ii) of Lemma 3.

REMARKS AND EXAMPLES. The hypotheses regarding Δ and the Δ_n's in Lemma 3 and Theorem 5 are there because Lemma 3 is false otherwise and because we are unable to prove or disprove Theorem 5 without this hypothesis; compare with our remarks following Theorem 4. We now give an example to show that (ii) of Lemma 3 can fail if $\Delta_1 \neq \Delta | H_1$. Let G be the group of real matrices $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, $x > 0$, $z > 0$; we abbreviate $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ as (x, y, z). See 15.28.b of [8]. Let $H_1 = \{(x, 0, 1) : x > 0\}$ and for $n \geqq 2$ let

$$H_n = \{(x, y, \exp(k \cdot 2^{-n})) : x > 0, y \in R, k \in Z\} .$$

The characteristic function f of $\{(x, y, z) : x > 1, |y| < 1, 1 < z < e\}$ is in $\mathfrak{L}_1(G)$; its left Haar integral is $\int_1^\infty \int_{-1}^1 \int_1^e x^{-2} z^{-1} dz dy dx = 2$. If $(a, b, c) \in G$, then $\int_{H_1} {}_{(a,b,c)} f d\lambda_1$ is the integral over H_1 of the characteristic function of $\{(x, y, z) : ax > 1, |ay + bz| < 1, 1 < cz < e\}$. If $a > 0$, $|b| < 1$, and $1 < c < e$, the intersection of this set with H_1 is $\{(x, 0, 1) : x > a^{-1}\}$ and therefore

$$\int_{H_1} {}_{(a,b,c)} f d\lambda_1 = \int_{a^{-1}}^\infty d\lambda_1 = \infty .$$

Thus $f^*(a, b, c) = \infty$ on the open set $\{(a, b, c) : a > 0, |b| < 1, 1 < c < e\}$ which is certainly not λ-locally null.

If one applies Theorem 5 to the real line R and its subgroups $H_n = \{k 2^{-n} : k \in Z\}$, one finds that

$$\lim_n 2^{-n} \sum_{k=-\infty}^{\infty} f\left(x + \frac{k}{2^n}\right) = \int_{-\infty}^{\infty} f(y) dy \text{ a.e.}$$

whenever f is in $\mathfrak{L}_1(R)$.

Groups G admitting nontrivial increasing sequences $\{H_n\}$ with dense union exist in profusion. For a compact Abelian group, this property holds if and only if the character group X contains a nontrivial decreasing sequence of subgroups whose only common element is the identity. Any nontorsion X has this property as does any X that is a sum or product of an infinite number of subgroups. Some groups without this property are finite products of $Z(p^\infty)$ groups. Thus finite products of the groups Δ_p of p-adic integers do not have nontrivial increasing sequences $\{H_n\}$. An allied question asks what groups contain increasing sequences of *finite* subgroups. No compact infinite Abelian torsion-free group enjoys this property. If G is a direct product of finite groups or groups with this property and there are at most $\mathfrak{c}$ factors, then G also has this property. Thus $\{-1, 1\}^{\aleph_0}$ and $T^{\mathfrak{c}}$ have this property. Finally, of course, there are nonabelian groups that contain increasing sequences of finite subgroups having dense union. Such an example is the group $\mathfrak{O}(2)$ of orthogonal transformations of the plane and its subgroups H_n of symmetries of the regular polygon with 2^n sides.

All the results of this paper are simple and uninteresting (though true) when the subgroups H_n are open. A locally compact Abelian group is the union of an increasing sequence of proper closed (respectively, open) subgroups if and only if it is not compactly generated; see Lemma 3.3 of [2].

The next technical lemma is needed for our last theorem.

LEMMA 4. *Let G be a compact group. Let f be a nonnegative Borel measurable function on G and let f^* and B_t^* be as in Lemma* 3. *For $u \geqq 0$, let $B_u = \{x \in G : f(x) > u\}$. If $0 < \alpha < 1$ and $t \geqq 0$, then*

(i) $(1-\alpha)t\lambda(B_t^*) \leqq \int_{B_{\alpha t}} f d\lambda$.

Proof. Let $g = f\xi_{B_{\alpha t}}$; then we have

$$\begin{aligned} f^*(x) &\leqq \sup_n \int_{H_n} f(xy)\xi_{B_{\alpha t}}(xy)d\lambda_n(y) \\ &\quad + \sup_n \int_{H_n} f(xy)\xi_{B'_{\alpha t}}(xy)d\lambda_n(y) \\ &\leqq g^*(x) + \alpha t . \end{aligned}$$

Thus, letting $C_u^* = \{x \in G : g^*(x) > u\}$, we have

$$B_t^* \subset C^*_{(1-\alpha)t} .$$

Applying (i) of Lemma 3 to g yields

$$(1-\alpha)t\lambda(B_t^*) \leqq (1-\alpha)t\lambda(C_{(1-\alpha)t}^*)$$
$$\leqq \int_{C_{(1-\alpha)t}^*} g d\lambda \leqq \int_G g d\lambda = \int_{B_{\alpha t}} f d\lambda .$$

THEOREM 6. *Let G be a compact group. Let f be a nonnegative Borel measurable function on G and let $f^*(x) = \sup_n \int_{H_n} {}_x f d\lambda_n$. If f is in $\mathfrak{L}_p$ where $1 < p < \infty$, then so is f^* and*

(i) $\|f^*\|_p \leqq \dfrac{p}{p-1} \|f\|_p$.[6]

If $f \in \mathfrak{L} \log^+ \mathfrak{L}$, then $f^* \in \mathfrak{L}_1$ and for every $\alpha \in]0, 1[$, we have

(ii) $\|f^*\|_1 \leqq \dfrac{1}{\alpha} + \dfrac{1}{1-\alpha} \displaystyle\int_G f \log^+ f d\lambda$.

If $f \in \mathfrak{L}_1$, then $f^* \in \mathfrak{L}_p$ for all $0 < p < 1$ and

(iii) $\|f^*\|_p \leqq (1-p)^{-1/p} \|f\|_1$.

This theorem is proved using (i) of Lemma 3 and Lemma 4 in exactly the same way that the Hardy-Littlewood maximal theorems (21.76 and 21.80 of [9]) are deduced from Lemmas 21.75 and 21.79 of [9].

Theorem 6 cannot be extended to locally compact noncompact groups as the following examples show. Consider the group R of real numbers and its subgroups $H_n = \{k2^{-n} : k \in Z\}$. If $f = \xi_{[0,1[}$, then $f^*(x) = 1$ for all $x \in R$. A more striking example is given by the function $g(x) = (1/x)\xi_{[1,\infty[}(x)$. Even though g belongs to $\mathfrak{L}_p(R)$ for all $p > 1$, $g^*(x) = \infty$ for all $x \in R$; g^* is not even locally in $\mathfrak{L}_p(R)$.

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