PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 1, 1967

ON EVANS' KERNEL

MITSURU NAKAI

In classical potential theory on the plane, two important kernels are considered: the hyperbolic kernel $\log(|1-\bar{\zeta}z|/|z-\zeta|)$ on $|z|<1$ and the logarithmic kernel $\log(1/|z-\zeta|)$ on $|z|<+\infty$. The former is extended to a general open Riemann surface of positive boundary as the Green's kernel.

The object of this note is to generalize the latter to an arbitrary open Riemann surface of null boundary, which we shall call Evans' kernel. The symmetry (Theorem 1) and the joint continuity (Theorem 2) of Evans' kernel are the main assertions of this note. It is also shown that Evans' kernel is obtained on every compact set in the product space as a uniform limit of Green's kernels of specified subsurfaces less positive constants (Theorem 3).

The hyperbolic and logarithmic kernels are characteristic of hyperbolic and parabolic simply connected Riemann surfaces, respectively. The corresponding rôle is played by the elliptic kernel $\log(1/[z,\zeta])$ for an elliptic simply connected Riemann surface, i.e., a sphere. The generalization of it, which we call Sario's kernel, is shown to be obtained in a natural manner from the Evan's kernel.

Wide applications of Evan's kernel are obviously promised, but we do not discuss these here at all.

1. Positive singularities. Throughout this note, we denote by R an open Riemann surface of null boundary, i.e., $R\in 0_G$ (cf. Ahlfors-Sario [1]). We denote by $\tilde{R}$ the one point compactification of Alexandroff and by ∞ the point at infinity, i.e., $\tilde{R}=R\cup\{\infty\}$ (cf. Kelly [4]).

Let $q\in\tilde{R}$. A *positive singularity* (or more precisely, normalized positive singularity) l_q at q is a positive harmonic function in a punctured open neighborhood $V(l_q)\subset R$ (i.e., $V(l_q)\cup\{q\}$ is an open neighborhood of q in $\tilde{R}$) such that

$$(1)\qquad \lim_{p\in V(l_q),\, p\to q} l_q(p)=+\infty$$

and

$$(2)\qquad \int_\alpha {}^*dl_q=-2\pi$$

for a (and hence for all) simple analytic curve $\alpha\subset V(l_q)$ which is the boundary of a neighborhood of q and is positively oriented with respect to this neighborhood. Here ${}^*dl_q$ is the conjugate differential of dl_q

(Ahlfors-Sario [1]).

Two singularities l_q and l'_q at $q \in \tilde{R}$ are said to be *equivalent* if $l_q - l'_q$ is bounded in a punctured neighborhood of q.

LEMMA 1. *There exists a positive singularity l_q for every $q \in \tilde{R}$. All l_q are equivalent by pairs for each fixed $q \in R$.*

In fact, let $q \in R$ and $\{U, z\}$ be a parametric disk at q, i.e., U is a neighborhood of q and z a conformal mapping of U onto $\{|z| < 1\}$ with $z(q) = 0$. Then $l_q(p) = \log(1/|z(p)|)$ on $V(l_q) = U - \{q\}$ is a positive singularity at q. Let l'_q be another positive singularity at q. Denote by $p = p(z)$ the inverse mapping of $z = z(p)$, and assume that $l'_q(p(z))$ is defined and positive in $\{0 < |z| < r\}$ $(0 < r < 1)$. Let ${}^*l'_q(p(z))$ be the multiple-valued conjugate of $l'_q(p(z))$ on $\{0 < |z| < r\}$, and consider $f(z) = e^{-(l'_q(p(z)) + i^*l'_q(p(z)))}$. In view of (2)

$$\int_\alpha {}^*dl'_q = -2\pi \; (\alpha = p(|z| = r'))$$

for every $0 < r < r'$. Hence $f(z)$ is single-valued in $\{0 < |z| < r\}$. It is also bounded, since $l'_q(p(z)) > 0$. Therefore $f(z)$ can be continued to all of $\{|z| < r\}$. Thus we can find a bounded analytic function $\varphi(z)$ in $\{|z| < r\}$ with $\varphi(0) \neq 0$ and $f(z) = z^n\varphi(z)$ $(n = 1, 2, \cdots)$. Hence

$$l'_q(p(z)) = -\log|f(z)| = -n\log|z| - \log|\varphi(z)| \, .$$

Clearly $\log|\varphi(z)|$ is harmonic in some $\{|z| < r''\}$ $(0 < r'' < r)$, and thus (2) implies that $n = 1$. Therefore $l'_q - l_q$ is bounded in a neighborhood of q.

For the existence of a positive singularity l_∞ at ∞, see Kuramochi [5], Nakai [6], or Sario-Noshiro [15].

There can exist two or more nonequivalent singularities at ∞. For example, let $R = \{|z| < +\infty\} - \{0\}$ and $l^\lambda_\infty(z) = \lambda\log(1/|z|)(0 < |z| < 1), (1-\lambda)\log|z|(|z| > 1)$. Since $\{0 < |z| < 1\} \cup \{1 < |z| < +\infty\}$ is a neighborhood of the Alexandroff point ∞ at infinity for R, all $l^\lambda_\infty(0 < \lambda < 1)$ are positive singularities at ∞, but the $l^\lambda_\infty - l^{\lambda'}_\infty$ are not bounded in any neighborhood of ∞ if $\lambda \neq \lambda'$.

2. Existence of Evans' kernel. The logarithmic kernel $\log(1/|z - \zeta|)$ on the plane $P = \{|z| < +\infty\}$ is a harmonic function in z on $P - \{\zeta\}$ which possesses positive and negative singularities at ζ and ∞, respectively, and is symmetric on $P \times P$. Having these in mind, we generalize the logarithmic kernel to an arbitrary open Riemann surface of null boundary as follows:

DEFINITION. An Evans' kernel $e(p, q)$ on R is a mapping of $R \times R$ onto $(-\infty, +\infty]$ satisfying the following four conditions:

(a) $e(p, q)$ is harmonic in p on $R - \{q\}$.

(b) $e(p, q)$, as a function of p, is a positive singularity at q,

(c) $-e(p, q)$, as a function of p, is a positive singularity at ∞, and $-e(p, q)$ and $-e(p, q')$ are equivalent for every pair $(q, q') \in R \times R$,

(d) $e(p, q)$ is symmetric, i.e., $e(p, q) = e(q, p)$ on $R \times R$.

The condition (b) means that there exists a positive singularity l_q at q such that

$$e(p, q) = l_q(p) + h_q(p) \tag{3}$$

in a punctured neighborhood $V(l_q)$ of q, where h_q is a harmonic function on $V(l_q) \cup \{q\}$. Since l_q is unique up to the equivalence, (3) has a definite meaning. The condition (c) means that there exists a positive singularity l_∞ at ∞ *independent of* q such that $\sup h_q < +\infty$ and

$$e(p, q) = -l_\infty(p) + h_q(p) \tag{4}$$

on R outside a compact set $K_q \subset R$. Since there can exist more than one nonequivalent positive singularity, $e(p, q)$ depends essentially on l_∞. For this reason, it would be better to call $e(p, q)$ an *l_∞-Evans' kernel*, indicating the dependence on l_∞.

We are now able to state

THEOREM 1. *On an arbitrary open Riemann surface R of null boundary there exists an l_∞-Evans' kernel which is unique up to an additive constant.*

The existence of a function satisfying (a), (b), and (c) is known (Evans [2], Selberg [16], Noshiro [10], Kuramochi [5], and Nakai [6]; see also Sario-Noshiro [15]). Such a function is usually called an Evans-Selberg's potential. Actually for each fixed $q \in R$, a function $\rho(p, q)$ with (a), (b), and (c) is obtained from $-l_\infty$ and l_q by the main existence theorem of Sario [11] (see Ahlfors-Sario [1; p. 154]). Thus the problem is to find a function $k(p)$ on R such that

$$\rho(p, q) + k(q) = \rho(q, p) + k(p)$$

for every $p, q \in R$. Instead of seeking such a $k(p)$, however, we will prove the theorem in § 3 and § 4 by an indirect procedure.

3. Let q_0 be an arbitrary but then fixed point in R. Consider open sets

$$R_n = \{p \mid p \in R, \rho(p, q_0) > -n\} \tag{5}$$

for each positive integer n. By (c), we conclude that $\bar{R}_n$ is compact in R. By the maximum principle, we also infer that R_n is connected. Clearly the relative boundary ∂R_n of R_n consists of a finite number of piecewise analytic Jordan curves. The sequence $\{R_n\}_1^\infty$ is an exhaustion of R, i.e., $\bar{R}_n \subset R_{n+1}$ and $R = \cup_1^\infty R_n$.

Let $g_n(p, q)$ be the *Green's kernel* on R_n, i.e., the mapping of $\bar{R}_n \times \bar{R}_n$ onto $[0, +\infty]$ such that $p \to g_n(p, q)$ is harmonic on $R_n - \{q\}$. It is a positive singularity at $q \in R_n$ and vanishes on ∂R_n. Moreover it is symmetric, i.e.,

$$g_n(p, q) = g_n(q, p) \tag{6}$$

on $\bar{R}_n \times \bar{R}_n$ (see Ahlfors-Sario [1]). Consider the kernel $u_n(p, q)$ on R_n defined by

$$u_n(p, q) = g_n(p, q) - n\ . \tag{7}$$

Since $R \in 0_G$, the increasing sequence $\{g_n(p, q)\}_1^\infty$ diverges to $+\infty$. However for $\{u_n(p, q)\}$, we obtain the following (cf. Tsuji [17]):

LEMMA 2. *The limit*

$$e(p, q) = \lim_{n\to\infty} u_n(p, q) \tag{8}$$

exists on $R \times R$ and is an l_∞-Evans' kernel. The convergence is uniform on $K \times \{q\}$ for all $q \in R$ and all compact sets $K \subset R - \{q\}$.

Let $q \in R$. By (c), there exists an integer $n(q)$ such that

$$|\rho(p, q) - \rho(p, q_0)| < c(q)$$

on $R - R_{n(q)}$, where $c(q)$ is a finite constant depending only on q. If $n \geqq n(q)$, then $\rho(p, q) - u_n(p, q)$ is harmonic on R_n and $\rho(p, q_0) - u_n(p, q) = 0$ on ∂R_n. Hence

$$|\rho(p, q) - u_n(p, q)| < c(q) \tag{9}$$

for every $p \in R_n$. Therefore

$$|u_{n+m}(p, q) - u_n(p, q)| < 2c(q) \tag{10}$$

for every $p \in R_n$ with $n \geqq n(q)$ and $m = 1, 2, \cdots$. Thus for an arbitrary fixed $q \in R$, there exists a subsequence of $\{u_n(p, q)\}_1^\infty$ which is uniformly convergent on each compact subset of $R - \{q\}$.

Let D be a countable dense subset of R. Using the diagonal process of Cantor, we can find a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_1^\infty$ such that for every fixed $q \in D$, $\{u_{n_k}(p, q)\}_{k=1}^\infty$ converges to a harmonic function, say $e_q(p)$, uniformly on each compact subset of $R - \{q\}$.

Next fix p arbitrarily in R. By (b), $u_n(p, q) = u_n(q, p)$ and thus $|u_{n+m}(p, q) - u_n(p, q)| = |u_{n+m}(q, p) - u_n(q, p)|$. Hence by (10), we obtain

$$|u_{n+m}(p, q) - u_n(p, q)| < 2c(p)$$

for every $q \in R_n$ with $n \geqq n(p)$ and $m = 1, 2, \cdots$. Fix k_0 with $n_{k_0} \geqq n(p)$. Then $\{u_{n_k}(p, q) - u_{n_{k_0}}(p, q)\}_{k_0}^{\infty}$ is a uniformly bounded sequence of harmonic functions in q and converges on a dense set D of R. Hence by Harnack's convergence theorem $\{u_{n_k}(p, q) - u_{n_{k_0}}(p, q)\}_{k_0}^{\infty}$, and a fortiori $\{u_{n_k}(p, q)\}_{k_0}^{\infty}$, converges uniformly on each compact subset of $R - \{p\}$. Set

$$h_p(q) = \lim_{k \to \infty} u_{n_k}(p, q) ,$$

which is a harmonic function on $R - \{p\}$.

Thus we conclude that

$$e(p, q) = \lim_{k \to \infty} u_{n_k}(p, q) \tag{11}$$

exists for every $(p, q) \in R \times R$. Again by (10), $|e(p, q) - u_{n_k}(p, q)| < c(q)$. By Harnack's theorem, the convergence in (11) is uniform on $K \times \{q\}$ for every $q \in R$ and every compact set $K \subset R - \{q\}$. Since $u_n(p, q) = u_n(q, p)$, $e(p, q)$ clearly satisfies (d). In view of (11), (a) is clearly satisfied by $e(p, q)$. From (9), it follows that

$$|\rho(p, q) - e(p, q)| < c(q) \tag{12}$$

for every $p \in R$. Since $\rho(p, q)$ satisfies (3) and (4), $e(p, q)$ also satisfies (b) and (c). Therefore $e(p, q)$ is an l_∞-Evans' kernel.

Finally we prove that (11) implies (8). Assume contrariwise that (8) is not valid. Let $\{\nu_k\}_{k=1}^{\infty}$ be the complementary subsequence of $\{n_k\}_{k=1}^{\infty}$, i.e., $\{\nu_k\}_{k=1}^{\infty} \cup \{n_k\}_{k=1}^{\infty} = \{n\}_1^{\infty}$. Since $\{u_{\nu_k}\}_{k=1}^{\infty}$ does not converge to $e(p, q)$ on $R \times R$, we can find a point $(p_1, q_1) \in R \times R$ and a subsequence $\{\mu_k\}_{k=1}^{\infty}$ of $\{\nu_k\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} u_{\mu_k}(p_1, q_1) \neq e(p_1, q_1) \tag{13}$$

exists. Since $\{u_{\mu_k}\}_{k=1}^{\infty}$ satisfies (9), by the same manner as above, we can find a subsequence $\{m_k\}_{k=1}^{\infty}$ of $\{\mu_k\}_{k=1}^{\infty}$ and an l_∞-Evans' kernel $e'(p, q)$ on R such that (11) is valid for $e'(p, q)$ and $\{u_{m_k}\}_{k=1}^{\infty}$. By (13), $e(p_1, q_1) \neq e'(p_1, q_1)$.

On the other hand, (12) is also true for $e'(p, q)$ and thus

$$|e(p, q) - e'(p, q)| < 2c(q)$$

for every $p \in R$. Therefore $p \to e(p, q) - e'(p, q)$ is a bounded harmonic function on R and consequently a constant $a(q)$ (see Ahlfors-Sario [1]).

By the symmetry, $q \longrightarrow e(p, q) - e'(p, q) = a(q)$ is also a bounded harmonic function and so $a(q)$ is a constant a, i.e., $e(p, q) = e'(p, q) + a$ on $R \times R$. Since $u_n(p, q_0) = \rho(p, q_0)$, $e(p, q_0) = e'(p, q_0)$ on R and thus $a = 0$. In particular $e(p_1, q_1) = e'(p_1, q_1)$, a contradiction.

4. To complete the proof of Theorem 1, we have only to show the uniqueness of the l_∞-Evans' kernel up to an additive constant. Let $e(p, q)$ and $e'(p, q)$ be l_∞-Evans' kernels. Consider the difference $E(p, q) = e(p, q) - e'(p, q)$. By (b) and (c), $p \longrightarrow E(p, q)$ is a bounded harmonic function on R, and so is $q \longrightarrow E(p, q)$. Similarly as above, we conclude that $E(p, q)$ is a constant.

5. **Joint continuity of Evans' kernel.** From the potential-theoretic view point, it is very important that the logarithmic kernel $\log (1/|z - \zeta|)$ is continuous on $P \times P = \{(z, \zeta) \mid |z|, |\zeta| < +\infty\}$ in the extended sense. The joint continuity of Green's kernel is well known. We can also prove the corresponding fact for Evans' kernel:

THEOREM 2. *Evans' kernel $e(p, q)$ on R is jointly continuous, i.e., e is a continuous mapping of $R \times R$ onto $(-\infty, +\infty]$.*

Specifically, $e(p, q)$ is finitely continuous on $R \times R$ outside the diagonal set, and for any relatively compact subsurface $V \subset R$, the decomposition

(14) $$e(p, q) = g_V(p, q) + v_V(p, q)$$

is valid on $V \times V$. Here g_V is the Green's kernel on V and v_V is a finitely continuous function on $V \times V$.

We shall use Heins' device (Heins [3]). Let $q_0 \in R$ and V be a relatively compact subsurface of R. We may assume that the relative boundary ∂V of V consists of a finite number of piecewise analytic Jordan curves. Assume that $q_0 \in V$. Set

$$d(q) = d(q; q_0, V) = \max_{p \in \partial V} | e(p, q) - e(p, q_0) | .$$

Observe that by (c), $e(p, q) - e(p, q_0)$ is a bounded harmonic function in p on $R - V$. Since $R \in 0_G$, $e(p, q) - e(p, q_0)$ takes its maximum and minimum on ∂V, and therefore

(15) $$| e(p, q) - e(p, q_0) | \leqq d(q)$$

for every $p \in R - V$. First we show that

(16) $$\lim_{q \to q_0} d(q) = 0 .$$

If this were not the case, then there would exist a sequence $\{q_n\}_1^\infty \subset V$ such that $\lim_n q_n = q_0$, $d(q_n) > 0$, and $\lim_n d(q_n) > 0$. Let W be a subsurface of R which is the same kind as V and such that $q_0 \in W \subset \bar{W} \subset V$. We may assume that $\{q_n\}_1^\infty \subset W$. Take the Green's kernel g_V on V and consider the function $v_n(p) = (e(p, q_n) - e(p, q_0))/d(q_n)$. This is harmonic on $R - W$ and, by (15), $|v_n(p)| \leqq 1$ for $p \in R - V$. Clearly $\mp (g_V(p, q_n) - g_V(p, q_0))/d(q_n) + 1 \pm v_n(p)$ are harmonic and non-negative on V. Therefore

$$|v_n(p)| \leqq 1 + b(q_n) \tag{17}$$

on $V - W$ and consequently on $R - W$, where

$$b(q_n) = \max_{p \in \partial W} |g_V(p, q_n) - g_V(p, q_0)|/d(q_n) .$$

Since $g_V(p, q)$ is continuous on $V \times V$ and $\{1/d(q_n)\}_1^\infty$ is bounded, we obtain $\lim_n b(q_n) = 0$. Hence from (17), it follows that $\{v_n\}_1^\infty$ is a sequence of uniformly bounded harmonic functions on $R - W$. Let $p \in R - W$ be arbitrary but fixed for the time being. Since $e(p, q_n) \to e(p, q_0)$ $(n \to \infty)$ and $\{1/d(q_n)\}_1^\infty$ is bounded, $\lim_n v_n(p) = 0$. Thus $\{v_n\}_1^\infty$ converges to zero uniformly on each compact subset of $R - W$ and in particular on ∂V. However this is impossible, since $\max_{p \in \partial V} |v_n(p)| = d(q_n)/d(q_n) = 1$. Hence (16) must be valid.

Let $(p_0, q_0) \in R \times R$ with $p_0 \neq q_0$. In particular, choose V in such a fashion that $p_0 \notin \bar{V}$, $q_0 \in V$. Let $p \notin \bar{V}$ and $q \in V$. Then from (15) it follows that

$$|e(p, q) - e(p_0, q_0)| \leqq |e(p, q_0) - e(p_0, q_0)| + d(q; q_0, V) .$$

By (16) and $\lim_{p \to p_0} e(p, q_0) = e(p_0, q_0)$, we conclude that

$$\lim_{(p,q) \to (p_0,q_0)} e(p, q) = e(p_0, q_0) ,$$

i.e., $e(p, q)$ is finitely continuous on $R \times R$ outside the diagonal set.

Finally consider

$$v_V(p, q) = v(p, q) = e(p, q) - g_V(p, q) \tag{18}$$

on $V \times V$. From what we have seen thus far, it follows that $v(p, q)$ is finitely continuous on $V \times V$ outside the diagonal set. Let $p_0 \in V$ and W be an open neighborhood of p_0 with $\bar{W} \subset V$. For any $\varepsilon > 0$, we can find an open neighborhood U of p_0 such that $\bar{U} \subset W$ and

$$v(p, p_0) - \varepsilon < v(p, q) < v(p, p_0) + \varepsilon \tag{19}$$

for every $(p, q) \in (\partial W) \times U$. For an arbitrary fixed $q \in U$, the functions of p on $\bar{W}$ involved in (19) are harmonic since positive singularities cancel, and (19) is valid on ∂W. Therefore by the maximum principle,

(19) is valid on $\overline{W}$. Thus in particular, (19) is true for every (p, q) in $U \times U$. Hence

$$|v(p, q) - v(p_0, p_0)| \leqq |v(p, p_0) - v(p_0, p_0)| + \varepsilon .$$

Since $\lim_{p \to p_0} v(p, p_0) = v(p_0, p_0)$, $\lim_{(p,q)\to(p_0,q_0)} v(p, q) = v(p_0, q_0)$. Therefore e is the sum of a finitely continuous function v and the Green's kernel which is also continuous on $V \times V$.

6. Approximation by Green's kernels.

As a complementary statement to Lemma 2, we shall prove

THEOREM 3. *Let $e(p, q)$ be an Evans' kernel on R and $g_\lambda(p, q)$ be the Green's kernel on $R_\lambda = \{p \mid p \in R, e(p, q_0) > -\lambda\}$ with a fixed $q_0 \in R$. Then*

$$e(p, q) = \lim_{\lambda \to \infty} (g_\lambda(p, q) - \lambda) \tag{20}$$

uniformly on each compact set of $R \times R$, i.e.,

$$\lim_{\lambda \to \infty} \sup_{(p,q) \in K \times K} |e(p, q) - (g_\lambda(p, q) - \lambda)| = 0$$

for every compact set $K \subset R$.

By a similar manner as in the proof of Lemma 2, we can show that $e'(p, q) = \lim_{\lambda \to \infty} (g_\lambda(p, q) - \lambda)$ exists on $R \times R$ and $e'(p, q)$ is an Evans' kernel such that $p \to e'(p, q)$ gives a positive singularity at ∞ equivalent to that of $p \to e(p, q)$. Moreover the convergence is uniform on $K \times \{q\}$ with an arbitrary $q \in R$ and an arbitrary compact set $K \subset R - \{q\}$. Since $p \to e(p, q) - e'(p, q)$ is bounded and harmonic on R, as in § 4, $e(p, q) - e'(p, q)$ is a constant on $R \times R$. Moreover $e(p, q_0) = g_\lambda(p, q_0) - \lambda = e'(p, q_0)$ on R_λ, and we conclude that $e(p, q) = e'(p, q)$ on $R \times R$, i.e., the identity (20) is valid.

Let $w_\lambda(p, q) = (e(p, q) - (g_\lambda(p, q) - \lambda))$ on R_λ. Fix an arbitrary $\lambda_0 > 0$ and let $\lambda > \lambda_0$. For an arbitrary fixed $q \in \overline{R}_{\lambda_0}$, $p \to w_\lambda(p, q)$ is harmonic on $\overline{R}_\lambda$ and for $p \in \partial R_\lambda$, $w_\lambda(p, q) = (e(p, q) - (g_\lambda(p, q) - \lambda)) = (e(p, q) - (g_\lambda(p, q_0) - \lambda)) = e(p, q) - e(p, q_0)$, since $g_\lambda(p, q) = g_\lambda(p, q_0) = 0$. Therefore $|w_\lambda(p, q)| \leqq \max_{p \in \partial R_\lambda} |e(p, q) - e(p, q_0)|$ for $p \in R_\lambda$, and thus

$$|w_\lambda(p, q)| \leqq \max_{(p,q) \in (\partial R_\lambda) \times \overline{R}_{\lambda_0}} |e(p, q) - e(p, q_0)| \tag{21}$$

for every $(p, q) \in R_\lambda \times \overline{R}_{\lambda_0}$. By Theorem 2, $|e(p, q) - e(p, q_0)|$ is finitely continuous on $(\partial R_\lambda) \times \overline{R}_{\lambda_0}$ and thus

$$M_\lambda = \max_{(p,q) \in (\partial R_\lambda) \times \overline{R}_{\lambda_0}} |e(p, q) - e(p, q_0)| < \infty . \tag{22}$$

By (c), $p \longrightarrow e(p, q) - e(p, q_0)$ is a bounded harmonic function on $R - R_\lambda$ for each fixed $q \in \bar{R}_{\lambda_0}$. Thus from $R \in 0_G$, it follows that

$$|e(p, q) - e(p, q_0)| \leqq \max_{p \in \partial R_\lambda} |e(p, q) - e(p, q_0)| \leqq M_\lambda$$

for every $(p, q) \in (R - R_\lambda) \times \bar{R}_{\lambda_0}$. Hence in particular

$$M_{\lambda'} \leqq M_\lambda \tag{23}$$

for all $\lambda' > \lambda$. Therefore by (21), (22), and (23), there exists a finite constant M and $\lambda_1 \in (\lambda_0, +\infty)$ such that

$$|w_\lambda(p, q)| < M \tag{24}$$

for every $(p, q) \in \bar{R}_{\lambda_0} \times \bar{R}_{\lambda_0}$ and $\lambda > \lambda_1$.

Set $f_\lambda(p, q) = w_\lambda(p, q) + M$. Then

$$0 \leqq f_\lambda(p, q) \leqq 2M \tag{25}$$

on $\bar{R}_{\lambda_0} \times \bar{R}_{\lambda_0}$. Hence $p \longrightarrow f_\lambda(p, q)$ and $q \longrightarrow f_\lambda(p, q)$ are nonnegative harmonic functions on R_{λ_0}. Therefore

$$\begin{aligned} k(p, p')^{-1} f_\lambda(p', q) \leqq f_\lambda(p, q) \leqq k(p, p') f_\lambda(p', q) \;, \\ k(q, q')^{-1} f_\lambda(p', q') \leqq f_\lambda(p', q) \leqq k(q, q') f_\lambda(p', q') \end{aligned} \tag{26}$$

for arbitrary points p, p', q, and q' in R_{λ_0}. Hence for $(s, t) \in R_{\lambda_0} \times R_{\lambda_0}$, $k(s, t)$ is given by

$$k(s, t) = k_{R_{\lambda_0}}(s, t) = \inf \{c \mid e^{-1} h(s) \leqq h(t) \leqq c h(s) \ \textit{for every}\ h \in HP(R_{\lambda_0})\} \;,$$

where $HP(R_{\lambda_0})$ is the class of all nonnegative harmonic functions on R_{λ_0}. From the Poisson formula, it follows that $1 \leqq k(s, t) < \infty$ and

$$\lim_{s \to t} k(s, t) = 1 \tag{27}$$

(cf. Nakai [7]). By (25) and (26), we obtain

$$|f_\lambda(p, q) - f_\lambda(p', q')| \leqq 2M(k(p, p')k(q, q') - 1)$$

and in turn

$$|w_\lambda(p, q) - w_\lambda(p', q')| \leqq 2M(k(p, p')k(q, q') - 1) \tag{28}$$

for every (p, q) and (p', q') in $R_{\lambda_0} \times R_{\lambda_0}$. From (27) and (28), it follows that the family $\{w_\lambda(p, q)\}_{\lambda > \lambda_1}$ is equicontinuous on $R_{\lambda_0} \times R_{\lambda_0}$. Therefore the convergence $\lim_{\lambda \to \infty} w_\lambda(p, q) = 0$ on $R_{\lambda_0} \times R_{\lambda_0}$ outside the diagonal implies the uniform convergence $\lim_{\lambda \to \infty} w_\lambda(p, q)$ on $\bar{R}_{\lambda_0/2} \times \bar{R}_{\lambda_0/2}$.

7. Sario's kernel. The most important potential-theoretic kernel on the extended plane $\tilde{P} = \{|z| \leqq +\infty\}$ is the elliptic kernel $\log(1/[z, \zeta])$, where

$$[z, \zeta] = |z - \zeta|/\sqrt{1 + |z|^2}\sqrt{1 + |\zeta|^2}\ .$$

For simplicity, let $s(z, \zeta) = \log(1/[z, \zeta])$ and $e(z, \zeta) = \log(1/|z - \zeta|)$. Observe that

$$s(z, \zeta) = \tfrac{1}{2}\log(1 + e^{-2e(z,0)})(1 + e^{-2e(\zeta,0)}) + e(z, \zeta)\ . \tag{29}$$

In view of this, the most natural generalization of the elliptic kernel to an arbitrary closed surface S is as follows:

$$s(p, q) = \tfrac{1}{2}\log(1 + e^{-2e(p,a)})(1 + e^{-2e(q,a)}) + e(p, q) \tag{30}$$

for $(p, q) \in S \times S$, where a is an arbitrary but then fixed point in S, ∞ a point in S different form a, and $e(p, q)$ is an Evans' kernel on $S - \{\infty\}$.

For an open Riemann surface $S \in 0_G$, the kernel $s(p, q)$ can also be defined by (30), where ∞ is taken as the Alexandroff point at infinity of S.

Even if $S \notin 0_G$, maintaining the formality (29), we may define

$$s(p, q) = \tfrac{1}{2}\log(1 + e^{-2g(p,a)})(1 + e^{-2g(q,a)}) + g(p, q)\ , \tag{31}$$

where $g(p, q)$ is the Green's kernel on S.

Then the kernel $s(p, q)$ on an arbitrary Riemann surface S enjoys most of the important properties of the elliptic kernel, and thus may be regarded as a generalization of the elliptic kernel. It satisfies the following:

(α) *$s(p, q)$ is bounded from below on $S \times S$,*

(β) *$s(p, q) = s(q, p)$ on $S \times S$,*

(γ) *$\Delta_p s(p, q)$ exists on $S - \{a, q\}$, is continuously extendable to S, and the resulting 2-form is independent of q,*

(δ) *for every subsurface $\Omega \subset S$ with $\Omega \notin 0_G$, there exists a finitely continuous function $v_\Omega(p,q)$ on $\Omega \times \Omega$ such that*

$$s(p, q) = g_\Omega(p, q) + v_\Omega(p, q) \tag{32}$$

on $\Omega \times \Omega$, where $g_\Omega(p, q)$ is the Green's kernel on Ω.

In general, a function with the four properties (α)—(δ) may be called *Sario's kernel* on S, since Sario [12, 13, 14] constructed such a function (see also Nakai [8, 9]). In our case, the formulas (30) and (31) enable us to prove (α)—(δ) quite rapidly.

The properties (β) and (γ) are direct consequences of (30) and (31). For open S, (δ) is again an easy consequence of the very definition of $s(p, q)$ and (14). For closed S, we have only to consider the case where Ω is a parametric disk at ∞ and $a \notin \bar{\Omega}$. Observe that there is only one positive singularity $g_\Omega(p, \infty)$ at ∞ up to the equivalence.

Let

$$v(p, q) = e(p, q) - g_\Omega(p, \infty) - g_\Omega(q, \infty) - g_\Omega(p, q) .$$

Both $p \longrightarrow v(p, q)$ and $q \longrightarrow v(p, q)$ are harmonic on $\bar{\Omega}$. Clearly

$$v(p, q) \geqq \min_{(p,q)\in(\partial\Omega)\times(\partial\Omega)} v(p, q)$$

for every $(p, q) \in \Omega \times \Omega$. Since $v(p, q) = e(p, q)(> -\infty)$ is continuous on $(\partial\Omega) \times (\partial\Omega)$, there exists a constant c such that $v(p, q) \geqq c > -\infty$. Similarly as in the proof of (28) we obtain

$$| v(p, q) - v(p', q') | \leqq | v(p', q') - c | (k_\Omega(p, p')k_\Omega(q, q') - 1) .$$

Thus $v(p, q)$ is finitely continuous on $\Omega \times \Omega$. From this, (32) follows.

8. Finally we prove (α). We only prove it for open $S \in 0_G$. If S is closed, then we have only to consider $S - \infty$. For $S \notin 0_G$, the same proceedure with the replacement of $e(p, q)$ by $g(p, q)$ and with an obvious modification gives the proof.

Take a relatively compact subsurface V of S containing a. Set

$$A_1 = \inf_{(p,q)\in\bar{V}\times\bar{V}} s(p, q) ,$$
$$A_2 = \inf_{(p,q)\in\bar{V}\times(S-\bar{V})} s(p, q) = \inf_{(p,q)\in(S-\bar{V})\times\bar{V}} s(p, q)$$

and

$$A_3 = \inf_{(p,q)\in(S-\bar{V})\times(S-\bar{V})} s(p, q) .$$

We have to show that $A_i > -\infty (i = 1, 2, 3)$.

In general, $s(p, q) > e(p, q) > -\infty$. Since $e(p, q)$ is continuous on $\bar{V} \times \bar{V}$, $A_1 \geqq \min_{(p,q)\in\bar{V}\times\bar{V}} e(p, q) > -\infty$.

Next consider the case $(p, q) \in (S - \bar{V}) \times \bar{V}$. Clearly

$$s(p, q) > e(p, q) - e(p, a) \equiv w(p, q) .$$

By (c), $p \longrightarrow w(p, q)$ is bounded and harmonic in $S - \bar{V}$. Since $S \in 0_G$,

$$w(p, q) \geqq \min_{p\in\partial V} w(p, q)$$

for every $(p, q) \in (S - \bar{V}) \times \bar{V}$. The function $w(p, q)(> -\infty)$ is continuous on $(\partial V) \times \bar{V}$ and thus

$$w(p, q) \geqq \min_{p\in\partial V} w(p, q) \geqq \min_{(p,q)\in(\partial V)\times\bar{V}} w(p, q) > -\infty$$

for all $(p, q) \in (S - \bar{V}) \times \bar{V}$. Therefore $A_2 \geqq \min_{(p,q)\in(\partial V)\times\bar{V}} w(p, q) > -\infty$.

Finally let $(p, q) \in (S - \bar{V}) \times (S - \bar{V})$ and observe that

$$s(p, q) > e(p, q) - e(p, a) - e(q, a) \equiv v(p, q) .$$

By (c), $p \to v(p, q)$ is bounded in a punctured neighborhood of ∞. Moreover it is harmonic in $S - \bar{V} - \{q\}$ and $v(q, q) = +\infty$. By $S \in 0_G$, we infer

$$v(p, q) \geqq \min_{p \in \partial V} v(p, q) \tag{33}$$

for every $(p, q) \in (S - \bar{V}) \times (S - \bar{V})$. Fix p arbitrarily in ∂V. Similarly as above, the minimum principle applied to the harmonic function $q \to v(p, q)$ gives

$$v(p, q) \geqq \min_{q \in \partial V} v(p, q) \tag{34}$$

for every $(p, q) \in (\partial V) \times (S - \bar{V})$. From (33) and (34), it follows that

$$v(p, q) \geqq \min_{(p,q) \in (\partial V) \times (\partial V)} v(p, q)$$

for all $(p, q) \in (S - \bar{V}) \times (S - \bar{V})$. Again since $v(p, q)(> -\infty)$ is continuous on $(\partial V) \times (\partial V)$, we conclude that

$$A_3 \geqq \min_{(p,q) \in (\partial V) \times (\partial V)} v(p, q) > -\infty .$$

References

1. L. V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Univ. Press, Princeton, 1960.
2. G. C. Evans, *Potentials and positively infinite singularities of harmonic functions*, Mh. Math. u. Phys. **43** (1936), 419-424.
3. M. Heins, *Lindelöfian maps*, Ann. of Math. **62** (1955), 418-446.
4. J. L. Kelly, *General Topology*, Van Nostrand, Princeton, 1955.
5. Z. Kuramochi, *Mass distributions on the ideal boundaries of abstract Riemann surfaces* I, Ôsaka Math. J. **8** (1956), 119-137.
6. M. Nakai, *On Evans potential*, Proc. Japan Acad. **38** (1962), 624-629.
7. ———, *Radon-Nikodym densities between harmonic measures on the ideal boundary of an open Riemann surface*, Nagoya Math. J. **27** (1965), 71-76.
8. ———, *Potentials of Sario's kernel*, J. d'Analyse Math. **17** (1966), 225-240.
9. ———, *Sario's potentials and analytic mappings*, Nagoya Math. J. **29** (1967), 93-101.
10. K. Noshiro, *Contributions to the theory of singularities of analytic functions*, Japanese J. Math. **19** (1948), 299-327.
11. L. Sario, *A linear operator method on arbitrary Riemann surfaces*, Trans. Amer. Math. Soc. **72** (1952), 281-295.
12. ———, *Value distribution under analytic mappings of arbitrary Riemann surfaces*, Acta Math. **109** (1963), 1-10.
13. ———, *General value distribution theory*, Nagoya Math. J. **25** (1963), 213-229.
14. ———, *A theorem on mappings into Riemann surfaces of infinite genus*, Trans. Amer. Math. Soc. **117** (1965), 276-284.
15. L. Sario and K. Noshiro, *Value Distribution Theory*, Van Nostrand, Princeton, 1966.
16. H. Selberg, *Über die ebenen Punktmengen von per Kapazität Null.*, Avh. Norske

Videnskaps-Akad. Oslo I Math.-Natur. (1937), No. 10.
17. M. Tsuji, *Existence of a potential function with a prescribed singularity on any Riemann surface*, Tôhoku Math. J. **4** (1952), 54-68.

Received July 26, 1966. The work was sponsored by the U.S. Army Research Office—Durham, Grant DA-AROD-31-124-G742, University of California, Los Angeles.

MATHEMATICAL INSTITUTE, NAGOYA UNIVERSITY
AND UNIVERSITY OF CALIFORNIA, LOS ANGELES