PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 3, 1967

ON THE STONE-WEIERSTRASS APPROXIMATION THEOREM FOR VALUED FIELDS

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Let X be a compact topological space, L a non-Archimedean rank 1 valued field and $\mathfrak{F}$ a uniformly closed L-algebra of L-valued continuous functions on X. Kaplansky has shown that if $\mathfrak{F}$ separates the points of X, then either $\mathfrak{F}$ consists of all L-valued continuous functions on X or else all of them which vanish on one point in X. In this paper analogous results are obtained, in the case that a group of transformations acts both on X and L, for the invariant L-valued continuous functions on X.

If L and K are fields such that $L \subset K$ and L/K is normal, we let $\mathrm{Aut}\,(L/K)$ denote the group of automorphisms of L which leave every element of K fixed, and we give $\mathrm{Aut}\,(L/K)$ the Krull topology; a basis for the open neighborhoods of the identity of $\mathrm{Aut}\,(L/K)$ is given by subgroups of the form

$$\{\sigma \in \mathrm{Aut}\,(L/K) : \sigma x = x \text{ if } x \in L_1\}$$

where L_1 is a finite extension of K contained in L.

Now suppose that L is a non-Archimedean field with a (multiplicative) rank 1 valuation, denoted $|\ \ |$ [1]. Suppose K is a subfield of L such that L/K is both normal and separable. Denote by L_c a completion of L and let K' be the closure of K in L_c. Put $L' = LK'$ (the composite field generated by L and K' in L_c) and note that K is dense in K'. It is clear that L'/K' is normal and separable. If $\sigma \in \mathrm{Aut}\,(L'/K')$, then, since K' is complete, $|\sigma x| = |x|$ for each $x \in L'$ so that σ is a continuous map of L' onto itself; furthermore the restriction of σ to L, $\sigma|_L \in \mathrm{Aut}\,(L/K)$. Finally suppose that X is a compact topological space for which there exists a continuous map $(\sigma, x) \rightarrow \sigma x$ of $\mathrm{Aut}\,(L'/K') \times X \rightarrow X$ satisfying $\sigma_1(\sigma_2 x) = (\sigma_1\sigma_2)x$ if $\sigma_1, \sigma_2 \in \mathrm{Aut}\,(L'/K')$, $x \in X$ and satisfying $ex = x$ if e is the identity of $\mathrm{Aut}\,(L'/K')$ and $x \in X$. It is immediate that if $\sigma \in \mathrm{Aut}\,(L'/K')$ then the map $x \rightarrow \sigma x$ of $X \rightarrow X$ is a homeomorphism of X. We shall call a set $Y \subset X$ *invariant* if $\mathrm{Aut}\,(L'/K')Y = Y$. Denote by $C_{L/K}(X)$ the set of L-valued continuous functions f on X satisfying $f(\sigma x) = \sigma f(x)$ for all $x \in X$ and $\sigma \in \mathrm{Aut}\,(L'/K')$; $C_{L/K}(X)$ is a K-algebra. If E is any valued field, denote by $C_E(X)$ the continuous E-valued functions on X and give $C_E(X)$ the sup-norm topology. Clearly $C_L(X) \supset C_{L/K}(X) \supset C_K(X)$.

THEOREM 1. *Suppose $\mathfrak{F}$ is a closed (in the sup-norm) K-sub-*

algebra of $C_{L/K}(X)$ *which separates the points of* X (i.e. *if* $x, y \in X$ *and* $x \neq y$, *there exists* $f \in \mathfrak{F}$ *such that* $f(x) \neq f(y)$). *Then either* $\mathfrak{F} = C_{L/K}(X)$ *or there exists* $x_0 \in X$ *such that*

$$\mathfrak{F} = \{f \in C_{L/K}(X) : f(x_0) = 0\} .$$

In the latter case the set $\{x_0\}$ *is invariant.*

Proof. Let $\mathfrak{F}'$ be the uniform closure of the K' algebra of functions generated by $\mathfrak{F}$ in $C_{L'}(X)$; since K is dense in K', $\mathfrak{F}$ is dense in $\mathfrak{F}'$ and hence it suffices to prove that $\mathfrak{F}' = C_{L'/K'}(X)$ or that $\mathfrak{F}' = \{f \in C_{L'/K'}(X) : f(x_0) = 0\}$. Thus we may assume without loss of generality that $K = K'$ and $L = L'$. We assume first that for each $x \in X$, there exists $f \in \mathfrak{F}$ such that $f(x) \neq 0$

LEMMA 2. *Assuming the hypotheses of Theorem* 1, *if* $x_0 \in X$ *and* $g \in C_{L/K}(X)$, *there exists* $f \in \mathfrak{F}$ *such that* $f(x_0) = g(x_0)$.

Proof. Put $L_1 = \{h(x_0) : h \in \mathfrak{F}\}$; clearly L_1 is a K-subalgebra of L containing a nonzero element of L. Suppose $c \in L_1$ and $c \neq 0$; c satisfies a polynomial equation $\sum_{i=0}^{n} a_i c^i = 0$, where the $a_i \in K$ and $a_0 \neq 0$. Then $a_0 \in L_1$ and hence $K = K a_0 \subset L_1$. It follows that L_1 is a subfield of L. Put

$$H = \{\sigma \in \mathrm{Aut}\,(L/K) : \sigma x_0 = x_0\} ;$$

H is a closed subgroup of $\mathrm{Aut}\,(L/K)$ which fixes every element of L_1 and also fixes $g(x_0)$. Now if $\sigma \in \mathrm{Aut}\,(L/K) - H$, then $x_0 \neq \sigma x_0$, and there exists $h \in \mathfrak{F}$ such that $h(x_0) \neq h(\sigma x_0)$ or $h(x_0) \neq \sigma h(x_0)$. Equivalently, if $\sigma \in \mathrm{Aut}\,(L/K)$ fixes every element of L_1, then $\sigma \in H$. Thus L_1 is the fixed field of the closed subgroup H. As H fixes $g(x_0)$, we have $g(x_0) \in L_1$, and there exists $f \in \mathfrak{F}$ such that $f(x_0) = g(x_0)$.

LEMMA 3. *Assuming the hypotheses of Theorem* 1, X *is totally disconnected.*

Proof. Since $\mathfrak{F}$ separates points, X is Hausdorff. Now take $x_0 \in X$ and an open neighborhood U of x_0. For each $y \notin U$, there exists $f_y \in \mathfrak{F}$ such that $f_y(x_0) \neq f_y(y)$. Put $\varepsilon_y = |f_y(x_0) - f_y(y)|$, and let

$$U_y = \{x \in X : |f_y(x) - f_y(x_0)| < \varepsilon_y/2\}$$

and

$$V_y = \{x \in X : |f_y(x) - f_y(y)| < \varepsilon_y/2\} ;$$

U_y and V_y are disjoint open and closed subsets of X with $x_0 \in U_y$.

The V_y cover the compact set $X - U$ and hence there exists a finite number, say $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$ whose union contains $X - U$. Then $\bigcap_{i=1}^{n} U_{y_i}$ is an open and closed neighborhood of x contained in U.

LEMMA 4. *Assuming the hypotheses of Theorem* 1, *suppose* V *is an open and closed invariant subset of* X. *Then the characteristic function of* V *is in* $\mathfrak{F}$.

Proof. By the Kaplansky-Stone-Weierstrass Theorem [2] and Lemma 3, the characteristic function of V is in the uniform closure of the L-subalgebra of $C_L(X)$ generated by $\mathfrak{F}$. Hence, if $\varepsilon > 0$, there exists $f \in C_L(X)$ such that $f = \sum_{i=1}^{m} a_i h_i$ where the $a_i \in L$ and the $h_i \in \mathfrak{F}$ and such that $|f(y) - 1| < \varepsilon$ if $y \in V$ while $|f(y)| < \varepsilon$ if $y \notin V$. Let $L_1 \subset L$ be the smallest normal extension field of K containing all of the a_i; L_1 is a finite algebraic extension of K and hence $\operatorname{Aut}(L_1/K)$ is finite. As $\operatorname{Aut}(L_1/K)$ is a homomorphic image of $\operatorname{Aut}(L/K)$, there exist representatives $\sigma_1, \sigma_2, \cdots, \sigma_n$ of $\operatorname{Aut}(L_1/K)$ in $\operatorname{Aut}(L/K)$ and the set of restrictions $\{\sigma_i|_{L_1} : 1 \leq i \leq n\}$ is $\operatorname{Aut}(L_1/K)$. If $\sigma \in \operatorname{Aut}(L/K)$, put $f^\sigma = \sum_{i=1}^{m} (\sigma a_i) h_i$. Then if $y \in X$,

$$
\begin{aligned}
f^\sigma(y) &= \sum_{i=1}^{m} (\sigma a_i) h_i(\sigma\sigma^{-1}y) \\
&= \sigma\Big(\sum_{i=1}^{m} a_i h_i(\sigma^{-1}y)\Big) \\
&= \sigma f(\sigma^{-1}y) .
\end{aligned}
$$

As $\sigma^{-1}V = V$, $|f^\sigma(y) - 1| < \varepsilon$ if $y \in V$, while $|f^\sigma(y)| < \varepsilon$ if $y \notin V$. Put $g = \prod_{i=1}^{n} f^{\sigma_i}$; then $g \in \mathfrak{F}$ and $|g(y) - 1| < \varepsilon$ if $y \in V$ while $|g(y)| < \varepsilon$ if $y \notin V$. Thus letting $\varepsilon \to 0$, we see that the characteristic function of V is in $\mathfrak{F}$.

Proof of Theorem 1 (concluded). Suppose $f \in C_{L/K}(X)$ and $\varepsilon > 0$. For each $x \in X$, there exists by Lemma 2, $g_x \in \mathfrak{F}$ such that $g_x(x) = f(x)$. Let U_x be an open and closed neighborhood of x such that $|g_x(y) - f(y)| < \varepsilon$ whenever $y \in U_x$. Put $V_x = \operatorname{Aut}(L/K)U_x$; clearly V_x is invariant. As V_x is the union of the open sets σU_x, $\sigma \in \operatorname{Aut}(L/K)$, V_x is open, and since it is the continuous image of the compact set $\operatorname{Aut}(L/K) \times U_x$, it is compact. If $y \in V_x$, there exists $\sigma \in \operatorname{Aut}(L/K)$ such that $\sigma y \in U_x$. Then

$$
\begin{aligned}
|g_x(y) - f(y)| &= |\sigma(g_x(y) - f(y))| \\
&= |g_x(\sigma y) - f(\sigma y)| < \varepsilon .
\end{aligned}
$$

The V_x are open sets which cover X. Hence a finite number, say $V_{x_1}, V_{x_2}, \cdots, V_{x_n}$ cover X. Put $D_1 = V_{x_1}$ and for $2 \leq i \leq n$, put

$$D_i = V_{x_i} - \bigcup_{j=1}^{i-1} V_{x_j} \, .$$

Each D_i is open and closed, and invariant; hence by Lemma 4, the characteristic function h_i of D_i is in $\mathfrak{F}$. In addition the D_i are disjoint and $\bigcup_{i=1}^{n} D_i = X$. Now put

$$g = \sum_{i=1}^{n} h_i g_{x_i} \, ,$$

so that $g \in \mathfrak{F}$. If $y \in X$, then there exists j such that $y \in D_j \subset V_{x_j}$; then $g(y) = g_{x_j}(y)$. As $|g_{x_j}(y) - f(y)| < \varepsilon$, $|g(y) - f(y)| < \varepsilon$. Letting $\varepsilon \to 0$ shows that $f \in \mathfrak{F}$. Finally, if there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathfrak{F}$, let $\mathfrak{F}_1$ be the K-algebra obtained from $\mathfrak{F}$ by adjoining the K-valued constant functions. Then if $g \in C_{L/K}(X)$ satisfies $g(x_0) = 0$, and $\varepsilon > 0$, there exists by what we have proved $f_1 \in \mathfrak{F}_1$ such that $|f_1(x) - f(x)| < \varepsilon$ for all $x \in X$. Then $f_1 = f + a$, where $f \in \mathfrak{F}$ and $a \in K$. Now $|a| = |f_1(x_0)| < \varepsilon$, hence $|f(x) - g(x)| < \varepsilon$ for all $x \in X$. Letting $\varepsilon \to 0$ shows that $g \in \mathfrak{F}$.

COROLLARY 5. *Suppose that $C_{L/K}(X)$ separates the points of X and that I is a closed ideal of the K-algebra $C_{L/K}(X)$. Then there exists a closed invariant set $Y \subset X$ such that*

$$I = \{f \in C_{L/K}(X) : f(Y) = \{0\}\} \, .$$

Proof. Put $Y = \bigcap_{f \in I} \{x : f(x) = 0\}$. Then Y is a closed invariant subset of X. If $x_1, x_2 \in X - Y$ and $x_1 \neq x_2$, then there exists $f \in I$ such that $f(x_1) \neq 0$. If $f(x_1) \neq f(x_2)$, let g be the constant function 1, while if $f(x_1) = f(x_2)$, choose $g \in C_{L/K}(X)$ such that $g(x_1) \neq g(x_2)$. Then in either case the function $h = gf \in I$ and $h(x_1) \neq h(x_2)$. Now let X_1 be the topological space obtained from X by identifying the points of Y, and let p be the projection from X to X_1. Then p is continuous and if $x_1, x_2 \in X$, we have $p(x_1) = p(x_2)$ if and only if either $x_1 = x_2$ or $x_1, x_2 \in Y$. A basis for the open neighborhoods of a point $x \in X_1$ is given by sets of the form $p(V)$, where V is an open neighborhood of $p^{-1}(x)$ in X. If $\sigma \in \mathrm{Aut}\,(L'/K')$ and $x \in X_1$, we define $\sigma x = p(\sigma p^{-1}(x))$; this is well defined and yields a continuous map $(\sigma, x) \to \sigma x$ of $\mathrm{Aut}\,(L'/K') \times X_1 \to X_1$. Denote by $C_{L/K}(X, Y)$ the K-algebra of $f \in C_{L/K}(X)$ which are constant on Y. If $f \in C_{L/K}(X, Y)$ define $pf \in C_{L/K}(X_1)$ by $(pf)(x) = f(p^{-1}(x))$; this is well defined and yields a norm preserving isomorphism between $C_{L/K}(X, Y)$ and $C_{L/K}(X_1)$. Put $pI = \{pf : f \in I\}$; pI is a uniformly closed K-subalgebra which separates the points of X_1, and every function $pf \in pI$ vanishes on $p(Y)$; hence by Theorem 1, pI consists of all $f \in C_{L/K}(X_1)$ which vanish on $p(Y)$. Thus I consists of all $f \in C_{L/K}(X)$ which vanish on Y.

COROLLARY 6. *Suppose that $C_{L/K}(X)$ separates the points of X. Then the maximal ideals of the K-algebra $C_{L/K}(X)$ are precisely the sets of the form*

$$\{f \in C_{L/K}(X) : f(x_0) = 0\}$$

where $x_0 \in X$.

The following theorem permits the extension of Theorem 1 and its corollaries to certain subsets of X.

THEOREM 7. *Suppose Y is a closed subset of X and $\mathrm{Aut}(L'/K')Y = X$. Then each continuous K-valued function f on Y, satisfying $f(\sigma y) = \sigma f(y)$ whenever $\sigma \in \mathrm{Aut}(L'/K')$ and both $y, \sigma y \in Y$, has a unique extension to a function $f_1 \in C_{L/K}(X)$.*

Proof. If $x \in X$, take $\sigma \in \mathrm{Aut}(L'/K')$ such that $\sigma x \in Y$ and define $f_1(x) = \sigma^{-1} f(\sigma x)$. This definition is independent of the choice of σ, and f_1 is the unique extension of f to X which satisfies $f_1(\sigma x) = \sigma f_1(x)$ for all $x \in X$ and $\sigma \in \mathrm{Aut}(L'/K')$. If f_1 were not continuous, there would exist a net $x_i \in X$ converging to $x_0 \in X$ such that the net $f_1(x_i)$ would not converge to $f_1(x_0)$. Suppose that $x_i = \sigma_i y_i$ where $\sigma_i \in \mathrm{Aut}(L'/K')$ and $y_i \in Y$. Since both $\mathrm{Aut}(L'/K')$ and Y are compact, we may assume, by taking subnets if necessary, that both $\lim y_i = y_0$ and $\lim \sigma_i = \sigma_0$ exist. Then $\sigma_0 y_0 = x_0$ and

$$\lim f_1(x_i) = \lim \sigma_i f(y_i) = \sigma_0 f(y_0) = f_1(x_0) .$$

This contradiction shows that f_1 is continuous.

We now consider a special case of the above results, which is of interest in applications. Suppose that K is a finite algebraic extension of a field of p-adic numbers Q_p and that $L = \tilde{K}$ the algebraic closure of K. We take X to be an invariant compact subset of $\tilde{K}$ (the action of $\mathrm{Aut}(\tilde{K}/K)$ is the usual one) and note that the map of $\mathrm{Aut}(\tilde{K}/K) \times X \to X$ given by $(\sigma, x) \to \sigma x$ is continuous. In fact given $\sigma_0 \in \mathrm{Aut}(\tilde{K}/K)$, $x_0 \in X$, and $\varepsilon > 0$, put

$$H = \{\sigma \in \mathrm{Aut}(\tilde{K}/K) : \sigma x_0 = \sigma_0 x_0\}$$

and

$$N = \{x \in X : |x - x_0| < \varepsilon\} ;$$

then both H and N are open and $HN = N$. We then obtain

THEOREM 8. *Suppose I is an ideal of $K[x]$; then the uniform closure of I in $C_{\tilde{K}/K}(X)$ is the set of functions $f \in C_{\tilde{K}/K}(X)$ which vanish at every zero of I.*

REFERENCES

1. E. Artin, *Theory of Algebraic Numbers,* Göttingen, 1959.
2. I. Kaplansky, *The Weierstrass Theorem in fields with valuations,* Proc. Amer. Math. Soc. **1** (1950), 356-357.

Received July 11, 1966. The preparation of this paper was sponsored in part by N.S.F. Grant GP 5497.

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