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A NOTE ON PRINCIPAL FUNCTIONS AND MULTIPLY-VALENT CANONICAL MAPPINGS

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L. Sario has constructed principal analytic functions on planar bordered Riemann surfaces by applying the method of linear operators to certain sets of singularity functions. Weakly λ-valent principal functions result from a similar construction, starting with singularity functions having flux equal to integral multiples of 2π. In fact, such λ-valent maps are characterized as integral powers of principal analytic functions already mentioned.

L. Sario has used linear operators to establish the existence of certain canonical mappings of planar bordered Riemann surfaces $\bar{W}$ onto slit disks [4]. These mappings $F_0(z)$ and $F_1(z)$, called principal analytic functions, are formed from principal harmonic functions, themselves constructed by applying the linear operator method of [5] to systems of singularity functions defined near certain point sets of $\bar{W}$. In particular, near γ, the border of $\bar{W}$, the singularity function $s_\gamma(z)$, which is constant on γ with flux 2π there, is chosen, while near ζ, a point of the surface $W = \bar{W} - \gamma$, the singularity function $s_\zeta(z) = \log|z - \zeta|$ is selected. By exhausting the planar bordered surface $\bar{W}$, one constructs the mappings $F_0(z)$ and $F_1(z)$ of $\bar{W}$ onto a plane disk, with radial or circular slits, possibly degenerate. It is easily established that, for $i = 0, 1$, $\Delta_\gamma(\arg F_i(z))$ is 2π, the flux on γ of the singularity function $s_\gamma(z)$, and that each $F_i(z)$ has a first order zero at $z = \zeta$. These conditions are easily seen to be a consequence of the selection of the singularity functions $s_\gamma(z)$ and $s_\zeta(z)$.

In this note, we investigate the nature of "canonical" maps $F_0^\lambda(z)$ and $F_1^\lambda(z)$ which result from starting with singularity functions $s_\gamma^\lambda(z)$ near γ and $s_\zeta^\lambda(z)$ near ζ. Here, $s_\gamma^\lambda(z)$ is constant for $z \in \gamma$ with flux $\int_\gamma ds_\gamma^{\lambda *} = 2\pi\lambda$ while $s_\zeta^\lambda(z) = \lambda \log|z - \zeta|$. If an approximation process similar to that of [4] is applied, canonical maps $F_0^\lambda(z)$ and $F_1^\lambda(z)$ result. Because $\Delta_\gamma \arg F_i(z) = \int_\gamma ds_\gamma^{\lambda *} = 2\pi\lambda$, it follows that the mappings $F_0^\lambda(z)$ and $F_1^\lambda(z)$ are λ-valent, at least near γ. Also, at the point ζ of W, these mappings have a λ-th order zero, and hence are λ-valent near ζ as well. It is then reasonable to ask whether the functions $F_0^\lambda(z)$ and $F_1^\lambda(z)$, with radial and circular slit behavior near the ideal boundary, are *globally* λ-valent in some sense.

For a bordered Riemann surface $\bar{V}$ with two border components δ and γ, constructions similar to those of [3], starting with singularity functions $s_\delta^\lambda(z)$ and $s_\gamma^\lambda(z)$, lead to similar questions concerning the nature of λ-valent mappings of $\bar{V}$ onto a slit annulus. The purpose of this note then, is to determine the geometric nature of multiply-valent canonical mappings based on the constructions already outlined. In terms of principal functions already known, we shall be able to establish a classification of such mappings based on the concept of weak λ-valence.

2. The λ-th principal analytic functions. We consider first an exhausting set of bordered surfaces $\{\bar{W}_n\}$, each of which has γ as one of its border components, and has remaining border components denoted $\beta_1, \cdots, \beta_{k(n)}$. On every $\bar{W}_n$, we construct λ-th principal analytic functions $F_{0n}^\lambda(z)$ and $F_{1n}^\lambda(z)$ such that (i) $|F_{in}^\lambda(z)| = \text{constant} = r_\gamma(F_{in}^\lambda)$ for $z \in \gamma$ with $\Delta_\gamma(\arg F_{in}^\lambda(z)) = 2\pi\lambda$, (ii) $F_{in}^\lambda(z)$ has a λ-th order zero at $z = \zeta$, and (iii) $F_{0n}^\lambda(z)$ $(F_{1n}^\lambda(z))$ maps each of $\bar{W}_n'$s remaining border components β_i onto a radial (circular) slit. Such mappings are constructed by selecting singularity functions $s_\gamma^\lambda(z)$ and $s_\zeta^\lambda(z)$ already defined in § 1, and selecting the singularity functions $s_{\beta_i}^\lambda(z)$ near β_i in the manner of [4]. For $i = 0, 1$, the functions $F_{in}^\lambda(z)$ result from an application of the linear operator method [5], and these are normalized by the condition $\lim_{z\to\zeta} F_{in}^\lambda(z)/(z-\zeta)^\lambda = 1$. The families $\{F_{0n}^\lambda(z)\}$ and $\{F_{1n}^\lambda(z)\}$ are normal, and the resulting limits $F_0^\lambda(z)$ and $F_1^\lambda(z)$ are called *λ-th principal analytic functions* on $\bar{W}$. It now seems reasonable to expect that these mappings are weakly λ-valent in the following sense.

DEFINITION. The mapping $F(z)$ is called *weakly λ-valent* if, for each $w \in F(\bar{W})$, the set $F^{-1}(w)$ consists of at most λ points $z \in \bar{W}$, and for some $w \in F(\bar{W})$, the set $F^{-1}(w)$ consists of exactly λ points. A weakly λ-valent mapping $F(z)$ of $\bar{W}$ into the point set S is called a *radial (circular) slit mapping of $\bar{W}$ into S* if each component of the set $\{w \in S; F^{-1}(w)$ contains at most $\lambda - 1$ points $z \in \bar{W}\}$ is a radial (circular) slit or point.

3. Properties of λ-th principal analytic functions. The following are some properties of the maps $F_0^\lambda(z)$ and $F_1^\lambda(z)$ which will prove useful.

(i) For $i = 0, 1$, the function $F_i^\lambda(z)$ has no zero on the surface $W - \zeta$.

(ii) If $\Delta(\zeta)$ is a parametric disc with boundary δ whose orientation is induced by $\Delta(\zeta)$, then for $i = 0, 1$, $\int_\delta d(\arg F_i^\lambda(z)) = 2\pi\lambda$.

(iii) If σ is a cycle contained in $W - \zeta$, then for $i = 0, 1$, λ divides the integer $(1/2\pi)\int_\sigma d(\arg F_i^\lambda(z))$.

Proof of (i). If Z_n is the number of zeros of $F_i^\lambda(z)$ in $\bar{W}_n$, with border $\gamma + \bar{\beta}_n$, we apply the argument principle and find:

$$Z_n = \frac{1}{2\pi i}\int_{\gamma+\bar{\beta}_n} \frac{dF_i^\lambda(z)}{dz}\Big/ F_i^\lambda(z)dz = \lim_m \frac{1}{2\pi i}\int_{\gamma+\bar{\beta}_n} \frac{dF_{im}^\lambda(z)}{dz}\Big/ F_{im}^\lambda(z)dz$$
$$= \lim_m \frac{1}{2\pi}\int_{\gamma+\bar{\beta}_n} d(\arg F_{im}^\lambda(z)) = \lambda\ .$$

Hence $F_i^\lambda(z)$ is never zero on the surface $W - \zeta$.

Proof of (ii). In the parametric disk $\Delta(\zeta)$, we may write $F_i^\lambda(z) = (z-\zeta)^\lambda f_i(z)$, where $f_i(z)$ is never zero in $\overline{\Delta(\zeta)}$. Hence it follows that $\int_\delta d(\arg F_i(z)) = 2\pi\lambda$.

Proof of (iii). We let σ be an arbitrary cycle of $W - \zeta$, and choose n large enough so that $\sigma \subset \bar{W}_n$. If a parametric disk $\Delta(\zeta)$ is removed from $\bar{W}_n$, the bordered $\bar{W}_n - \Delta(\zeta)$ results. Hence there are integers a, b, and c_i such that σ is homologous (in $\bar{W}_n - \Delta(\zeta)$) to $a\gamma + b\delta + \sum_1^{k(n)} c_j\beta_j$; and $\int_\sigma d(\arg F_i^\lambda(z))$ may be written as

$$\begin{aligned}\int_\sigma d(\arg F_i^\lambda(z)) &= a\int_\gamma d(\arg F_i^\lambda(z)) + b\int_\delta d(\arg F_i^\lambda(z)) \\ &\quad + \sum_1^{k(n)} c_j \int_{\beta_j} d(\arg F_i^\lambda(z)) \\ &= 2\pi\lambda a - 2\pi\lambda b + \lim_m \sum_1^{k(n)} c_j \int_{\beta_j} d(\arg F_{im}^\lambda(z)) \\ &= 2\pi\lambda a - 2\pi\lambda b\ .\end{aligned}$$

Thus it follows that λ divides $(1/2\pi)\int_\sigma d(\arg F_i^\lambda(z))$.

The following theorem, characterizing the nature of $F_0^\lambda(z)$ and $F_1^\lambda(z)$, is our main result.

THEOREM 1. *The λ-th principal analytic function $F_0^\lambda(z)$ $(F_1^\lambda(z))$ is the λ-th power of the principal analytic function $F_0(z)$ $(F_1(z))$ of* [4].

Proof. We consider only the mapping $F_0^\lambda(z)$ because the arguments for $F_1^\lambda(z)$ are entirely analogous. According to property (iii) of this section, λ divides $(1/2\pi)\int_\sigma d(\arg F_0^\lambda(z))$ for each cycle σ in $W - \zeta$. Hence it follows from the theorem of the appendix that $F_0^\lambda(z)$ has an analytic λ-th root, say $G(z)$, in $W - \zeta$. But ζ is a removable singularity for $G(z)$, and we call $G(z)$, with $G(\zeta) = 0$, an analytic λ-th root of $F_0^\lambda(z)$ in W, and in fact in $\bar{W}$.

In the neighborhood $\varDelta$ of ζ, we have $F_0^\lambda(z) = (G(z))^\lambda = (z-\zeta)^\lambda H(z)$, where $H(\zeta)$ has been normalized to 1. If we let $\tilde{H}(z)$ be some analytic λ-th root of $H(z)$, then the set of functions $\{\tilde{H}(z)\exp 2\pi ik/\lambda; k = 0, 1, \cdots, \lambda-1\}$, according to the corollary in the appendix, represents all λ-th roots. But $(\tilde{H}(\zeta))^\lambda = 1$, hence one of the quantities $\tilde{H}(\zeta)\exp 2\pi ik/\lambda$ is 1. We assume that this occurs for $k=0$, that is, $\tilde{H}(\zeta)=1$, and we take $G(z)$, with $(G(z))^\lambda = F_0^\lambda(z)$, as that branch for which $\tilde{H}(\zeta) = 1$. In particular, in $\varDelta(\zeta)$, $G(z) = (z-\zeta)\tilde{H}(z)$, and $G'(\zeta) = 1$.

We claim now that the functional $\varPhi(G) = 2\pi \log r(G) - A(G)$ of [4] has the value $\varPhi(F_0)$, where F_0 is the univalent principal radial slit mapping of [4]. To see this, we need only compute the deviation $\varPhi(F_0) - \varPhi(G)$, and according to Theorem 3 of [4], this is only $D_{\bar{W}}(\log|G(z)/F_0(z)|)$. But $\log|G(z)/F_0(z)|$ has a removable singularity at ζ, hence we write

$$\begin{aligned}\lambda^2 D_{\bar{W}}\left(\log\left|\frac{G(z)}{F_0(z)}\right|\right) &= D_{\bar{W}}\left(\log\frac{|F_0^\lambda(z)|}{|F_0(z)|^\lambda}\right)\\ &= \int_\gamma \log\frac{|F_0^\lambda(z)|}{|F_0(z)|^\lambda}\, d\left(\arg\frac{F_0^\lambda(z)}{(F_0(z))^\lambda}\right)\\ &\quad + \int_\beta \frac{|F_0^\lambda(z)|}{|F_0(z)|^\lambda}\, d\left(\arg\frac{F_0^\lambda(z)}{(F_0(z))^\lambda}\right)\\ &= \log\frac{r_\gamma(F_0^\lambda)}{(r_\gamma(F_0))^\lambda}(2\pi\lambda - 2\pi\lambda) = 0\ .\end{aligned}$$

It now follows from reasoning similar to the proof of Theorem 3 in [4] that $G(z)$, and only $G(z)$, maximizes $\varPhi$ among analytic functions F satisfying (i) $F(z) = \text{const}$ for $z \in \gamma$ and $\int_\gamma d(\arg F(z)) = 2\pi$. (ii) $F(\zeta) = 0$ and $F'(\zeta) = 1$, and (iii) $\int_{\beta_i} d(\arg F(z)) = 0$. But $F_0(z)$ of [4] also uniquely maximizes $\varPhi$ in the same class of functions. Hence $F_0(z) = G(z)$, and this completes the proof of Theorem 1.

If we apply the corollary in the appendix, we find

COROLLARY 1. *The set of mappings* $\{F_0(z)\exp(2\pi i/\lambda)k; k = 0, 1, \cdots, \lambda-1\}$ *represents all analytic* λ*-th roots of* $F_0^\lambda(z)$. *Also, the set of mappings* $\{F_1(z)\exp(2\pi i/\lambda)k; k = 0, 1, \cdots, \lambda-1\}$ *represents all analytic* λ*-th roots of* $F_1^\lambda(z)$.

COROLLARY 2. *The mappings* $F_0^\lambda(z)$ *and* $F_1^\lambda(z)$ *are respectively weakly* λ*-valent radial and circular slit disk mappings of* $\bar{W}$.

COROLLARY 3. *For positive integers* λ *and* μ, *the relations*

$$F_i^{\lambda\mu}(z) = (F_i^{\lambda}(z))^{\mu} = (F_i^{\mu}(z))^{\lambda} = (F_i(z))^{\lambda\mu}$$

hold for $i = 0, 1$.

Similar results may be obtained for λ-th principal analytic functions defined on $\bar{V}$, a bordered Riemann surface with two border components γ and δ. Again the construction of such functions is suggested by a known construction in the univalent case [3]. One starts with singularity functions $s_\gamma^\lambda(z)$ and $s_\delta^\lambda(z)$ defined near γ and δ of the approximating bordered $\bar{V}_n$, and takes as further singularity functions $s_{\beta_i}^\lambda(z)$, the functions $s_{\beta_i}(z)$ of [3]. Here, $s_\gamma^\lambda(z)$ is constant on γ with flux $2\pi\lambda$, while $s_\delta^\lambda(z)$ is constant on δ with flux $-2\pi\lambda$. An application of the linear operator method [5] to each $\bar{V}_n$ establishes the existence of the normal families $\{F_{0n}^\lambda(z)\}$ and $\{F_{1n}^\lambda(z)\}$, all subject to the condition $F_{in}^\lambda(\zeta) = 1$. Principal λ-th analytic functions $F_0^\lambda(z)$ and $F_1^\lambda(z)$ now result upon taking limits on n. We state the following characterization of these functions in terms of the principal functions $F_0(z)$ and $F_1(z)$ of [3].

THEOREM 2. *The mapping* $F_0^\lambda(z)\,(F_1^\lambda z))$ *is the* λ*-th power of the univalent principal mapping* $F_0(z)\,(F_1(z))$ *of* [3].

Appendix. We state, without proof, a well known characterization of those analytic functions which, on an open planar Riemann surface W, have analytic λ-th roots. Since such a surface may be conformally embedded in the complex plane, standard techniques of complex analysis [1] may be employed.

THEOREM. *Let* $f(z)$ *be analytic and never zero on the open planar Riemann surface* W *and let* λ *be a positive integer. Then* $f(z)$ *has an analytic* λ*-th root in* W *if and only if, for each cycle* $\sigma \subset W$, λ *divides the integer* $(1/2\pi)\int_\sigma d(\arg f(z))$.

COROLLARY. *Let* $f(z)$ *be an analytic function which is never zero in* W. *If, for each cycle* $\sigma \subset W$, λ *divides the integer* $(1/2\pi)\int_\sigma d(\arg f(z))$, *then* $f(z)$ *has exactly* λ *analytic* λ*-th roots in* W.

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