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PROJECTIVE AND INJECTIVE DISTRIBUTIVE LATTICES

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This paper is concerned with the properties of projective and injective distributive lattices.

By considering the minimal Boolean extension of a distributive lattice L, the question of the injectivity of L is transferred to the category of Boolean algebras, where a characterization is known. The result is that L is injective-in the category of distributive lattices-if and only if it is a complete Boolean algebra.

The first section deals with a method of defining E-free sequences of elements in a distributive lattice. Roughly speaking, these are elements which satisfy a given set E of inequalities and no others except consequences of E.

We prove that a finite distributive lattice is projective if and only if the sum of any two meet irreducible elements is meed irreducible. For the general case we show that a distributive lattice is projective if and only if it is generated by an E-free sequence, where E is a certain set of one-sided inequalities.

The last section concerns the projectivity of Boolean algebras, chains, and direct products.

1. E-free distributive lattices.

Throughout this paper $\{x_i\}, i \in I$, will denote a sequence of distinct variables.

DEFINITION 1.1. An *inequality* in $\{x_i\}, i \in I$, is a formula of the form

$$x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} . \tag{1}$$

DEFINITION 1.2. Suppose $\{a_i\}, i \in J$, is a sequence of elements of a distributive lattice and $I \subseteqq J$. Then $\{a_i\}, i \in J$, *satisfies* the inequality (1) if $a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}$. If E is a set of inequalities in $\{x_i\}, i \in I$, then $\{a_i\}, i \in J$, *satisfies* E if it satisfies all members of E.

THEOREM 1.3. *If $\{a_i\}, i \in J$, is a sequence of elements of a distributive lattice $L, I \subseteqq J$, and $f: L \to M$ is a homomorphism into a distributive lattice M, then if $\{a_i\}, i \in J$, satisfies a set E of inequalities in $\{x_i\}, i \in I$, then $\{f(a_i)\}, i \in J$, satisfies E.*

Proof. This follows from the fact that homomorphisms preserve order.

DEFINITION 1.4. Let E be a set of inequalities in (x_i), $i \in I$, and e an inequality in (x_i), $i \in J$, where $I \subseteqq J$. Then e is said to be a *consequence* of E if and only if whenever $\{a_i\}$, $i \in J$, is a sequence in a distributive lattice L which satisfies E, then it satisfies e.

DEFINITION 1.5. If E is a set of inequalities in $\{x_i\}$, $i \in I$, then a sequence $\{a_i\}$, $i \in J$, $I \subseteqq J$, is said to be *E-free* if and only if:

(i) $\{a_i\}$, $i \in J$, satisfies E.

(ii) If $\{a_i\}$, $i \in J$, satisfies an inequality e in (x_i), $i \in I$, then e is a consequence of E.

It is clear that if $\{a_i\}$, $i \in J$, is E-free and e is a consequence of E, then $\{a_i\}$, $i \in J$, satisfies e.

THEOREM 1.6. *Let $\{a_i\}$, $i \in I$ be a sequence in a distributive lattice. Then there exists a set E of inequalities in $\{x_i\}$, $i \in I$, such that $\{a_i\}$, $i \in I$, is an E-free sequence.*

Proof. Let

$$E = \{x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \mid a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}\} .$$

Now $\{a_i\}$, $i \in I$, is E-free for it satisfies E and if it satisfies an inequality e in $\{x_i\}$, $i \in I$, then $e \in E$. It is trivial that $e \in E$, implies that e is a consequence of E.

LEMMA 1.7. *A mapping f of a set G of generators of a distributive lattice L into a distributive lattice M can be extended to a homomorphism $f'\colon L \rightarrow M$ if and only if for any finite nonempty subsets S, T of G, whenever*

$$\pi(S) \leqq \Sigma(T) \tag{2}$$

then

$$\pi(f(S)) \leqq \Sigma(f(T)) \text{ where } f(S) = \{f(s) \mid s \in S\} . \tag{3}$$

Proof. The necessity follows immediately. Now if $a \in L$, then $a = \sum_{i=1}^{n} \pi(S_i)$ where S_i is a finite nonempty subset of G for $1 \leqq i \leqq n$. Define $f'\colon L \rightarrow M$ by $f(a) = \sum_{i=1}^{n} \pi(f(S_i))$. Since $\sum_{i=1}^{n} \pi(S_i) = \sum_{j=1}^{m} \pi(T_j)$ is equivalent to a collection of relations of the form (2), which by hypothesis are preserved by f, the function f' is well defined. It is now easy to show that f' is a homomorphism and an extension of f.

THEOREM 1.8. *If $\{a_i\}$, $i \in J$, generates a distributive lattice L and E is a set of inequalities in $\{x_i\}$, $i \in I$, $I \subseteqq J$, then $\{a_i\}$, $i \in J$, is*

E-free if and only if

(i) $\{a_i\}, i \in J$, *satisfies* E.

(ii) *whenever* $\{b_i\}, i \in J$, *is a sequence of elements of a distributive lattice* M *such that* $\{b_i\}, i \in J$, *satisfies* E, *then there exists a homomorphism* $f: L \to M$ *such that* $f(a_i) = b_i$.

Proof. For the necessity of (ii), let M be such a distributive lattice. By Lemma 1.7, we need only show that

$$a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m} \quad \text{implies} \tag{4}$$

$$b_{i_1} \cdot \cdots \cdot b_{i_n} \leqq b_{j_1} + \cdots + b_{j_m} . \tag{5}$$

But if (4) holds then $\{a_i\}, i \in J$, satisfies

$$e = x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} .$$

Since $\{a_i\}, i \in J$ is E-free, e is a consequence of E. But M satisfies E so (5) holds. Conversely, suppose $\{a_i\}, i \in J$ satisfies an inequality e in $\{x_i\}, i \in J$. To show e is a consequence of E, let $\{b_i\}, i \in J$, be a sequence in a distributive lattice M which satisfies E. By hypothesis there is a homomorphism $f: L \to M$ such that $f(a_i) = b_i$. Thus, by Theorem 1.3, $\{b_i\}, i \in J$, satisfies E. So e is a consequence of E.

THEOREM 1.9. (*Existence*) *If* E *is a set of inequalities* $\{x_i\}$, $i \in I$, *and* $J \supseteqq I$, *then there exists an* E*-free sequence* $\{A_i\}, i \in J$.

Proof. For each inequality $e = x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m}$ in $\{x_i\}, i \in I$, let

$$\mathfrak{S}(e) = \{s \in 2^J \mid s(i_1) = \cdots = s(i_n) = 1,\ s(j_1) = \cdots = s(j_m) = 0\}$$

and if E is a set of such inequalities, let $\mathfrak{S}(E) = \bigcup_{e \in E} \mathfrak{S}(e)$. Let $A_i = \{s \in 2^J \mid s(i) = 1,\ s \notin \mathfrak{S}(E)\}$. Finally, for each $i \in J$, set $A_{i,1} = A_i$ and $A_{i,0} = 2^J - A_i$.

We first show that

(i) If $\bigcap_i A_{i,s(i)} = \varnothing$ then $s \in \mathfrak{S}(E)$.

(ii) Let $\{a_i\}, i \in J$, be a sequence of members of a ring of subsets of a set X and let $e = x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m}$.
Then $\{a_i\}, i \in J$, satisfies e if and only if $\bigcap_i a_{i,i(s)} = \varnothing$ for all $s \in \mathfrak{S}(e)$, where $a_{i,1} = a_i$ and $a_{i,0} = X - a_i$.

For (i) if $s \notin \mathfrak{S}(E)$ then if $s(i) = 1$, $s \in A_{i,1}$ and if $s(i) = 0$, $s \in A_{i,0}$. Hence $s \in \bigcap_i A_{i,s(i)}$. For (ii) first suppose $\{a_i\}, i \in J$, satisfies e. Then $a_{i_1} \cap \cdots \cap a_{i_n} \cap a'_{j_1} \cap \cdots \cap a'_{j_m} = \varnothing$ and hence $\bigcap_i a_{i,s(i)} = \varnothing$ whenever $s \in \mathfrak{S}(e)$. Conversely, suppose there exists

$$p \in a_{i_1} \cap \cdots \cap a_{i_n} \cap a'_{j_1} \cap \cdots \cap a'_{j_m} .$$

Define $s \in 2^J$ by $s(i) = 1$ if and only if $p \in a_i$. Then $s \in \mathfrak{S}(e)$ but $p \in \bigcap_i a_{i,s(i)}$.

Suppose

$$x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \in E .$$

Then $\bigcap_{k=1}^n A_{i_k} \subseteqq \bigcup_{l=1}^m A_{j_l}$, for if $s \notin \mathfrak{S}(E)$ and $s(i_1) = \cdots = s(i_n) = 1$ then $s(j_l) = 1$ for some $l \in \{1, \cdots, m\}$. Hence $\{A_i\}, i \in J$, satisfies E. Now suppose $\{A_i\}, i \in J$, satisfies an inequality e in $\{x_i\}, i \in J$ and $\{a_i\}, i \in J$, is a sequence in a distributive lattice M which satisfies E. We can assume that M is a ring of sets. Note that if we apply (ii) to $\{A_i\}, i \in J$, then (i) shows that every member s of $\mathfrak{S}(e)$ is in $\mathfrak{S}(E)$. Again by (ii), $\{a_i\}, i \in J$, will satisfy e provided every member of $\mathfrak{S}(e)$ is a member of $\mathfrak{S}(e')$ for some $e' \in E$. But this follows since $\mathfrak{S}(e) \subseteqq \mathfrak{S}(E)$.

DEFINITION 1.10. A distributive lattice is said to be *E-free* if it is generated by an E-free sequence.

By Theorem 1.6 every distributive lattice is E-free for some set E, and any $\varnothing$-free distributive lattice is free.

THEOREM 1.11. (*Uniqueness*) *Let E be a set of inequalities in $\{x_i\}, i \in I$. If L and M are distributive lattices generated by E-free sequences $\{a_i\}, i \in J$, and $\{b_i\}, i \in J$, where $I \subseteqq J$, then $L \cong M$.*

Proof. Follows immediately from Theorem 1.8.

The following type of theorem is easily proved: Suppose L is generated by the E-free sequence $\{a_i\}, i \in I$, where the inequalities of E are of the form $x_i \leqq x_j$. If P and Q are finite nonempty subsets of $\{a_i\}, i \in I$, and $\pi(P) \leqq \Sigma(Q)$, then there exist $a_p \in P$ and $a_q \in Q$ and a finite sequence $a_p \leqq a_{i_1} \leqq \cdots \leqq a_{i_n} \leqq a_q$ such that all of the inequalities $x_p \leqq x_{i_1}, \cdots, x_{i_n} \leqq x_q$ are in E. Also it can be shown that if e is a consequence of E then it is a consequence of a finite subset of E.

Again suppose E is a set of inequalities in $\{x_i\}, i \in I$ and $\{A_i\}, i \in I$, is the E-free sequence as in the proof of Theorem 1.9. Let L be the ring of sets generated by $\{A_i\}, i \in I$. Setting $X' = \{s \in 2^I \mid s \notin \mathfrak{S}(E)\}$, the following theorem can be proved by direct computation.

THEOREM 1.12. *F is a prime filter in L if and only if F is the filter generated by $\{A_i \mid s(i) = 1\}$ for some $s \in X'$.*

Thus we obtain the following characterization of the Stone space of the E-free distributive lattice L.

THEOREM 1.13. *The Stone space of* L *is* X' *with* $\{A_i \mid i \in I\}$, *as a subbasis for its topology.*

2. Definitions. The definitions in this section are of a universal nature, so we consider an arbitrary category of algebras.

DEFINITION 2.1. An algebra A is a *retract* of an algebra A_1 if there exist homomorphisms $f: A_1 \to A$ and $g: A \to A_1$ such that $fg = I_A$, the identity function on A.

DEFINITION 2.2. An algebra A is *injective* if for every pair of algebras A_1 and A_2, every homomorphism $h: A_2 \to A$, and every monomorphism $g: A_2 \to A_1$, there exists a homomorphism $f: A_1 \to A$ such that $fg = h$.

DEFINITION 2.3. An algebra A is *projective* if for every pair of algebras A_1 and A_2, every homomorphism $h: A \to A_2$, and every epimorphism $f: A_1 \to A_2$, there exists a homomorphism $g: A \to A_1$ such that $fg = h$.

The terms retract, injective, and projective, when prefixed by $(0, 1)$-, will be taken in the category of distributive lattices with a smallest and a greatest element, and homomorphisms which preserve 0 and 1. Otherwise, the category will be distributive lattices. It is immediate that retracts of injective (projective) distributive lattices are injective (projective).

3. Injective distributive lattices. We make use of the following theorem, proved by Halmos [3; p. 141] in the category of Boolean algebras: A Boolean algebra is injective if and only if it is complete.

LEMMA 3.1. *A complete Boolean algebra* L *is* $(0, 1)$-*injective.*

Proof. Let M be a distributive lattice with 0 and 1. Nerode has shown that there exists a Boolean algebra A, and a $(0, 1)$-monomorphism $\varphi: M \to A$ such that $\varphi(M)$ Boolean generates A. A is unique to within isomorphism, and is called the minimal Boolean extension of M [4].

Now let L_1 and L_2 be distributive lattices with 0 and 1, $h: L_2 \to L$ a $(0, 1)$-homomorphism, and $g: L_2 \to L_1$ a $(0, 1)$-monomorphism. Let B, B_1 and B_2 be the minimal Boolean extensions of L, L_1, L_2 and φ, φ_1, φ_2 the corresponding $(0, 1)$-monomorphisms. By a theorem of Nerode [4, p. 399] there exists Boolean homomorphisms $h': B_2 \to B$ and $g': B_2 \to B_1$ such that $h'\varphi_2 = \varphi h$ and $g'\varphi_2 = \varphi_1 g$.

$$\begin{tikzcd}
L_1 \arrow[drr, dashed, "f"] \arrow[ddd, "\varphi_1"] & & \\
& L_2 \arrow[ul, "g"] \arrow[r, "h"] \arrow[d, "\varphi_2"] & L \arrow[d, "\varphi"'] \\
& B_2 \arrow[r, "h'"'] \arrow[dl, "g'"] & B \\
B_1 \arrow[urr, dashed, "f'"'] & &
\end{tikzcd}$$

Furthermore since g is one-to-one, so is g'. By hypothesis, L is a Boolean algebra, so $\varphi: L \to B$ is an isomorphism. Now since B is complete, it is injective in the category of Boolean algebras. Therefore, there is a Boolean homomorphism $f': B_1 \to B$ such that $f'g' = h'$. Since φ is an isomorphism, we can define $f: L_1 \to L$ by $f = \varphi^{-1}f'\varphi_1$. Then $fg = \varphi^{-1}f'\varphi_1 g = \varphi^{-1}f'g'\varphi_2 = \varphi^{-1}h'\varphi_2 = \varphi^{-1}\varphi h = h$. Clearly f preserves 0 and 1.

THEOREM 3.2. *A distributive lattice is injective if and only if it is a complete Boolean algebra.*

Proof. Suppose first that L is a complete Boolean algebra. Let L_1 and L_2 be distributive lattices, $h: L_2 \to L$ a homomorphism and $g: L_2 \to L_1$ a monomorphism. Let $L_1' = L_1 \cup \{0', 1'\}$ where $0' < x < 1'$ for all $x \in L_1$, and $L_2' = L_2 \cup \{0^*, 1^*\}$ where $0^* < x < 1^*$ for all $x \in L_2$. Define $h': L_2' \to L$ by $h' \mid L_2 = h$, $h'(0^*) = 0_L$ and $h'(1^*) = 1_L$. Define $g': L_2' \to L_1'$ by $g' \mid L_2 = g$, $g'(0^*) = 0'$ and $g'(1^*) = 1'$. Since L is a complete Boolean algebra, it is $(0, 1)$-injective so there is a $(0, 1)$-homomorphism $f': L_1' \to L$ such that $f'g' = h'$. Define $f: L_1 \to L$ by $f = f' \mid L_1$. Then if $x \in L_2$, $fg(x) = f'g'(x) = h'(x) = h(x)$.

Conversely, suppose L is injective and B is the complete Boolean algebra of all subsets of the collection of prime filters of L. Then there exists a monomorphism $g: L \to B$. Since L is injective there exists a homomorphism $f: B \to L$ such that $fg = I_L$. Thus L is the homomorphic image of a Boolean algebra and is therefore a Boolean algebra. For completeness, let $S \subseteqq L$. Then $\Sigma_B\{g(s) \mid s \in S\} = p$ exists in B. It is easily verified that $\Sigma_L(S) = f(p)$.

4. Basic properties of projective distributive lattices.

LEMMA 4.1. *A distributive lattice is projective if and only if it is a retract of a free distributive lattice.*

Proof. Essentially as in [3, p. 137].

Shanin [5, p. 91] has shown the topological dual of the statement that free Boolean algebras contain no uncountable chains. This implies the same condition of free distributive lattices, so we have:

THEOREM 4.2. *There are no uncountable chains in a projective distributive lattice L.*

Proof. Since L is projective, there is a free distributive lattice F and a monomorphism $g: L \to F$. If C was an uncountable chain in L, $\{G(c) \mid c \in C\}$ would be an uncountable chain in F.

In the category of Boolean algebras, every projective Boolean algebra satisfies the ω-chain condition. For distributive lattices there is an even stronger condition.

DEFINITION 4.3. A subset S of a distributive lattice L is said to be a-disjointed ($a \in L$) if $xy = a$ whenever x and y are distinct elements of S.

LEMMA 4.4. *Let* $\{Z_i\}, i = 1, \cdots, m$ and $\{T_i\}, i = 1, 2, \cdots$ *be sequences of finite sets such that*

$$\begin{array}{c} Z_1 \nsubseteq T_1, \cdots, Z_m \nsubseteq T_1 \\ Z_1 \nsubseteq T_2, \cdots, Z_m \nsubseteq T_2 . \\ \cdot \quad \cdots \quad \cdot \end{array}$$

Then there exists i, j such that $i \neq j$, and $Z_1 \nsubseteq T_i \cup T_j, \cdots, Z_m \nsubseteq T_i \cup T_j$.

Proof. The sequence $\{Z_1 - T_i\}, i = 1, 2, \cdots$ contains only finitely many distinct sets since Z_1 is finite. So there exists a subsequence $\{T_{2,i}\}, i = 1, 2, \cdots$ of $\{T_i\}, i = 1, 2, \cdots$ such that $Z_1 - T_{2,1} = Z_1 - T_{2,2} = \cdots$. Hence $Z_1 \nsubseteq T_{2,1} \cup T_{2,2} \cup \cdots$. Proceeding by induction, suppose $\{T_{n,i}\}, i = 1, 2, \cdots$ is a sequence such that $Z_n - T_{n,1} = Z_n - T_{n,2} = \cdots$. Now the sequence $\{Z_{n+1} - T_{n,i}\}, i = 1, 2, \cdots$ contains only finitely many distinct sets, so there is a subsequence $\{T_{n+1,i}\}, i = 1, 2, \cdots$ of $\{T_{n,i}\}, i = 1, 2, \cdots$ such that $Z_{n+1} - T_{n+1,1} = Z_{n+1} - T_{n+1,2} = \cdots$. Hence $Z_{n+1} \nsubseteq T_{n+1,1} \cup T_{n+1,2} \cup \cdots$. In particular, $Z_m \nsubseteq T_{m,1} \cup T_{m,2} \cup \cdots$. Now for each $n \in \{1, \cdots, m\}$, $Z_n \nsubseteq T_{n,1} \cup T_{n,2} \cup \cdots$ and since $\{T_{m,i}\}, i = 1, 2, \cdots$ is a subsequence of $\{T_{n,i}\}, i = 1, 2, \cdots$ we have $Z_n \nsubseteq T_{m,1} \cup T_{m,2}$ for all $n \in \{1, \cdots, m\}$.

LEMMA 4.5. *Let F be a free distributive lattice generated by an independent set G. Then $\pi(S_1) + \cdots + \pi(S_n) \leqq \pi(T_1) + \cdots + \pi(T_m)$, where S_i and T_j are finite nonempty subsets of G if and only if for each $i \in \{1, \cdots, n\}$ there exists $j \in \{1, \cdots, m\}$ such that $T_j \subseteqq S_i$.*

Proof. The sufficiency follows immediately. On the other hand if there exists $p \in \{1, \cdots, n\}$ such that for each $j \in \{1, \cdots, m\}$ there is an element $t_j \in T_j - S_p$, then $\pi(S_p) \leqq t_1 + \cdots + t_m$, contradicting independence.

THEOREM 4.6. *In a free distributive lattice L, every a-disjointed subset is finite.*

Proof. Let L be generated by the independent set G and suppose $D = \{d_i \mid i = 1, 2, \cdots\}$ is an infinite a-disjointed subset. There are finite nonempty subsets $S_{i,j}$ of G such that $d_i = \sum_{j=1}^{p(i)} \pi(S_{i,j})$. Let $a = \sum_{j=1}^{m} \pi(Z_j)$ where Z_j is a finite nonempty subset of G for $1 \leqq j \leqq m$. We can assume $a \notin D$. Thus we have

(i) $a < d_i$ $(i = 1, 2, \cdots)$.

(ii) $d_i d_j = a$ whenever $i \neq j$.

(iii) There exists a positive integer n such that for each $k \in \{1, \cdots, p(n)\}$, $Z_r \subseteqq S_{n,k}$ for some $r \in \{1, \cdots, m\}$.

If (iii) does not hold then for each i there is an $S_{i,j}$ such that $Z_r \not\subseteqq S_{i,j}$ for all $r \in \{1, \cdots, m\}$. By Lemma 4.4 there exists S_{i,j_i} and S_{k,j_k} $(i \neq k)$ such that $Z_r \not\subseteqq S_{i,j_i} \cup S_{k,j_k}$ for all $r \in \{1, \cdots, m\}$. But $\pi(S_{i,j_i} \cup S_{k,j_k}) \leqq d_i d_j = a$, so by Lemma 4.5 there is an r such that $Z_r \subseteqq S_{i,j_i} \cup S_{k,j_k}$, a contradiction.

By (iii) we have $d_n \leqq a$, contradiction (i).

COROLLARY 4.7. *In a projective distributive lattice every a-disjointed set is finite.*

Proof. Similar to the proof of Theorem 4.2.

EXAMPLE 4.8. Neither the ring of all sets of integers nor the ring of finite sets of integers is projective since the singletone form an infinite disjointed set.

EXAMPLE 4.9. The field of all finite and co-finite sets of integers is Boolean projective [3; p. 139, Corollary 2], but not projective.

5. Some characterizations of projective distributive lattices.

THEOREM 5.1. *A distributive lattice L generated by an E-free sequence $\{a_i\}$, $i \in I$, is projective if and only if for each i there exist finite nonempty subsets $S_{i,1}, \cdots, S_{i,p(i)}$ of I such that*

(i) $a_i = \sum_{k=1}^{p(i)} P(S_{i,k})$ *where* $P(S_{i,k}) = \pi\{a_l \mid l \in S_{i,k}\}$.

(ii) *If*

$$x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \in E$$

then for each $f \in \boldsymbol{P}(i_1, \cdots, i_n)$ *there exist* q, r *such that* $S_{j_q,r} \subseteqq$

$\bigcup_{t=1}^{n} S_{i_t,f(i_t)}$ *where* $\boldsymbol{P}(i_1, \cdots, i_n)$ *is the set of all function on* $\{i_1, \cdots, i_n\}$ *such that for each* $t, f(i_t)$ *is a positive integer* $\leqq p(i_t)$.

Proof. Suppose L is projective and F is the free distributive lattice generated by the independent sequence $\{b_i\}, i \in I$. Let $h: F \rightarrow L$ be an epimorphism such that $h(b_i) = a_i$. By hypothesis there exists a homomorphism $g: L \rightarrow F$ such that $hg = I_L$. Let $g(a_i) = \sum_{k=1}^{p(i)} P_F(S_{i,k})$ where for each i and k, $S_{i,k}$ is a finite nonempty subset of I and $P_F(S_{i,k}) = \{b_l \mid l \in S_{i,k}\}$. Then

$$a_i = hg(a_i) = \sum_k h(P_F(S_{i,k})) = \sum_k P(S_{i,k}) .$$

Next suppose $x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \in E$. Since $\{a_i\}, i \in I$ is E-free, we have $a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}$, so $g(a_{i_1}) \cdot \cdots \cdot g(a_{i_n}) \leqq g(a_{j_1}) + \cdots + g(a_{j_m})$. Thus for each $f \in \boldsymbol{P}(i_1, \cdots, i_n)$

$$\begin{aligned} P_F\Big(\bigcup_{t=1}^{n} S_{i_t,f(i_t)}\Big) &\leqq \Big(\sum_k P_F(S_{i_1,k})\Big) \cdot \cdots \cdot \Big(\sum_k P_F(S_{i_n,k})\Big) \\ &\leqq \sum_k P_F(S_{j_1,k}) + \cdots + \sum_k P_F(S_{j_m,k}) . \end{aligned}$$

By Lemma 4.5, we have that for each f, there exist q, r such that $S_{j_q,r} \subseteqq \bigcup_{t=1}^{n} S_{i_t,f(i_t)}$.

Now suppose (i) and (ii) hold and again let F be the free distributive lattice generated by the independent sequence $\{b_i\}, i \in I$, and $h: F \rightarrow L$ an epimorphism such that $h(b_i) = a_i$. Define a sequence $\{c_i\}, i \in I$ in F by $c_i = \sum_{k=1}^{p(i)} P_F(S_{i,k})$ for each $i \in I$. We will show that $\{c_i\}, i \in I$ satisfies E. If $x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \in E$, then by hypothesis, for each $f \in \boldsymbol{P}(i_1, \cdots, i_n)$, there exist q, r such that $S_{j_q,r} \subseteqq \bigcup_{t=1}^{n} S_{i_t,f(i_t)}$. So $P_F(\bigcup_{t=1}^{n} S_{i_t f(i_t)}) \leqq P_F(S_{j_q,r}) \leqq c_{j_q} \leqq c_{j_1} + \cdots + c_{j_m}$. Thus,

$$\begin{aligned} c_{i_1} \cdot \cdots \cdot c_{i_n} &= \Big(\sum_k P_F(S_{i_1,k})\Big) \cdot \cdots \cdot \Big(\sum_k P_F(S_{i_n,k})\Big) \\ &= \sum_f P_F\Big(\bigcup_{t=1}^{n} S_{i_t,f(i_t)}\Big) \leqq c_{j_1} + \cdots + c_{j_m} . \end{aligned}$$

Since $\{c_i\}, i \in I$ satisifies E, by Theorem 1.8, there exists a homomorphism $g: L \rightarrow F$ such that $g(a_i) = c_i$. Hence $hg(a_i) = h(c_i) = \sum_{k=1}^{p(i)} P(S_{i,k}) = a_i$. By Lemma 4.1, L is projective.

It may be remarked that for each i, the sets $S_{i,1}, \cdots, S_{i,p(i)}$ may be chosen so that no one of them contains any other one.

COROLLARY 5.2. *Suppose* L *is a distributive lattice generated by a sequence* $\{a_i\}, i \in I$. *Then* L *is projective if and only if for each* i, *there exists finite nonempty subsets* $S_{i,1}, \cdots, S_{i,p(i)}$ *of the distinct* *elements of* $\{a_i\}, i \in I$, *such that*

(i) $a_i = \sum_{k=1}^{p(i)} \pi(S_{i,k})$.

(ii) *If* $a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}$, *then for each* $f \in \boldsymbol{P}(i_1, \cdots, i_n)$, *there exist* q, r *such that* $S_{j_q, r} \subseteqq \bigcup_{t=1}^{n} S_{i_t, f(i_t)}$, *where* $\boldsymbol{P}(i_1, \cdots, i_n)$ *is the set of all functions on* $\{i_1, \cdots, i_n\}$ *such that for each* t, $f(i_t)$ *is a positive integer* $\leqq p(i_t)$.

Proof. This follows immediately from Theorem 5.1 by defining E as in Theorem 1.6.

In Theorem 5.1 and Corollary 5.2, we shall refer to condition (i) as the projective representation of the element a_i, and to (ii) as the projective criterion for $x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} (a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m})$. Observe that in (ii) of Corollary 5.2, if $a_{i_p} = a_{j_q}$ for some i_p and j_q, then the criterion is automatically satisfied. From this we again see that free distributive lattices are projective.

DEFINITION 5.3. An inequality $e = x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m}$ in $\{x_i\}, i \in I$, is said to be *one-sided* if $n = 1$ or $m = 1$. If $G = \{a_i \mid i \in I\}$, is a subset of a distributive lattice then G is said to be *lower semi-independent* (*upper semi-idependent*) if whenever G satisfies e then there exists $p \in \{1, \cdots, n\}$ such that $a_{i_p} \leqq a_{j_1} + \cdots + a_{j_m}$ (there exists $q \in \{1, \cdots, m\}$ such that $a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_q}$).

For the following theorem $\{x_i\}, i \in I$, will be, as before, a sequence of distinct variables. Fix a definite simple ordering of I. If $i_1 < \cdots < i_n$ and $X = \{x_{i_1}, \cdots, x_{i_n}\}$, then $\pi(X)$ will denote the expression $x_{i_1} \cdot \cdots \cdot x_{i_n}$ and $\Sigma(X)$ will denote $x_{i_1} + \cdots + x_{i_n}$.

THEOREM 5.4. *Let* L *be a distributive lattice generated by* $\{a_i\}$, $i \in I$. *If* L *is projective then* $\{a_i\}, i \in I$, *is* E*-free for some set* E *of one-sided inequalities. Specifically,* L *is projective if and only if* $\{a_i\}, i \in I$, *is* E*-free for some set* E *of inequalities of the form* $E_1 \cup E_2$ *where*

$$E_1 = \bigcup_{i \in I} \{\pi(X_{i,j}) \leqq x_i \mid j = 1, \cdots, p(i)\} \text{ and}$$

$$E_2 = \bigcup_{i \in I} \{x_i \leqq \Sigma(Y_{i,j}) \mid j = 1, \cdots, q(i)\} \text{ and}$$

(i) $X_{i,j}$ *is a finite subset of* $\{x_i \mid i \in I\}$, $1 \leqq j \leqq p(i)$.

(ii) *For each* i: $Y_{i,1}, \cdots, Y_{i,q(i)}$ *are all possible sets of the form* $\{x_{i_1}, \cdots, x_{i_n}\}$ *where* $x_{i_j} \in X_{i,j}$.

(iii) *If* $x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \in E$ (*so that* $n = 1$ *or* $m = 1$), *then for each* $f \in \boldsymbol{P}(i_1, \cdots, i_n)$, *there exist* q, r *such that* $X_{j_q, r} \subseteqq \bigcup_{t=1}^{n} X_{i_t, f(i_t)}$.

Proof. For sufficiency, let

$$S_{i,j} = \{k \mid x_k \in X_{i,j}\}, \quad \text{and} \quad T_{i,j} = \{k \mid x_k \in Y_{i,j}\} .$$

Since $\{a_i\}$, $i \in I$, satisfies E, for each i, we have $\pi\{a_k \mid k \in S_{i,j}\} \leqq a_i$ for $j = 1, \cdots, p(i)$. Therefore $\sum_{j=1}^{p(i)} \pi\{a_k \mid k \in S_{i,j}\} \leqq a_i$. Similarly, $a_i \leqq \sum\{a_k \mid k \in T_{i,j}\}$ for $j = 1, \cdots, q(i)$. A simple calculation shows that $a_i = \sum_{j=1}^{p(i)} P(S_{i,j})$ where $P(S_{i,j}) = \pi\{a_k \mid k \in S_{i,j}\}$. Since $X_{j_q,r} \subseteqq \bigcup_{t=1}^{n} X_{i_t,f(i_t)}$ implies $S_{j_q,r} \subseteqq \bigcup_{t=1}^{n} S_{i_t,f(i_t)}$, L is projective by Theorem 5.1.

For the necessity we use Theorem 5.2. Thus, let $a_i = \sum_{k=1}^{p(i)} \pi(S_{i,j})$ be a projective representation for each i. Then for each i we have $\pi(S_{i,j}) \leqq a_i$ for $j = 1, \cdots, p(i)$ and $a_i \leqq \Sigma(T_{i,j})$ for $j = 1, \cdots, q(i)$ where $T_{i,1}, \cdots, T_{i,q(i)}$ are all possible sets of the form $\{a_{i_1}, \cdots, a_{i_n}\}$ and $a_{i_j} \in S_{i,j}$. Setting $X_{i,j} = \{x_k \mid a_k \in S_{i,j}\}$ and $Y_{i,j} = \{x_k \mid a_k \in T_{i,j}\}$, define E as in the statement of the theorem. Consequently (i) and (ii) are satisfied. For (iii) suppose $x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m} \in E$. By the definitions of $S_{i,j}$, $T_{i,j}$, $X_{i,j}$, and $Y_{i,j}$, we have $a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}$. But L is projective so for each $f \in \boldsymbol{P}(i_1, \cdots, i_n)$ there exist q, r such that $S_{j_q,r} \subseteqq \bigcup_{t=1}^{n} S_{i_t,f(i_t)}$. Hence $X_{j_q,r} \subseteqq \bigcup_{t=1}^{n} X_{i_t,f(i_t)}$.

It remains to show that $\{a_i\}$, $i \in I$, is E-free. First, $\{a_i\}$, $i \in I$, obviously satisfies E. Now suppose $\{a_i\}$, $i \in I$, satisfies an inequality $e = x_{i_1} \cdot \cdots \cdot x_{i_n} \leqq x_{j_1} + \cdots + x_{j_m}$. Then $a_{i_1} \cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}$. Since L is projective, for each $f \in \boldsymbol{P}(i_1, \cdots, i_n)$ there exist q, r such that $S_{j_q,r} \subseteqq \bigcup_{t=1}^{n} S_{i_t,f(i_t)}$. Hence

$$X_{j_q,r} \subseteqq \bigcup_{t=1}^{n} X_{i_t,f(i_t)} . \tag{6}$$

To show e is a consequence of E, let $\{b_i\}$, $i \in I$, be a sequence in a distributive lattice that satisfies E. Let $B_{i,j} = \{b_k \mid x_k \in X_{i,j}\}$. Then $b_i = \sum_{k=1}^{p(i)} \pi(B_{i,k})$ and by (6), for each $f \in \boldsymbol{P}(i_1, \cdots, i_n)$ there exist q, r such that $B_{j_q,r} \subseteqq \bigcup_{t=1}^{n} B_{i_t,f(i_t)}$. Hence $\pi(\bigcup_{t=1}^{n} B_{i_t,f(i_t)}) \leqq \pi(B_{j_q,r}) \leqq b_{j_q}$. Thus

$$\begin{aligned} b_{i_1} \cdot \cdots \cdot b_{i_n} &= \Big(\sum_k \pi(B_{i_1,k})\Big) \cdot \cdots \cdot \Big(\sum_k \pi(B_{i_n,k})\Big) \\ &= \sum_f \pi\Big(\bigcup_f^{n} B_{i_t,f(i_t)}\Big) \leqq b_{j_1} + \cdots + b_{j_m} . \end{aligned}$$

So e is a consequence of E, and $\{a_i\}$, $i \in I$, is E-free.

EXAMPLE 5.5. Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of distinct variables and $E = \{x_1 x_2 \leqq x_3 + x_4\}$. Then any E-free distributive lattice is nonprojective for $x_1 x_2 \leqq x_3 + x_4$ is not a consequence of one-sided inequalities.

EXAMPLE 5.6. Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of distinct variables and

$$E = \{x_1x_2 \leqq x_3,\ x_1x_2 \leqq x_4,\ x_3 \leqq x_4 + x_5,\ x_1 \leqq x_4 + x_5\} .$$

Then the distributive lattice generated by the E-free sequence $\{a_i\}$, $i \in I$, is projective for a projective representation is: $a_1 = a_1a_4 + a_1a_5$, $a_2 = a_2$, $a_3 = a_3a_4 + a_3a_5 + a_1a_2$, $a_4 = a_4 + a_1a_2$, and $a_i = a_i$ for $i \geqq 5$.

6. Meet and join irreducible elements.

DEFINITION 6.1. An element a of the lattice L is called *meet irreducible* (M.I.) if whenever $xy \leqq a$ then $x \leqq a$ or $y \leqq a$. *Join irreducible* (J.I.) elements are defined dually.

In a distributive lattice the following are equivalent

(i) a is M.I.

(ii) If $a_1 \cdot \cdots \cdot a_n \leqq a$ then $a_i \leqq a$ for some $i \in \{1, \cdots, n\}$.

(iii) If $a_1 \cdot \cdots \cdot a_n = a$ then $a_i = a$ for some $i \in \{1, \cdots, n\}$.

THEOREM 6.2. *In a projective distributive lattice the sum of any two meet irreducible elements is meet irreducible and the product of any two join irreducible elements is join irreducible.*

Proof. Let $L = \{a_i,\} \ i \in I$ and suppose a_1 and a_2 are M.I. and

$$a_3a_4 \leqq a_1 + a_2 . \tag{7}$$

Let $a_i = \sum_{j=1}^{p(i)} \pi(S_{i,j})$ be a projective representation for each i. If $a_3 \not\leqq a_1 + a_2$ and $a_4 \not\leqq a_1 + a_2$ then there exist integers m, n such that $\pi(S_{3,m}) \not\leqq a_1 + a_2$ and $\pi(S_{4,m}) \not\leqq a_1 + a_2$. For otherwise, for either $k = 1$ or $k = 2$, $\pi(S_{k,t}) \leqq a_1 + a_2$ for all $t \in \{1, \cdots, p(k)\}$ and so $a_k \leqq a_1 + a_2$. Now choosing $f \in \boldsymbol{P}(3, 4)$ such that $f(3) = m$ and $f(4) = n$, we have

$$\pi(S_{t,f(t)}) \not\leqq a_1 + a_2 \text{ for } t = 3, 4 . \tag{8}$$

By the criterion (applied to (7) and for the given f), there exist $q \in \{1, 2\}$ and $r \in \{1, \cdots, p(q)\}$ such that $S_{q,r} \subseteqq S_{3,f(3)} \cup S_{4,f(4)}$. Hence $\pi(S_{3,f(3)})\pi(S_{4,f(4)}) \leqq \pi(S_{q,r}) \leqq a_q$. But a_q is M.I. so for either $n = 3$ or $n = 4$, $\pi(S_{n,f(n)}) \leqq a_q \leqq a_1 + a_2$, contradicting (8).

Since the dual of a projective distributive lattice is projective, the second statement follows.

7. Finite projective distributive lattices. We will now choose a special set of generators and apply Theorem 5.2. In particular, recall that in a finite distributive lattice every element is a product of M.I. elements.

THEOREM 7.1. *In a finite distributive lattice L, the following are equivalent.*

(i) *L is projective.*

(ii) *L is generated by a lower semi-independent set.*

(ii′) *L is generated by an upper semi-independent set.*

(iii) *The sum of any two meet irreducible elements is meet irreducible.*

(iii′) *The product of any two join irreducible elements is join irreducible.*

Proof. (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iii′) follows from Theorem 6.2. (iii) $\Rightarrow$ (ii): Let G be the set of M.I. elements. Then G generates L. If

$$a_{i_1}\cdot \cdots \cdot a_{i_n} \leqq a_{j_1} + \cdots + a_{j_m}$$

where $a_{i_s}, a_{j_t} \in G$ then by (iii) $a_{j_1} + \cdots + a_{j_m}$ is M.I., so there exists $p \in \{1, \cdots, n\}$ such that $a_{i_p} \leqq a_{j_1} + \cdots + a_{j_m}$. (iii′) $\Rightarrow$ (ii′): This is the dual of (iii) $\Rightarrow$ (ii). (ii′) $\Rightarrow$ (i). For each $a_i \in G$, it will be proved that a projective repesentation is $a_i = \pi(S_{i,1}) + \cdots + \pi(S_{i,p(i)})$ where the $S_{i,j}$ are all possible sets such that $\pi(S_{i,j}) \leqq a_i$. Equality holds since one of these sets is $\{a_i\}$. To show the criterion is satisfied, suppose $a_{i_1}\cdot \cdots \cdot a_{i_n} \leqq a_{j_1}\cdot \cdots \cdot a_{j_m}$. Then there exists q such that $a_{i_1}\cdot \cdots \cdot a_{i_n} \leqq a_{j_q}$. Let $f \in \boldsymbol{P}(i_1, \cdots, i_n)$, then

$$\pi\Bigl(\bigcup_{t=1}^{n} S_{i_t, f(i_t)}\Bigr) = \prod_{t=1}^{n} \pi(S_{i_t, f(i_t)}) \leqq a_{i_1}\cdot \cdots \cdot a_{i_n} \leqq a_{j_q} .$$

But by the definition of $S_{j_q,1}, S_{j_q,2}, \cdots$ there is an r such that $S_{j_q,r} = \bigcup_{t=1}^{n} S_{i_t, f(i_t)}$. (ii) $\Rightarrow$ (i): Since (ii′) $\Rightarrow$ (i), by duality if (ii) then the dual of L is projective and hence L is projective.

The hypothesis of finiteness is essential for the J.I. elements in ring of subsets of the integers are the singletons and $\varnothing$. So (iii′) is satisfied but we have seen (Ex. 4.8) that this lattice is not projective.

EXAMPLE 7.2. Let f be the free distributive lattice generated by the independent set $\{a_1, a_2, a_3\}$. Then the sublattice

$$L_1 = \{a_1a_2a_3, a_1a_2, a_1a_3, a_1(a_2 + a_3), a_1, a_2 + a_3, a_1 + a_2 + a_3\}$$

is not projective for a_1 and $a_2 + a_3$ are J.I. in L_1 but $a_1a_2 + a_1a_3$ is not J.I. in L_1.

By considering the partially ordered set of nonzero J.I. elements of a finite distributive lattice we obtain the following theorem [1, p. 139].

THEOREM 7.3. *Every finite distributive lattice is isomorphic with the family of all hereditary subsets of partially ordered set.*

Conversely, the family of all hereditary subsets of a finite partially ordered set is a lattice.

By adding a 0 and 1 to the partially ordered set, we find that every finite distributive lattice is isomorphic with the family of all nonempty proper hereditary subsets of a partially ordered set with $0, 1$. Conversely, every such family is a distributive lattice. In contrast, for finite projective distributive lattices we have:

THEOREM 7.4. *Every finite projective distributive lattice is isomorphic with the family of all nonempty proper hereditary subsets of a finite lattice. Conversely, the family of all nonempty proper hereditary subsets of a finite lattice is a projective distributive lattice.*

Proof. Let L be a finite projective distributive lattice and P the set of all nonzero J.I. elements. Then L is isomorphic with the family of hereditary subsets of P. Let $M = \{0\} \cup P \cup \{1^*\}$ where $1^* > x$ for all $x \in L$. Now M will be a lattice if $S \subseteqq M$ implies $\pi(S)$ exists. If $S = \emptyset$ then $\pi(S) = 1^* \in M$ and if $S \neq \emptyset$ it is sufficient to consider $S = \{x, y\}$. If x or y equals 0 then $\pi(S) = 0$ and if x or y equals 1^* then $\pi(S)$ equals y or x respectively. Thus, suppose $x, y \in P$. Since x, y are J.I. and L is projective $x \cdot_L y \in P \cup \{0\}$ and therefore is equal to $x \cdot_M y$. Hence, M is a lattice. Finally, the lattice of hereditary subsets of P is isomorphic with the lattice of nonempty proper hereditary subsets of M under the correspondence $H \in P \leftrightarrow \{0\} \cup H \in M$.

Now suppose L is the family of all nonempty proper hereditary subsets of a finite lattice M. Clearly L is a distributive lattice. Let L_1 be the set of all hereditary subsets of M; then $L_1 = \{\emptyset, M\} \cup L$. By the proof of [1, Th. 5, p. 139], the set of all nonzero J.I. elements of L_1 is isomorphic with the collection of principal ideals of M. Therefore, the set of J.I. elements of L is isomorphic with the set consisting of $\emptyset$ and all proper principal ideals of M, and is therefore closed under products. Hence L is projective.

8. Applications and Examples.

THEOREM 8.1. *Boolean algebras and Boolean rings are projective if and only if they are finite.*

Proof. Infinite Boolean algebras and rings contain infinite disjointed sets, and hence can not be projective. On the other hand, a finite Boolean ring is a Boolean algebra and every finite Boolean algebra is isomorphic with the collection of all subsets of a finite set. Clearly the J.I. elements-the singletons and $\emptyset$-are closed under products.

THEOREM 8.2. *A chain is projective if and only if it is countable.*

Proof. Theorem 4.2 shows the necessity. Now suppose $C = \{a_i \mid i = 1, 2, \cdots\}$ is a chain. It will be shown that C is a retract of the free distributive lattice F generated by the independent set $\{b_i \mid i = 1, 2, \cdots\}$. Let $f: F \to C$ be an epimorphism such that $f(b_i) = a_i$ for $i = 1, 2, \cdots$. Define,inductively, a function $g: C \to F$ by $g(a_1) = b_1$ and

$$g(a_n) = b_n \pi\{g(a_i) \mid a_i > a_n,\ i < n\} + \Sigma\{g(a_i) \mid a_i < a_n,\ i < n\} .$$

Then g is a homomorphism and $fg = I_C$.

EXAMPLE 8.3. Let C be the chain of nonnegative integers. Then $C \times C$ is not projective.

Proof. $C \times C$ is generated by the elements $a_i = (i, 0)$ and $b_j = (0, j)$ where $i > 0, j > 0$. If $C \times C$ is projective then there exists a projective representation:

$$\begin{aligned} a_i &= \pi(S_{i,1}) + \cdots + \pi(S_{i,p(i)}) \ (i > 0) \\ b_j &= \pi(T_{j,1}) + \cdots + \pi(T_{j,p(j)}) \ (j > 0) . \end{aligned}$$

Now for some r, say $r = 1$, $S_{i,r}$ is of the form $S_{i,r} = \{a_{i_1}, \cdots, a_{i_n}\}$ where $i \leqq i_1 < \cdots < i_n$. For if not, by distributivity, we have $a_i \leqq c_1 + \cdots + c_{p(i)}$, where for each r, $c_r \in S_{i,r}$ and either $c_r = b_j$ for some j or $c_r = a_k$ for some $k < i$. This is impossible, as is seen by comparing first coordinates. Similarly, we may assume $T_{j,1} = \{b_{j_1}, \cdots, b_{j_m}\}$, where $j \leqq j_1 < \cdots < j_m$.

Now let p be an integer larger than the subscripts of all elements a_i or b_j which occur in the projective representation of a_1. Since $a_p b_p \leqq a_1$, by the projective criterion, $S_{p,1} \cup T_{p,1} \supseteqq S_{1,i}$ for some i. This is a contradiction.

THEOREM 8.4. *The direct product $\prod_{i \in I} L_i$ of finite distributive lattices is projective if and only if L_i is projective for each $i \in I$ and $|L_i| = 1$ for all but finitely many $i \in I$.*

Proof. Suppose the condition holds. Then it is sufficient to show that if L_1 and L_2 are projective then $L_1 \times L_2$ is projective. It is easily verified that the M.I. elements of $L_1 \times L_2$ are those of the form (x, y), $(x, 1)$ and $(1, y)$ where x and y are M.I. in L_1 and L_2 respectively. But since the M.I. elements of $L_1(L_2)$ are closed under sums, the M.I. elements of $L_1 \times L_2$ also have this property. Hence $L_1 \times L_2$ is projective. Conversely, each L_i is a retract of $\prod_{i \in I} L_i$ and is therefore projective. If $L_i > 1$ for infinitely many $i \in I$, then $\prod_{i \in I} L_i$ has an infinite disjointed subset and could therefore not be projective.

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