PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 1, 1967

AN L^1 ALGEBRA FOR CERTAIN LOCALLY COMPACT TOPOLOGICAL SEMIGROUPS

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This paper may be considered as another chapter in the theory of convolution algebras inaugurated by Hewitt and Zuckerman. The interest here is in finding an L^1 theory for locally compact commutative topological semigroups which extends the known l_1 theory for discrete commutative semigroups and L^1 theory for locally compact topological groups. Let S be a locally compact commutative semigroup and m a nonnegative regular Borel measure on S such that if $x \in S$ and $E \subset S$ with $m(E) = 0$ then $m(Ex^{-1}) = 0$ $(Ex^{-1} = [y : yx \in E])$. When $L^1(S, m)$ is defined as the Banach space of all bounded complex measures $\mu \in M(S)$ which are absolutely continuous with respect to m, then $L^1(S, m)$ is a convolution algebra as a subalgebra of $M(S)$. It is shown that there is a one to one correspondence between the measurable semicharacters on S and the multiplicative linear functionals on $L^1(S, m)$ analogus to the group situation. Extensions of the above results to those S with a measure m satisfying the above condition in a local sense are also obtained.

This research was motivated by the results of Hewitt and Zuckerman [2, 3] and the desire to find that condition that the counting measure on a discrete semigroup and Haar measure on a locally compact group possess that cause their respective L^1 spaces to be convolution algebras. The notation and terminology used here can be found in [4]. In addition, we use $S \backslash A$ to indicate the complement of the set A in S and $\varnothing$ to denote the empty set.

2. Existence of $L^1(S, m)$ as an algebra.

In this section we define for a locally compact topological semigroup S possessing nonnegative regular Borel measure m the Banach space $L^1(S, m)$ and give sufficient conditions on m so that $L^1(S, m)$ is a Banach algebra. It is clear that if S is a group (it will then be a locally compact topological group) and m is Haar measure on S then $L^1(S, m)$ is the usual group algebra of S. It is also clear that if S is a discrete commutative semigroup and m is the counting measure on S, then $L^1(S, m)$ is Hewitt and Zuckermans $l_1(S)$.

The space of all continuous complex valued functions on S vanishing at infinity is denoted by $C_0(S)$ and its dual space $C_0(S)^*$ by $M(S)$, the corresponding bounded complex measures via [1, 14.4]. The space $M(S)$ is to be considered as a Banach algebra with multiplication given by the convolution of measures, that is if $\mu, \nu \in M(S)$ then $\mu * \nu(f) = \iint f(xy)\mu(dx)\nu(dy)$. At times this identification will be glossed over and we will write $\mu(E)$ for $\mu(\psi_E)$ etc. For two complex valued Borel measures μ and ν on S, ν is said to be absolutely continuous with respect to μ (written $\nu \ll \mu$) if for each Borel set $E \subset S$, $\mu(E) = 0$ implies $\nu(E) = 0$.

DEFINITION. Let S be a locally compact topological semigroup and m a regular Borel measure on S. We define

$$L^1(S, m) = [\mu : \mu \in M(S) \quad \text{and} \quad \mu \ll m] .$$

PROPOSITION 2.1. The space $L^1(S, m)$ is a Banach Space.

NOTATION. For any subset $E \subset S$ and $x \in S$. Let

$$Ex^{-1} = [y : y \in S \cdot yx \in E]$$

and

$$x^{-1}E = [y ; y \in S \cdot xy \in E] .$$

There follows immediately:

LEMMA 2.2. *If $x \in S$ and E is open (closed) then $x^{-1}E$ and Ex^{-1} are both open (closed). If $x \in S$ and E is a Borel set then so are $x^{-1}E$ and Ex^{-1}.*

THEOREM 2.3. *Let S be a locally compact topological semigroup and m a regular Borel measure on S. If for each $E \subset S$ with $m(E) = 0$, $m(Ex^{-1}) = 0$ for almost all $x \in S$, then $L^1(S, m)$ is a subalgebra of $M(S)$.*

Proof. Let μ and ν belong to $L^1(S, m)$ and let E be any set in S with $m(E) = 0$. Now there is a set F of measure 0 such that $m(Ex^{-1}) = 0$ for all $x \notin F$. Since

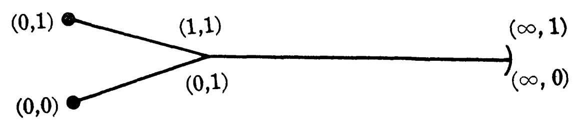
$$\begin{aligned} \mu * \nu(E) &= \iint \psi_E(xy)\mu(dx)\nu dy) = \int \mu(Ey^{-1})\nu(dy) \\ &= \int_F \mu(Ey^{-1})\nu(dy) + \int_{S\backslash F} \mu(Ey^{-1})\nu(dy) , \end{aligned}$$

and $\mu(Ey^{-1}) = 0$ for $y \in S\backslash F$, and $\nu(F) = 0$, it follows that $\mu * \nu(E) = 0$ and $\mu * \nu \ll m$. Thus, $\mu * \nu L^1(S, m)$ and $L^1(S, m)$ is a subalgebra of $M(S)$.

It is obvious that the requirement in the above theorem on the measure m is always satisfied by Haar measure on a locally compact group and by the counting measure on a discrete semigroup. However, the following condition is weaker than left invariance of a measure and also satisfies the hypothesis in the above theorem. If m is a nonnegative regular Borel measure such that $m(Ex) \geqq m(E)$ for all measurable sets E, then $m(Ex^{-1}) \leqq m((Ex^{-1})x) \leqq m(E)$ since $(Ex^{-1})x \subset E$; thus, if $m(E) = 0$ then $m(Ex^{-1}) = 0$. There follows:

COROLLARY 2.4. *If S is a locally compact topological semigroup with a nonnegative regular Borel measure m, for which $m(Ex) \geqq m(E)$ whenever E is a Borel set, then $L^1(S, m)$ is a Banach subalgebra of $M(S)$.*

We note that the condition on m in the corollary above does not force S to be a subset of a group. For let T be the cartesian product of $[0, \infty)$ with a two element group $\{0, 1\}$ with coordinate wise operations. Let $\mathscr{R}$ be the relation on T for which $(x, y)\mathscr{R}(a, b)$ where $0 \leqq x < \infty$, $0 \leqq a < \infty$, $y, b \in \{0, 1\}$ if and only if $x = a$ and $x \geqq 1$ or $x = a$ and $y = b$. The resulting semigroup $S = T/\mathscr{R}$ (Figure 1)



can have induced the measure m given by $m(E) = (\lambda \times \mu)(\varnothing^{-1}(E))$ where $\varnothing : T \to S$ is the natural mapping, λ is Lebesgue measure on $[0, \infty]$ and μ is Haar measure on $\{0, 1\}$. Then if

$$E = [(x, y) : 0 \leqq x \leqq 1,\ y = 1],\ m(E) = 1$$

but $m[E + (1, 0)] = 2$.

The conditions on the measure m in the preceding theorem and corollary are very strong and the class of such semigroups under discussion needs to be expanded. Let S be a locally compact topological semigroup satisfying

(1) S is a union of disjoint locally compact subsemigroups S_α, and

(2) on each S_α, there exists a nonnegative regular Borel measure m_α such that if $E \subset S_\alpha$ and $m_\alpha(E) = 0$, then $m_\alpha(Ex^{-1} \cap S_\alpha) = 0$ for almost all $x \in S_\alpha$, and

(3) a set $E \subset S$ is measurable if and only if $E \cap S_\alpha$ is measurable (m_α) and $m(E) = \sum m_\alpha(E \cap S_\alpha)$.

THEOREM 2.5. *Let S and m be as above. The Banach space $L^1(S, m)$ is isomorphic (algebraically and topologically) to the com-*

pletion of the weak direct sum of the subspaces $L^1(S_\alpha, m_\alpha)$ (i.e., *the closure of the weak direct sum of the* $L^1(S_\alpha, m_\alpha)$ *in* $M(S)$).

The proof of the above is straightforward.

3. Complex homomorphisms on $L^1(S, m)$. In this section we consider the multiplicative linear functions on the Banach algebra $L^1(S, m)$ and show that there is a correspondence with the m-measurable bounded complex-valued homomorphisms on S, when m is further restricted.

Let S be a locally compact abelian topological semigroup and let m be a nonnegative regular Borel measure on S such that (i) if $E \subset S$ and $m(E) = 0$ then $m(Ex^{-1}) = 0$, (ii) each compact set in S has finite measure, and (iii) each nonempty open set has positive measure. The conditions (ii) and (iii) are imposed in order to obtain (3.3) and (3.6). They are always true in a locally compact topological group with m as Haar measure and also in a discrete semigroup with counting measure. Before the investigation of the relationship between the multiplicative linear functionals on $L^1(S, m)$ and the bounded complex valued homomorphisms on $S(\theta(xy) = \theta(x)\theta(y))$, the following are necessary tools to be used.

LEMMA 3.1. *Let* $\mu \in L^1(S, m)$ *and* $x \in S$. *Denote by* $\bar{x}$ *the point measure at* x *in* S. *Then,* $\mu * \bar{x} \in L^1(S, m)$.

Proof. If $E \subset S$ and $m(E) = 0$, then

$$\begin{aligned}\mu * \bar{x}(E) &= \iint \psi_E(zy)\mu(dz)\bar{x}(dy)\\ &= \int \psi_E(zx)\mu(dz) = \int \psi_{Ex^{-1}}(z)\mu(dz) = \mu(Ex^{-1}) .\end{aligned}$$

Since $\mu \ll m$ and $m(Ex^{-1}) = 0$, $\mu(Ex^{-1}) = 0$ and $\mu * \bar{x} \in L^1(S, m)$.

We state without proof:

LEMMA 3.2. *Let A and B Borel sets in S and $x \in S$, then*

(1) $x(x^{-1}A) \subset A$,

(2) (a) $x^{-1}(xA) \supset A$

(b) $x^{-1}(xA) = A$ *for all Borel sets* $A \Leftrightarrow S$ *is a cancellation semigroup,*

(3) $x^{-1}(S \backslash B) = S \backslash (x^{-1}B)$

(4) $[\bar{x} * \mu](A) = \mu(x^{-1}A)$

(5) $[\bar{x} * |\mu|](S \backslash xB) \leqq |\mu|(S \backslash B)$.

Equality in (1) above need not hold even in a cancellation semigroup, for let $S = (0, 1]$, $A = [1/2, 3/4]$ and $x = (1/2)$, then $x(x^{-1}A) = \{1/2\}$.

DEFINITION. A character τ on S is a nontrivial (i.e. not identially 0) bounded homomorphism of S into the multiplicative semigroup of complex numbers.

The set of all m-measurable characters will be denoted by S^*. Two characters which agree almost everywhere with respect to m are to be considered as the same character.

THEOREM 3.3. *Each $\tau \in S^*$ gives rise to a multiplicative linear functional h on $L^1(S, m)$ via*

$$h(\mu) = \int \tau d\mu .$$

Conversely, each multiplicative linear functional h on $L^1(S, m)$ defines an m-measurable character τ on S and $h(\mu) = \int \tau d\mu$.

Before proceeding with the proof, let us note that if S is a group and m is Haar measure then the above theorem is a well known result, since each measurable character is continuous [1]. Further, the identification of elements of S^* which agree a.e. (m) is necessary in order to have a one-to-one correspondence of the measurable characters and the maximal ideal space of the Banach algebra $L^1(S, m)$. In certain classes of semigroups with specially chosen measures to be considered in a forthcoming paper, it will be shown that the measurable characters are continuous a.e. (m).

Proof of theorem. Let $\tau \in S^*$ and $h(\mu) = \int \tau d\mu$, then it is known that h is a bounded linear functional on $L^1(S, m)$ and it need only be shown that $h(\mu * \nu) = h(\mu)h(\nu)$. If $\mu, \nu \in L^1(S, m)$,

$$\begin{aligned} h(\mu * \nu) = \int \tau d(\mu * \nu) &= \iint \tau(xy)\mu(dx)\nu(dy) \\ &= \iint \tau(x)\tau(y)\mu(dx)\nu(dy) = h(\mu)h(\nu) . \end{aligned}$$

Conversely, if h is a bounded multiplicative linear functional on $L^1(S, m)$, then there is a bounded measurable function $\varnothing \in L^\infty(S, m)$ such that $h(\mu) = \int \varnothing d\mu$. Let $\mu \in L^1(S, m)$ with $h(\mu) \neq 0$ and set $\tau(x) = h(\mu * \bar{x})/h(\mu)$. If follows readily that τ is a well defined bounded complex valued homomorphism on S independent of the choice of μ.

There remains to prove the measurability of τ. Fix $\lambda \in L^1(S, m)$ such that $h(\lambda) = 1$ and let E be a σ-compact subset of S such that

$|\lambda|(S\backslash E) = 0$. Since $(\bar{x} * |\lambda|)(S\backslash E) \leqq |\lambda|(S\backslash E) = 0$ by (3.2) (5),

$$\tau(x) = h(\bar{x} * \lambda) = \int \varnothing(y)(\bar{x} * \lambda)(dy) = \int \psi_{xE}(y)\varnothing(y)(\bar{x} * \lambda)(dy)$$
$$= \int \psi_{xE}(xy)\varnothing(xy)\lambda(dy) = \int \psi_{x^{-1}(xE)}(y)\varnothing(xy)\lambda(dy) ,$$

but, by (3.2), (2), $x^{-1}(xE) \supset E$ and $|\lambda|(S\backslash E) = 0$ imply

$$\int \psi_{x^{-1}(xE)\backslash E}(y)\varnothing(xy)\lambda(dy) = 0$$

so that

$$\tau(x) = \int \psi_E(y)\varnothing(xy)\lambda(dy) .$$

Let F be a compact subset of S with $m(F) > 0$. The function $f: (x, y) \rightarrow \psi_F(x)\psi_E(y)\varnothing(xy)$ is Borel measurable on $S \times S$ and vanishes outside $F \times E$ a σ-compact set Now

$$\iint \psi_F(x)\psi_E(y) \,|\varnothing(xy)|\, |\lambda|(dy)m(dx) \leqq \|\varnothing\|_\infty \|\lambda\| m(F) < \infty ,$$

thus, $f \in L^1(S \times S, m \times |\lambda|)$ and hence

$$x \longrightarrow \int \psi_F(x)\psi_E(y)\varnothing(xy)\lambda(dy) = \psi_F(x)\int \psi_E(y)\varnothing(xy)\lambda(dy) = \psi_F(x)\tau(x)$$

is m-measurable on S. It follows that τ is m-measurable on F for each compact $F \subset S$; hence, τ is measurable on S.

Let $\lambda, \mu \in L^1(S, m)$ with $h(\lambda) \neq 0$, then

$$h(\lambda)\int \tau(x)\mu(dx) = \int h(\bar{x} * \lambda)\mu(dx)$$
$$= \iiint \varnothing(zw)\bar{x}(dz)\lambda(dw)\mu(dx)$$
$$= \iint \varnothing(xw)\lambda(dw)\mu(dx)$$
$$= \int \varnothing(t)(\lambda * \mu)(dt) = h(\lambda * \mu) = h(\lambda)h(\mu) .$$

Hence

$$h(\mu) = \int \tau(x)\mu(dx)$$

for all $\mu \in L^1(S, m)$.

Let S be a locally compact abelian topological semigroup satisfying:

(1) S is a union of disjoint locally compact subsemigroups $\{S_\alpha : \alpha \in I\}$, I an index set, such that if $\alpha, \beta \in I$ then there is a $\lambda \in I$

such that $S_\alpha S_\beta \subset S_\gamma$, and

(2) on each S_α, there exists a nonnegative regular Borel measure m_α such that (i) if $E \subset S_\alpha$ and $m_\alpha(E) = 0$ then $m_\alpha(Ex^{-1} \cap S_\alpha) = 0$ for all $x \in S_\alpha$, (ii) if F is compact and $F \subset S_\alpha$ then $m_\alpha(F) < \infty$, (iii) if U is open in S_α then $m_\alpha(U) > 0$ and (iv) if $S_\alpha S_\beta \subset S_\beta$ then for each $y \in S_\beta$ the mapping $\varnothing_{(\alpha,\beta,y)}; S_\alpha \to S_\beta$ given by $\varnothing_{(\alpha,\beta,y)}(x) = xy$ is measurable i.e. E m_β-measurable in S_β implies $\varnothing^{-1}_{(\alpha\,\beta,y)}(E)$ m_α-measurable in S_α for all $y \in S_\beta$), and

(3) a measure m is defined on S by (i) $E \subset S$ is measurable if and only if $E \cap S_\alpha$ is m_α-measurable for all $\alpha \in I$, and (ii) $m(E) = \sum m_\alpha(E \cap S_\alpha)$.

THEOREM 3.4. *Let S and m be as above. If $L^1(S, m)$ is a subalgebra of $M(S)$ such that $\bar{x} * L^1(S, m) \subset L^1(S, m)$ for each $x \in S$, then there is a one to one correspondence between the m-measurable characters on S and the multiplicative linear functionals on $L^1(S, m)$ as in* (3.3).

Proof. Let τ be any m-measurable character on S and for $\mu \in L^1(S, m)$ set $h(\mu) = \int_S \tau(x)\mu(dx)$. Then h is a bounded linear functional on $L^1(S, m)$ and since τ is a homomorphism for any $\mu, v \in L_1(S, m)$,

$$\int_S \tau(x)\mu(dx)\int_S \tau(y)v(dy) = \int_S\int_S \tau(xy)\mu(dx)v(dy) = \int_S \tau(z)(\mu * v)dz$$

thus $h(\mu)h(v) = h(\mu * v)$ and h is a multiplicative linear functional.

On the other hand, let h be a multiplicative linear functional on $L^1(S, m)$, then $h \mid L^1(S_\alpha, m_\alpha)$ is a multiplicative linear functional on $L^1(S_\alpha, m_\alpha)$ and the corresponding character on S_α is m_α-measurable by (3.3). If $\mu \in L^1(S, m)$ such that $h(\mu) \neq 0$ then $\tau(x) = h(\bar{x} * \mu)/h(\mu)$ is a character on S and is independent of the choice of such a μ. Further, if $h/L^1(S_\alpha, m_\alpha) \neq 0$ then τ agrees with τ_α (the m_α-measurable character induced via 3.3)) on S_α. In order to show that τ is m-measurable, it suffices to show that $\tau \mid S_\alpha$ is m_α-measurable for all α. It is clear that if $\tau \mid S_\alpha \equiv 0$ then τ is m_α-measurable on S_α. Let $\tau \mid S_\alpha \neq 0$, then there is a β such that $h \mid L^1(S_\beta, m_\beta) \neq 0$ and hence a $\mu \in L^1(S_\beta, m_\beta)$ such that $h(\mu) \neq 0$, $h(\bar{x} * \mu) \neq 0$. Now the support of $\bar{x} * \mu \subset S_\gamma$ for some γ (i.e. that γ such that $S_\alpha S_\beta \subset S_\gamma$). It follows that there is a $y \in S_\gamma$ such that $\tau(y) \neq 0$. Now for each $x \in S_\alpha$, $\tau(x) = \tau(xy)/\tau(y)$ and hence is measurable on S_α since τ is measurable on S_γ and multiplication by y is measurable by (2) (iv).

As in the last part of (3.3), $h(\mu) = \int \tau d\mu$ for all $\mu \in L^1(S, m)$.

We note that in order to identify S^* as the union of the S^*_α, we must be able to show that the m-measurable characters are all extensions of m_α-measurable characters on S_α's and that each m_α-measurable character on an S_α has an extension to an m-measurable character on S. In a subsequent communication on duality in semigroups, we prove that this happens for compact regular abelian semigroups and for locally compact regular abelian semigroups satisfying the first axiom of countability.

Let S and m be as given preceding (3.1) and let $\mathscr{P}$ denote the Jacobson radical of $L^1(S, m)$ [3, 4].

LEMMA (3.5). *The radical $\mathscr{P}$ consists of all $\mu \in L^1(S, m)$ such that $\int \tau d\mu = 0$ for all $\tau \in S^*$. In particular, $L^1(S, m)$ is semisimple if and only if for each $\mu \in L^1(S, m)$, there is a $\tau \in S^*$ such that*

$$\int \tau d\mu \neq 0 .$$

Proof. Since the identically 1 function is an element of $S^*, L^1(S, m)$ is not a radical algebra $\left(\left[\mu: \int d\mu = 0\right] \text{ is proper regular maximal ideal}\right)$. The present lemma thus follows from (3.3) and known results.

THEOREM (3.6). *Let S and m be as above and let S have an identity element 1. Then S^* separates points of S if $L^1(S, m)$ is semisimple.*

Proof. If a and $b \in S$ and are such that $\tau(a) = \tau(b)$ for all $\tau \in S^*$, then $h(\bar{a} * \mu) = h(\bar{b} * \mu)$ for all multiplicative linear functionals h and all $\mu \in L^1(S, m)$. Let W and V be open sets in S having compact closure with $a \in W, b \in V, V \cap W = \varnothing$. The identity 1 of S is in $Wa^{-1} \cap Vb^{-1}$, which is open by (2.2). Let U be open with $\bar{U}$ compact and $1 \in U \subset Wa^{-1} \cap Vb^{-1}$. Since $m(U) > 0$ and $m(U) < \infty$, the measure $\mu = m \mid U$ and 0 on $S \backslash U$ is an element of $L^1(S, m)$. It follows that $(\bar{a} * \mu)(aU) = \mu(a^{-1}(aU)) = \mu(a^{-1}(aU) \cap U) = \mu(U) = m(U)$ since $a^{-1}(aU) \supset U$ by (3.2), 2(*a*). and

$$(\bar{b} * \mu)(aU) = \mu(b^{-1}aU) = m(b^{-1}(aU) \cap U) .$$

If $y \in b^{-1}(aU) \cap U$, then $yb \in aU \subset W$ and $yb \in V$, but $V \cap W = \varnothing$ hence $b^{-1}aU \cap U = \varnothing$ so $(\bar{b} * \mu)(aU) = 0$. Now $L^1(S)$ is semisimple so $\bar{a} * \mu = \bar{b} * \mu$ and thus $m(U) = 0$, a contradiction. Hence, S^* separates points of S.

The converse of the above theorem seems to be very difficult. However, the converse is true for certain semigroups. This will be presented in a forthcoming paper on L^1 algebras of semigroups and duality theorems.

The extension of the above theorem to certain semigroups satisfying the hypotheses preceding (3.4) can be obtained and will also be presented in a forthcoming paper.

References

1. Edwin Hewitt and K. A. Ross, *Abstract Harmonic Analysis* I, Springer-Verlag, Berlin, 1963.
2. Edwin Hewitt and H. S. Zuckerman, *Finite dimensional convolution algebras*, Acta. Math. **93** (1955), 67-119.
3. Edwin Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **83** (1956), 70-97.
4. Lynn H. Loomis, *An introduction to abstract harmonic analysis,*, Van Nostrand, New York, 1953.

Received Jaunary 25, 1966. This research was sponsored in part by the National Science Foundation under Grant GP 1849.

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