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A HELLINGER INTEGRAL REPRESENTATION FOR BOUNDED LINEAR FUNCTIONALS

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The function space considered is that consisting of the complex-valued, quasicontinuous functions on a real interval $[a, b]$, anchored at a, and having the LUB norm. It is shown that each bounded linear functional on this Banach space has a Hellinger integral representation. A formula for the norm of the functional is given in terms of the integrating functions involved in its representation. A new existence criterion for the Hellinger integral is uncovered on the way to the representation theorem.

2. **Definitions.** In this section certain definitions and notational conventions are adopted for use in the succeeding sections. Throughout the paper, $[a, b]$ will denote a given interval and the word function will mean map from $[a, b]$ into the complex numbers.

DEFINITION 2.1. If c is any number in $(a, b]$, then R_c denotes a function such that $R_c(t) = 0$ if t is in $[a, c)$ and $R_c(t) = 1$ if $c \leqq t \leqq b$. If c is in $[a, b)$, then L_c denotes a function such that $L_c(t) = 0$ if $a \leqq t \leqq c$ and $L_c(t) = 1$ is t is in $(c, b]$. The functions L_c and R_c are called unit step functions. A linear combination of unit step functions is called a step function. Notice that each step function vanishes at a.

DEFINITION 2.2. We now specify the function space, $Q_0[a, b]$, which plays the central role. Its elements are the quasicontinuous functions anchored at a and they may be defined in two ways. First, $Q_0[a, b]$ is the set of all functions which vanish at a and which have a limit from the right at each t in $[a, b)$ and a limit from the left at each t in $(a, b]$. Second, let $B[a, b]$ be the Banach space of bounded functions, with LUB norm. Then $Q_0[a, b]$ is the closure, in $B[a, b]$, of the linear space of all step functions. So $Q_0[a, b]$ is a Banach space with norm $\|x\| = LUB\,|x(t)|$ for all t in $[a, b]$. Also, each bounded linear functional on $Q_0[a, b]$ is determined by its values on the step functions, since the latter form a dense linear subspace.

For proof of the equivalence of these two formulations of $Q_0[a, b]$, see [1, Lemma 4.16].

DEFINITION 2.3. Suppose g is any subset of $[a, b]$. If x is a function, then x_g denotes a function such that $x_g(t) = x(t)$ if t is in g

and $x_g(t) = 0$ if t is in $[a, b]$ but not in g. If F is a linear functional defined on $Q_0[a, b]$ and it is true that x_g is in $Q_0[a, b]$ for each x in $Q_0[a, b]$, then F_g denotes a linear functional such that $F_g(x) = F(x_g)$ for each x in $Q_0[a, b]$.

DEFINITION 2.4. "v has bounded slope variation with respect to u" means that v is a function, u is a real-valued, increasing function, and there exists a nonnegative number B such that if $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ with $n > 1$, then

$$\sum_{p=1}^{n-1} \left| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right| \leqq B .$$

The least such number B is denoted by $V_a^b(dv/du)$ and is called the slope variation of v with respect to u over $[a, b]$.

DEFINITION 2.5. Suppose each of u, v, and w is a function and u is increasing. "The Hellinger integral $\int_a^b dwdv/du$ exists" means that $\int_a^b dwdv/du$ is a number and for each positive number ε there exists a subdivision D of $[a, b]$ such that if $\{t_p\}_{p=0}^n$ is any refinement of D then

$$\left| \int_a^b \frac{dwdv}{du} - \sum_{p-1}^{n} \frac{[w(t_p(- w(t_{p-1})] \cdot [v(t_p) - v(t_{p-1})]}{u(t_p) - u(t_{p-1})} \right| < \varepsilon$$

Clearly, this integral has a unique value.

DEFINITION 2.6. If u is an increasing function and v is a function and c is in $[a, b)$ then "$D_u^+ v(c)$ exists" means that

$$\operatorname*{limit}_{t \to c+} \frac{v(t) - v(c)}{u(t) - u(c)}$$

exists and equals $D_u^+ v(c)$. The notation $D_u^- v(c)$ is used in a corresponding manner for numbers c in $(a, b]$.

3. **Lemmas.** This section contains results which are used in the proofs given for theorems in § 4.

LEMMA 3.1. *If n is an integer greater than 2 and $k_0, k_1, \cdots, k_n$ is a sequence of complex numbers and $e_1, e_2, \cdots, e_n$ is a sequence of positive real numbers then*

$$\sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_p}{e_{p+1}} - \frac{k_p - k_{p-1}}{e_p} \right|$$
$$\geqq \frac{1}{e_n} \left(\sum_{q=1}^{n} e_q \right) \left| \frac{k_n - k_0}{\sum_{q=1}^{n} e_q} - \frac{k_{n-1} - k_0}{\sum_{q=1}^{n-1} e_q} \right| + \sum_{p=1}^{n-2} \left| \frac{k_{p+1} - k_0}{\sum_{q=1}^{p+1} e_q} - \frac{k_p - k_0}{\sum_{q=1}^{p} e_q} \right|$$

Proof by induction. For the case $n = 3$,

$$\left|\frac{k_3 - k_2}{e_3} - \frac{k_2 - k_1}{e_2}\right| + \left|\frac{k_2 - k_1}{e_2} - \frac{k_1 - k_0}{e_1}\right|$$
$$= \left|\frac{k_3 - k_2}{e_3} - \frac{k_2 - k_1}{e_2}\right| + \frac{e_1}{e_2}\left|\frac{k_2 - k_0}{e_1 + e_2} - \frac{k_1 - k_0}{e_1}\right|$$
$$+ \left|\frac{k_2 - k_0}{e_1 + e_2} - \frac{k_1 - k_0}{e_1}\right| .$$

But by the triangle inequality, the sum of the first two terms of the right-hand member is greater than or equal to

$$\left|\frac{(k_3 - k_0) - (k_2 - k_0)}{e_3} - \frac{(k_2 - k_0) - (k_1 - k_0)}{e_2} + \frac{(k_2 - k_0)e_1}{e_2(e_1 + e_2)} - \frac{k_1 - k_0}{e_2}\right|$$
$$= \left|\frac{k_3 - k_0}{e_3} - \frac{(k_2 - k_0)(e_1 + e_2 + e_3)}{e_3(e_1 + e_2)}\right| .$$

Thus it may be seen that the conclusion is true for this case.

For the final step in the induction we begin by noting that

$$\sum_{p=1}^{n}\left|\frac{k_{p+1} - k_p}{e_{p+1}} - \frac{k_p - k_{p-1}}{e_p}\right| \geqq \sum_{p=1}^{n-1}\left|\frac{k_{p+1} - k_0}{\sum_{q=1}^{p+1} e_q} - \frac{k_p - k_0}{\sum_{q=1}^{p} e_q}\right|$$
$$+ \frac{1}{e_{n+1}}\cdot\left(\sum_{q=1}^{n+1} e_q\right)\left|\frac{k_{n+1} - k_0}{\sum_{q=1}^{n+1} e_q} - \frac{k_n - k_0}{\sum_{q=1}^{n} e_q}\right|$$

is true provided the last term of the left-hand member is greater than or equal to the sum of the last term of the right-hand member and

$$\left(1 - \frac{1}{e_n}\cdot\sum_{q=1}^{n} e_q\right)\left|\frac{k_n - k_0}{\sum_{q=1}^{n} e_q} - \frac{k_{n-1} - k_0}{\sum_{q=1}^{n-1} e_q}\right| .$$

But this is true provided the sum of the last term of the left-hand member and

$$\frac{1}{e_n}\cdot\left(\sum_{q=1}^{n-1} e_q\right)\left|\frac{k_n - k_0}{\sum_{q=1}^{n} e_q} - \frac{k_{n-1} - k_0}{\sum_{q=1}^{n-1} e_q}\right|$$

is greater than or equal to the last term of the right-hand member. This last sum, is, by the triangle inequality, greater than or equal to

$$\left| \frac{(k_{n+1} - k_0) - (k_n - k_0)}{e_{n+1}} - \frac{(k_n - k_0) - (k_{n-1} - k_0)}{e_n} \right.$$

$$\left. + \frac{(k_n - k_0) \cdot \sum_{q=1}^{n-1} e_q}{e_n \cdot \sum_{q=1}^{n} e_q} - \frac{k_{n-1} - k_0}{e_n} \right|$$

$$= \left| \frac{k_{n+1} - k_0}{e_{n+1}} - \frac{(k_n - k_0) \cdot \sum_{q=1}^{n+1} e_q}{e_{n+1} \cdot \sum_{q=1}^{n} e_q} \right| .$$

Thus each of the inequalities is true. Hence Lemma 3.1.

LEMMA 3.2. *If n is an integer greater than 2 and $k_0, k_1, \cdots, k_n$ is a number sequence and $s_0, s_1, \cdots, s_n$ is an increasing real number sequence, then*

$$\sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_p}{s_{p+1} - s_p} - \frac{k_p - k_{p-1}}{s_p - s_{p-1}} \right| \geqq \sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_0}{s_{p+1} - s_0} - \frac{k_p - k_0}{s_p - s_0} \right| .$$

This result follows immediately from Lemma 3.1 by the transformation: $s_p - s_{p-1} = e_p$ for $p = 1, 2, \cdots, n$.

LEMMA 3.3. *If v has bounded slope variation with respect to u then $D_u^- v(t)$ exists for each t in $(a, b]$ and $D_u^+ v(t)$ exists for each t in $[a, b)$.*

Proof. Suppose c is in $[a, b)$ and $\text{limit}_{t \to c+} (v(t) - v(c))/(u(t) - u(c))$ does not exist. Then there exists a positive number ε such that if r is in (c, b) then there exists a number s in (c, r) for which

$$\left| \frac{v(r) - v(c)}{u(r) - u(c)} - \frac{v(s) - v(c)}{u(s) - u(c)} \right| \geqq \varepsilon .$$

It may be seen, then, that if n is an integer greater than 2 there exists an increasing number sequence $s_0, s_1, \cdots, s_n$ with $s_0 = c$ and each term in $[c, b]$ such that

$$\sum_{p=1}^{n-1} \left| \frac{v(s_{p+1}) - v(c)}{u(s_{p+1}) - u(c)} - \frac{v(s_p) - v(c)}{u(s_p) - u(c)} \right| \geqq (n-1)\varepsilon .$$

But from this inequality and Lemma 3.2 it follows that

$$\sum_{p=1}^{n-1} \left| \frac{v(s_{p+1}) - v(s_p)}{u(s_{p+1}) - u(s_p)} - \frac{v(s_p) - v(s_{p-1})}{u(s_p) - u(s_{p-1})} \right| \geqq (n-1)\varepsilon .$$

Since there exists an integer n for which $(n-1)\varepsilon > V_a^b(dv/du)$, this is a contradiction. Hence $D_u^+v(c)$ exists for each c in $[a, b)$. An argument similar to that just given shows that $D_u^-v(c)$ exists for each c in $(a, b]$. Hence Lemma 3.3.

LEMMA 3.4. *Suppose v has bounded slope variation with respect to u. If t is in $(a, b]$, then $\int_a^b dR_t dv/du$ exists and is equal to $D_u^-v(t)$. If t is in $[a, b)$, then $\int_a^b dL_t dv/du$ exists and is equal to $D_u^+v(t)$.*

This lemma follows readily from Lemma 3.3 and the observation that, in each of the two equations implied by Lemma 3.4, each approximant for the right-hand member is an approximant for the left-hand member.

LEMMA 3.5. *If v has bounded slope variation with respect to u then the functional F, given by*

$$F(x) = \int_a^b \frac{dxdv}{du},$$

is linear on its domain, the dv/du-integrable functions x, and these form a linear space.

Proof of lemma is not given.

LEMMA 3.6. *If S is a step function and v has bounded slope variation with respect to u then*

$$\int_a^b \frac{dSdv}{du} \quad \textit{exists.}$$

This lemma follows from Definition 2.5 and Lemmas 3.4 and 3.5.

LEMMA 3.7. *If a normed linear space A may be written as a direct sum $A = B \oplus C$ of two of its subspaces in such a way that*

$$\| a \| = \text{Max}\{\| Pr_1(a) \|, \| Pr_2(a) \|\}$$

for each a in A, then

$$\| F \| = \| F \circ Pr_1 \| + \| F \circ Pr_2 \|,$$

for each bounded linear functional F on A.

Proof of this lemma is not given.

LEMMA 3.8. *Suppose h is subset of $[a, b]$ and f and g are mutually exclusive subsets of h whose union is h. Suppose, moreover, that if x is any function in $Q_0[a, b]$, then each of x_f, x_g, and x_h is in $Q_0[a, b]$. If F is a bounded linear functional from $Q_0[a, b]$ then each of F_f, F_g, and F_h is a bounded linear functional and*

$$\|F_f\| + \|F_g\| = \|F_h\| \leqq \|F\| .$$

This lemma is a mere application of Lemma 3.7.

4. **Theorems.** In this section a representation for the bounded linear functionals on $Q_0[a, b]$ in terms of the Hellinger integral is developed and a formula for their norms is given.

THEOREM 4.1. *If x is in $Q_0[a, b]$ and v has bounded slope variation with respect to u, then $\int_a^b dxdv/du$ exists and*

$$\left|\int_a^b \frac{dxdv}{du}\right| \leqq \left\{V_a^b \frac{dv}{du} + |D_u^- v(b)|\right\} \|x\| .$$

Proof. Let $S_1, S_2, S_3, \cdots$ be a sequence of step functions such that $\|S_p - x\| < 1/p$ if p is a positive integer. Suppose n is an integer greater than 1, $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ and q is a positive integer. Then, using summation by parts,

$$\begin{aligned}
&\sum_{p=1}^{n} \frac{[S_q(t_p) - S_q(t_{p-1})]\,[v(t_p) - v(t_{p-1})]}{u(t_p) - u(t_{p-1})} \\
&\quad = -\sum_{p=1}^{n-1} S_q(t_p)\left\{\frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})}\right\} \\
&\qquad + S_q(t_n)\frac{v(b) - v(t_{n-1})}{u(b) - u(t_{n-1})} .
\end{aligned}$$

It is thus evident that the left-hand member of this equation is, in absolute value, less than or equal to

$$\|S_q\|\left\{V_a^b \frac{dv}{du} + \left|\frac{v(b) - v(t_{n-1})}{u(b) - u(t_{n-1})}\right|\right\} .$$

From this and Lemmas 3.3 and 3.6 one may conclude that

$$\left|\int_a^b \frac{dS_q dv}{du}\right| \leqq \|S_q\|\left\{V_a^b \frac{dv}{du} + |D_u^- v(b)|\right\}$$

(It is to be noted that this inequality holds true with S_q replaced by any other function in $Q_0[a, b]$ for which the integral exists). If m is an integer greater than q, then, since $\|S_q - S_m\| < 2/q$, it follows that

$$\left|\int_a^b \frac{d(S_q - S_m)dv}{du}\right| \leqq \frac{2}{q}\left\{V_a^b \frac{dv}{du} + |D_u^- v(b)|\right\}.$$

Consequently, the sequence

$$\left\{\int_a^b \frac{dS_q dv}{du}\right\}_{q=1}^{\infty}$$

is a Cauchy sequence and so has a sequential limit. Call this limit I. We now show that the approximants to $\int_a^b dxdv/du$ tend, under refinement, to I.

There exists a number B such that

$$V_a^b \frac{dv}{du} + \left|\frac{v(b) - v(t)}{u(b) - u(t)}\right| < B$$

for each t in $[a, b)$. Since $\| x - S_p \| < 1/p$ for $p = 1, 2, \cdots$, it follows that

$$\left|\sum_{i=1}^{m} \frac{\{x(s_i) - S_p(s_i) - [x(s_{i-1}) - S_p(s_{i-1})]\}[v(s_i) - v(s_{i-1})]}{u(s_i) - u(s_{i-1})}\right| < \frac{B}{p}$$

for any subdivision $\{s_i\}_{i=0}^m$ of $[a, b]$ and any positive integer p. For each positive integer p there exists a subdivision D_p of $[a, b]$ such that if $\{s_i\}_{i=0}^m$ is any refinement of D_p then

$$\sum_{i=1}^{m} \frac{[S_p(s_i) - S_p(s_{i-1})][v(s_i) - v(s_{i-1})]}{u(s_i) - u(s_{i-1})} - \int_a^b \frac{dS_p dv}{du}\Bigg| < \frac{B}{p}.$$

Since

$$\left|\int_a^b \frac{dS_p dv}{du} - I\right| \leqq \frac{2B}{p} \qquad \text{for } p = 1, 2, \cdots$$

it follows that, for each positive integer p,

$$\left|\sum_{i=1}^{m} \frac{[x(s_i) - x(s_{i-1})][v(s_i) - v(s_{i-1})]}{u(s_i) - u(s_{i-1})} - I\right| < \frac{4B}{p}$$

provided $\{s_i\}_{i=0}^m$ is a refinement of D_p. Hence $\int_a^b dxdv/du$ exists and its value is I. That the integral satisfies the inequality of the conclusion may be seen from the parenthetical note above. Hence Theorem 4.1.

THEOREM 4.2. *Suppose v has bounded slope variation with respect to u and F is the functional defined by*

$$F(x) = \int_a^b \frac{dxdv}{du}$$

for each x in $Q_0[a, b]$. Then F is a bounded linear functional whose norm is $V_a^b(dv/du) + |D_u^- v(b)|$.

Proof. It is clear from Lemma 3.5 and Theorem 4.1 that F is linear and bounded and that the norm of F does not exceed $V_a^b(dv/du) + |D_u^- v(b)|$. We now construct a function z in $Q_0[a, b]$ such that $\|z\| = 1$ and $F(z)$ equals the sum of $|D_u^- v(b)|$ and the approximant for $V_a^b(dv/du)$ corresponding to a preassigned subdivision of $[a, b]$.

Suppose $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ with $n > 1$. Define d_p, for $p = 1, 2, \cdots, (n-1)$, by

$$d_p = \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})}$$

if this expression is not zero and $d_p = 1$ if the expression is zero. For $p = 1, 2, \cdots, (n-1)$, let z_p be a function such that

$$z_p(t) = \begin{cases} -\dfrac{u(t) - u(t_{p-1})}{u(t_p) - u(t_{p-1})} \cdot \dfrac{|d_p|}{d_p} & \text{for } t \text{ in } [t_{p-1}, t_p] \\ -\dfrac{u(t_{p+1}) - u(t)}{u(t_{p+1}) - u(t_p)} \cdot \dfrac{|d_p|}{d_p} & \text{for } t \text{ in } [t_p, t_{p+1}] \\ 0 \quad \text{for } t \text{ in } [a, b] \text{ but not in } [t_{p-1}, t_{p+1}] . \end{cases}$$

If $D_u^- v(b) = 0$, let $z_n = R_b$. If $D_u^- v(b) \neq 0$ let $z_n = (D_u^- v(b)/|D_u^- v(b)|)R_b$. Finally, let $z = \sum_{p=1}^n z_p$.

Each of $z, z_1, z_2, \cdots, z_n$ is in $Q_0[a, b]$ and it may be verified that

$$\int_a^b \frac{dz_p dv}{du} = \left[\frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})}\right] \frac{|d_p|}{d_p}$$

for $p = 1, 2, \cdots, (n-1)$ and $\int_a^b (dz_n dv/du) = |D_u^- v(b)|$. Hence,

$$\int_a^b \frac{dzdv}{du} = \sum_{p=1}^{n-1} \left| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right| + |D_u^- v(b)|$$

If t is in $[a, t_1]$, then

$$|z(t)| = \left| -\frac{u(t) - u(a)}{u(t_1) - u(a)} \cdot \frac{|d_1|}{d_1} \right| \leqq 1 .$$

If t is in $[t_{n-1}, b)$, then

$$|z(t)| = \left| -\frac{u(b) - u(t)}{u(b) - u(t_{n-1})} \cdot \frac{|d_{n-1}|}{d_{n-1}} \right| \leqq 1 .$$

If p is one of $1, 2, \cdots, (n-2)$ and t is in $[t_p, t_{p+1}]$ then

$$|z(t)| = \left| -\frac{u(t) - u(t_p)}{u(t_{p+1}) - u(t_p)} \frac{|d_{p+1}|}{d_{p+1}} - \frac{u(t_{p+1}) - u(t)}{u(t_{p+1}) - u(t_p)} \frac{|d_p|}{d_p} \right| \leqq 1 .$$

And $|z(b)| = 1$. Hence $\|z\| = 1$.

It may be inferred from the foregoing that the norm of F is not less than $V_a^b(dv/du) + |D_u^- v(b)|$. Hence Theorem 4.2.

THEOREM 4.3. *If F is a bounded linear functional from $Q_0[a, b]$ then there exist two functions u and v, with v having bounded slope variation with respect to u, such that*

$$F(x) = \int_a^b \frac{dxdv}{du}$$

for each x in $Q_0[a, b]$.

Proof. Suppose c is in $(a, b]$. If r and s are numbers such that $a < r < s < c$, then, by Lemma 3.8, $\|F_{(r,c)}\| \geqq \|F_{(s,c)}\| \geqq 0$. Consequently, $\text{limit}_{t \to c-} \|F_{(t,c)}\|$ exists. Let λ denote the function such that $\lambda(c) = \text{limit}_{t \to c-} \|F_{(t,c)}\|$ for each number c in $(a, b]$ and $\lambda(a) = 0$. Similarly, let ρ denote the function such that $\rho(c) = \text{limit}_{t \to c+} \|F_{(c,t)}\|$ for each c in $[a, b)$ and $\rho(b) = 0$.

Now it may be seen from the definition of λ and Lemma 3.8 that if $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$, then

$$\sum_{p=0}^n \lambda(t_p) \leqq \|F\| .$$

A similar statement is true of ρ. Thus there exists a countable subset M of $[a, b]$ such that if t is in $[a, b]$ but not in M then $\lambda(t) = \rho(t) = 0$.

Let u denote an increasing function such that (1) if t is in (a, b) and $\lambda(t) > 0$, then $u(t) - u(t-) > 0$, and (2) if t is in $[a, b)$ and $\rho(t) > 0$, then $u(t+) - u(t) > 0$. For each t in $[a, b]$ let u_t denote the function such that $u_t(s) = 0$ for $a \leqq s \leqq t$ and $u_t(s) = u(s) - u(t)$ for $t \leqq s \leqq b$. Let v denote the function such that $v(t) = -F(u_t)$ for each t in $[a, b]$.

Suppose $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ and $n > 1$. Then, by the definition of v and the linearity of F there exists a number sequence $\{d_p\}_{p=1}^{n-1}$, with $|d_p| = 1$ for $p = 1, 2, \cdots, (n-1)$, such that

$$\sum_{p=1}^{n-1} \left| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right|$$
$$= F\left(\sum_{p=1}^{n-1} \left[\frac{u_{t_{p+1}} - u_{t_p}}{u(t_{p+1}) - u(t_p)} - \frac{u_{t_p} - u_{t_{p-1}}}{u(t_p) - u(t_{p-1})} \right] d_p \right) .$$

It may be verified that the norm of the function which is the argument of F in the right-hand member of the equation is 1. Consequently the left-hand member is less than or equal to $\|F\|$. Thus it may be inferred that v has bounded slope variation with respect to u.

Let G denote the bounded linear functional such that

$$G(x) = \int_a^b \frac{dxdv}{du}$$

for each x in $Q_0[a, b]$. Suppose c is in $(a, b]$. By Lemma 3.4

$$G(R_c) = D_u^- v(c)$$

$$G(R_c) = \operatorname*{limit}_{t \to c-} \frac{v(c) - v(t)}{u(c) - u(t)}$$

$$= \operatorname*{limit}_{t \to c-} F\left(\frac{u_t - u_c}{u(c) - u(t)}\right).$$

For t in (a, c), one has

$$\left|\frac{u_t(s) - u_c(s)}{u(c) - u(t)} - R_c(s)\right| \leqq \begin{cases} 0 \text{ if } s \text{ is in } [a, t] \\ \dfrac{u(c-) - u(t)}{u(c) - u(t)} \text{ if } s \text{ is in } (t, c) \\ 0 \text{ if } c \leqq s \leqq b \end{cases}$$

so that

$$\left|\frac{v(c) - v(t)}{u(c) - u(t)} - F(R_c)\right| = \left|F_{(t,c)}\left(\frac{u_t - u_c}{u(c) - u(t)} - R_c\right)\right|$$

$$\leqq \|F_{(t,c)}\| \cdot \frac{u(c-) - u(t)}{u(c) - u(t)} \leqq \|F_{(t,c)}\|.$$

Now $\operatorname{limit}_{t \to c-} \|F_{(t,c)}\| = \lambda(c)$. But if $\lambda(c) > 0$, then $u(c) - u(c-) > 0$ so that

$$\operatorname*{limit}_{t \to c-} \frac{u(c-) - u(t)}{u(c) - u(t)} = 0.$$

So, whether $\lambda(c)$ is positive or zero, one has that

$$\operatorname*{limit}_{t \to c-} \left|\frac{v(c) - v(t)}{u(c) - u(t)} - F(R_c)\right| = 0.$$

Hence $F(R_c) = G(R_c)$ for each c in $(a, b]$. A similar argument shows that $F(L_c) = G(L_c)$ for each c in $[a, b)$. Therefore $F(S) = G(S)$ for every step function S. Thus, $F = G$. Hence Theorem 4.3. Clearly, the norm of F is given by the expression appearing in Theorem 4.2.

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