PACIFIC JOURNAL OF MATHEMATICS
Vol. 20, No. 1, 1967

ON CHARACTERIZING THE GAMMA AND THE NORMAL DISTRIBUTION

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We will characterize the gamma distribution by the nature of the joint distribution of the two quotients X_1/X_3, X_2/X_3 for three identically gamma distributed random variables.

It is well known that if two independent identically distributed random variables X_1, X_2 have the gamma distribution given by the density

$$(1) \qquad f(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} & \text{for } x > 0 \end{cases} \qquad \begin{pmatrix} a > 0 \\ p > 0 \end{pmatrix}$$

then their quotient

$$(2) \qquad Y = X_1/X_2$$

has the beta distribution of the second kind given by the density

$$(3) \qquad g(y) = \begin{cases} 0 & \text{for } y \leqq 0 \\ \dfrac{1}{B(p, p)} \cdot \dfrac{y^{p-1}}{(1+y)^{2p}} & \text{for } y > 0 . \end{cases}$$

However, this property does not characterize the gamma distribution uniquely. There exist pairs of independent positive identically distributed random variables X_1, X_2 whose common distribution function $F(x)$ differs from the one given by the density (1), but where the quotients (2) are distributed according to the density (3). Some such distribution functions $F(x)$ are given by the following densities

$$(4) \qquad \begin{aligned} f_1(x) &= \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{a^p}{\Gamma(p)} x^{-p-1} e^{-a/x} & \text{for } x > 0 \end{cases} \\ f_2(x) &= \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{2\Gamma\left(\dfrac{2p+1}{2}\right)}{\Gamma\left(\dfrac{p}{2}\right)\Gamma\left(\dfrac{p+1}{2}\right)} \cdot \dfrac{x^{p-1}}{(1+x^2)^{p+1/2}} & \text{for } x > 0 \end{cases} \end{aligned}$$

$$f_3(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{2\Gamma\left(\dfrac{2p+1}{2}\right)}{\Gamma\left(\dfrac{p}{2}\right)\Gamma\left(\dfrac{p+1}{2}\right)} \cdot \dfrac{x^p}{(1+x^2)^{p+1/2}} & \text{for } x > 0 \,. \end{cases}$$

Now let X_1, X_2, X_3, be three independent positive identically distributed random variables whose common distribution function is $F(x)$. It is interesting that although the distribution of one quotient (2) does not characterize $F(x)$ uniquely, the joint distribution of two such quotients

$$Y_1 = X_1/X_3,\ Y_2 = X_2/X_3 \tag{5}$$

characterize (by some assumptions) $F(x)$ uniquely, up to a change of the scale.

The condition of identical distributions of X_1, X_2, X_3 may be omitted. The distribution functions $F_k(x)$ of the independent positive random variables $X_k (k = 1, 2, 3)$ are uniquely characterized (by some assumptions) by the joint distribution of the pair of qoutients (5), up to a change of the scale.

The normal distribution is also characterized in the same way. There are also some generalizations of these problems.

1. Some lemmas.

LEMMA 1. *Let X_1, X_2, X_3 be three independent real random variables, and let*

$$Z_1 = X_1 - X_3,\ Z_2 = X_2 - X_3 \,. \tag{6}$$

If the characteristic function of the pair (Z_1, Z_2) does not vanish, then the distribution of (Z_1, Z_2) determines the distributions of X_1, X_2, X_3 up to a change of the location.

Proof. Denote $\varphi(t_1, t_2)$ the characteristic function of the pair (Z_1, Z_2), and $\varphi_k(t)$ the characteristic functions of X_k (k = 1, 2, 3). Then there is

$$\begin{aligned} \varphi(t_1, t_2) &= E\exp[i(t_1Z_1 + t_2Z_2) \\ &= E\exp[i[t_1(X_1 - X_3) + t_2(X_2 - X_3)]] \\ &= E\exp[i[t_1X_1 + t_2X_2 + (-t_1 - t_2)X_3]] \\ &= \varphi_1(t_1)\varphi_2(t_2)\varphi_3(-t_1 - t_2) \,. \end{aligned} \tag{7}$$

The condition of nonvanishing of $\varphi(t_1, t_2)$ is equivalent to nonvanishing of any of the functions $\varphi_k(t)$(k = 1, 2, 3).

Let U_1, U_2, U_3 be another three independent real random variables, having characteristic functions $\psi_k(t) = E\,e^{itU_k}$ $(k = 1, 2, 3)$ and satisfying the assumptions of Lemma 1. Let $V_1 = U_1 - U_3$, $V_2 = U_2 - U_3$ and $\psi(t_1, t_2) = E \exp[i(t_1 V_1 + t_2 V_2)]$. Making the same considerations as for formula (7) it is easy to see that there is

$$\psi(t_1, t_2) = \psi_1(t_1)\psi_2(t_2)\psi_3(-t_1 - t_2)\,. \tag{8}$$

Let the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution. Then their characteristic functions are equal. Using formulae (7) and (8) one can obtain the following equation

$$\psi_1(t_1)\psi_2(t_2)\psi_3(-t_1 - t_2) = \varphi_1(t_1)\varphi_2(t_2)\varphi_3(-t_1 - t_2) \quad (-\infty < t_1 < +\infty,\ -\infty < t_2 < +\infty)\,. \tag{9}$$

Put

$$\psi_k(t) = \varphi_k(t)\cdot p_k(t) \qquad (k = 1, 2, 3)\,. \tag{10}$$

Putting (10) into the equation (9) one obtains the following equation

$$p_1(t_1)p_2(t_2)p_3(-t_1 - t_2) = 1 \begin{pmatrix} -\infty < t_1 < +\infty \\ -\infty < t_2 < +\infty \end{pmatrix} \tag{11}$$

in which $p_k(t)$ are unknown functions; they are complex functions, continuous on the whole line $-\infty < t < +\infty$, satisfying the condition

$$p_k(0) = 1 \qquad (k = 1, 2, 3)\,. \tag{12}$$

In order to solve the equation (11) let us put $t_1 = t$, $t_2 = 0$ and, later $t_2 = t$, $t_1 = 0$. Then, using (12),

$$p_1(t)\cdot p_3(-t) = 1, \qquad p_2(t)\cdot p_3(-t) = 1\,. \tag{13}$$

Putting $p_1(t)$ and $p_2(t)$ given by (13) into the equation (11) and changing signs, we obtain

$$p_3(t_1 + t_2) = p_3(t_1)\cdot p_3(t_2)\,. \tag{14}$$

The only continuous function which satisfies the equation (14) and condition (12) is the exponential function

$$p_3(t) = e^{ct} \qquad (-\infty < t < +\infty) \tag{15}$$

where c is a complex number. Putting (15) into (13) it is easy to see that

$$p_1(t) = p_2(t) = p_3(t) = e^{ct}\,. \tag{16}$$

Putting (16) into (10) there is

$$\psi_k(t) = e^{ct}\varphi_k(t) \qquad (k = 1, 2, 3)\,. \tag{17}$$

Since the known property of characteristic functions $\varphi(-t) = \overline{\varphi(t)}$ the formulae (17) become

$$\psi_k(t) = e^{ibt}\varphi_k(t) \qquad (k = 1, 2, 3) \tag{18}$$

where b is a real constant. This means that the distributions of X_k are the same as of U_k up to a change of the location.

Remark 1. The assumption of nonvanishing of the joint characteristic function of the differences (6) may be replaced by the assumption that all X_k have analytic characteristic functions. All considerations are valid in such a case for t complex, being inside a circle $|t| < t_0$ $(t_0 > 0)$, where the characteristic functions do not vanish. Because of the analyticity of the characteristic functions the formulae (18) may be spread on the whole real line.

Remark 2. If the assumption of nonvanishing of the characteristic functions is omitted then the theorem becomes false. In order to show this the following example is given.

Examplė. Let (X_1, X_2, X_3) and (U_1, U_2, U_3) be two three's of independent real random variables having their characteristic functions $\varphi_k(t)$ and $\psi_k(t)$ respectively. Let be

$$\begin{cases} \varphi_1(t) = \varphi_2(t) = \psi_1(t) = \psi_2(t) = \begin{cases} 0 & \text{for } |t| > 1 \\ 1 - |t| & \text{for } |t| \leqq 1 \end{cases} \\ \varphi_3(t) = \begin{cases} 0 & \text{for } |t| > 2 \\ 1 - |t|/2 & \text{for } |t| \leqq 2 \end{cases} \\ \psi_3(t) = \varphi_3(t) \text{ for } |t| \leqq 2,\ \psi_3(t+4) = \psi_3(t)\,. \end{cases} \tag{19}$$

It is easy to see that in such a case the equation (9) holds though $\psi_3(t)$ is not the same as $\varphi_3(t)$. Hence there exist trios (X_1, X_2, X_3) and (U_1, U_2, U_3) of independent real random variables whose distributions are not the same, but where the distributions of the corresponding pairs of differences $(X_1 - X_3, X_2 - X_3)$, $(U_1 - U_3, U_2 - U_3)$ are the same.

Remark 3. Lemma 1 remains true if the three independent real random variables (X_1, X_2, X_3) are replaced by n such random variables $(X_1, X_2, \cdots, X_n)(n \geqq 3)$, and the pair of differences $(X_1 - X_3, X_2 - X_3)$ by $n - 1$ differences $(X_1 - X_n, X_2 - X_n, \cdots, X_{n-1} - X_n)$.

Remark 4. The differences in formula (6) may be replaced by sums.

Remark 5. Lemma 1 remains true if the real random variables X_k are replaced by n-dimensional real random vectors.

LEMMA 2. *Let X_1, X_2, X_3 be three independent positive random variables, and let (Y_1, Y_2) be their quotients given by formula* (5). *If the joint characteristic function of the pair* $(\ln Y_1, \ln Y_2)$ *does not vanish, then the distribution of (Y_1, Y_2) determines the distributions of X_1, X_2, X_3 up to change of the scale.*

Proof. The proof of Lemma 2 is obvious because $\ln X_k$ $(k = 1, 2, 3)$ satisfy the assumptions of Lemma 1.

REMARK 6. The positive random variables X_k in Lemma 2 may be replaced by symmetrical about the origin real random variables satisfying the condition $P(X_k = 0) = 0$.

2. Characterizing the gamma distribution. The problem of characterizing the gamma distribution of two independent random variables X_1, X_2 by the distribution of their quotient was first posed by J. G. Mauldon [9], he showed that there is no such characterization. Further investigations on this problem were made by I. Kotlarski [1]. A full treatment of the problem has been made by R. G. Laha [6] (on this subject see also E. Lukacs, R. G. Laha [7], p. 59). The authors of [1] and [6] searched for the properties of the set of distribution functions $F(x)$ for which the quotient (2) is distributed according to (3), where X_1, X_2 are independent positive random variables identically distributed according to $F(x)$.

THEOREM 1. *Let X_1, X_2, X_3 be three independent positive random variables, let (Y_1, Y_2) be the pair of the quotients given by* (5). *The necessary and sufficient condition for X_k to be gamma distributed with parameters p_k and a (a – common, $k = 1, 2, 3$) is that the joint distribution of (Y_1, Y_2) is the bivariate beta distribution of the second kind given by the density*

$$g(y_1, y_2) = \begin{cases} \dfrac{\Gamma(p_1 + p_2 + p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} \cdot \dfrac{y_1^{p_1-1} y_2^{p_2-1}}{(1 + y_1 + y_2)^{p_1+p_2+p_3}} & \text{for} \quad \begin{matrix} y_1 > 0 \\ y_2 > 0 \end{matrix} \\ 0 & \textit{elsewhere} \end{cases} \tag{20}$$

Proof. The characteristic function of $\ln X_k$ where X_k is gamma distributed with parameters p_k, a, is given by

$$\varphi_k(t) = Ee^{it\ln X_k} = a^{-it}\frac{\Gamma(p_k + it)}{\Gamma(p_k)}. \tag{21}$$

The characteristic function of the pair ($\ln Y_1$, $\ln Y_2$) is

$$\begin{aligned}\varphi(t_1, t_2) &= \varphi_1(t_1)\varphi_2(t_2)\varphi_3(-t_1 - t_2) \\ &= \frac{\Gamma(p_1 + it_1)}{\Gamma(p_1)} \cdot \frac{\Gamma(p_2 + it_2)}{\Gamma(p_2)} \cdot \frac{\Gamma(p_3 - it_1 - it_2)}{\Gamma(p_3)}\end{aligned} \tag{22}$$

it is easy to see that the characteristic function of ($\ln Y_1$, $\ln Y_2$) where (Y_1, Y_2) is distributed according to the density (20) is also given by the right side of the formula (22), This ends the proof.

THEOREM 2. *Let X_1, X_2, X_3 be three independent positive random variables, let (U_1, U_2) be given by formulae*

$$U_1 = \frac{X_1}{X_1 + X_2}, \qquad U_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3} \tag{23}$$

The necessary and sufficient condition for X_k to be gamma distributed with parameters p_k and a (a – common, $k = 1, 2, 3$) is that U_1, U_2 are independent beta distributed random variables, U_1 with parameters (p_1, p_2), and U_2 with parameters $(p_1 + p_2, p_3)$.

Proof. The necessary condition is obvious. In order to prove the sufficient condition let us put

$$U_1 = \frac{Y_1}{Y_1 + Y_2}, \qquad U_2 = \frac{Y_1 + Y_2}{1 + Y_1 + Y_2} \tag{24}$$

where Y_1 and Y_2 are given by (5). It is easy to see that if U_1, U_2 are independent, U_1 being distributed accoring to the beta law of the first kind with the density

$$g_1(u) = \begin{cases} \dfrac{1}{B(p_1, p_2)} u^{p_1-1}(1 - u)^{p_2-1} & \text{for } 0 < u < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{25}$$

and U_2 has a similar distribution with parameters $(p_1 + p_2, p_3)$, then (Y_1, Y_2) is distributed according to the density (20). Using Theorem 1 it is easy to see that X_k are gamma distributed with parameters (p_k, a), (a – common, $k = 1, 2, 3$). This ends the proof.

3. Characterizing the normal distribution. The problem of characterizing the normal distribution of two independent symmetrical about the origin random variables by the distribution of their quotient has been considered by several authors. J. G. Mauldon [9] showed that there is no such characterization. Further investigation on this problem were made by R. G. Laha [3, 4, 5], I. Kotlarski [2], G. P. Steck [10].

Though there is no characterization of two independent symmetrical about the origin random variables by the distribution of their quotient, there is a characterization of three such random variables by the joint distribution of two their quotients. In this Section two theorems on characterizing the normal distribution in such a way are presented.

THEOREM 3. *Let X_1, X_2, X_3 be three independent real symmetrical about the origin random variables satisfying the condition $P(X_k = 0) = 0$, $(k = 1, 2, 3)$. Let (Y_1, Y_2) be the pair of quotients given by formula* (5). *The necessary and sufficient condition for X_k to be normal distributed with a common standard deviation σ is that the joint distribution of (Y_1, Y_2) is the bivariate Cauchy distribution given by the density*

$$g(y_1, y_2) = \frac{1}{2\pi} \cdot \frac{1}{(1 + y_1^2 + y_2^2)^{3/2}} \quad \begin{pmatrix} -\infty < y_1 < +\infty \\ -\infty < y_2 < +\infty \end{pmatrix} \tag{26}$$

Proof. The characteristic function of $\ln |X_k|$ where X_k is normal distributed with zero mean and standard deviation σ is

$$\varphi_k(t) = E \exp it \ln |X_k| = (\sigma\sqrt{2})^{it} \frac{\Gamma((1 + it)/2)}{\sqrt{\pi}} \tag{27}$$

The characteristic function of the pair $(\ln |Y_2|, \ln |Y_2|)$ is

$$\begin{aligned} \varphi(t_1, t_2) &= \varphi_1(t_1)\varphi_2(t_2)\varphi_3(-t_1 - t_2) \\ &= \frac{1}{\pi^{3/2}} \Gamma\left(\frac{1 + it_1}{2}\right)\Gamma\left(\frac{1 + it_2}{2}\right)\Gamma\left(\frac{1 - i(t_1 + t_2)}{2}\right). \end{aligned} \tag{28}$$

It is easy to see that the characteristic function of $(\ln |Y_1|, \ln |Y_2|)$ where $(Y_1\ Y_2)$ is distributed according to the density (26) is also given by the right side of the formula (28). This ends the proof.

THEOREM 4. *Let X_1, X_2, X_3 be three independent real symmetrical about the origin random variables satisfying the condition $P(X_k = 0) = 0$ $(k = 1, 2, 3)$. Denote*

$$V_1 = \frac{X_1}{\sqrt{X_1^2 + X_2^2}}, \qquad V_2 = \frac{\sqrt{X_1^2 + X_2^2}}{\sqrt{X_1^2 + X_2^2 + X_3^2}} \tag{29}$$

The necessary and sufficient condition for X_k to be normal distri-

buted with a common standard deviation is that V_1 and V_2 are independent distributed according to the densities

$$(30)\qquad \begin{aligned} h_1(v) &= \begin{cases} \dfrac{1}{\pi}\cdot\dfrac{1}{\sqrt{1-v^2}} & \text{for } |v|<1 \\ 0 & \text{elsewhere} \end{cases} \\ h_2(v) &= \begin{cases} \dfrac{v}{\sqrt{1-v^2}} & \text{for } 0<v<1 \\ 0 & \text{elsewhere}\,. \end{cases} \end{aligned}$$

Proof. The necessary condition is obvious. In order to prove the sufficient condition let us put

$$(31)\qquad V_1 = \frac{Y_1}{\sqrt{Y_1^2+Y_2^2}}\,, \qquad V_2 = \frac{\sqrt{Y_1^2+Y_2^2}}{\sqrt{1+Y_1^2+Y_2^2}}$$

where Y_1, Y_2 are given by formula (5). It is easy to see that if (V_1, V_2) are independently distributed according to (30), then (Y_1, Y_2) is distributed according to the density (26). Using Theorem 3 it is easy to see that all X_k are normal distributed with a common standard deviation. This ends the proof.

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Received February 26, 1965.

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