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ON UNIQUELY DIVISIBLE SEMIGROUPS ON THE TWO-CELL

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A topological semigroup S is a Hausdorff space together with a continuous associative multiplication on S. A semigroup S is said to be uniquely divisible if each element of S has unique roots of each positive integral order in S. The present paper concerns uniquely divisible semigroups on the two-cell.

The main result of this paper is a statement of equivalent conditions for a commutative uniquely divisible semigroup on the two-cell to be the continuous homomorphic image of the cartesian product of two threads. This result is applied to determine the structure of commutative uniquely divisible semigroups on the two-cell whose idempotent set consists of a zero and an identity.

A *U-semigroup* is a semigroup which is iseomorphic (topologically isomorphic) to the real unit interval $[0, 1]$ under usual multiplication. A *thread* is a semigroup on an arc such that one endpoint is a zero and the other endpoint is an identity.

For a semigroup S, $E(S)$ denotes the set of all idempotent elements of S. The statement "$E(S) = \{0, 1\}$" means that the only idempotents of S are a zero (0) and an identity (1).

Throughout this paper N denotes the set of all positive integers and R denotes the set of all positive rational numbers. Hereafter the statement "S is an UDS" means that S is an uniquely divisible topological semigroup.

If S is an UDS, $x \in S$, and $n \in N$, then $x^{1/n}$ denotes the unique nth. root of x in S. If $r \in R$, $r = m/n; m, n \in N$, and $x \in S$, then $x^r = (x^{1/n})^m$. It is not difficult to show that x^r is unique for each $r \in R$. Define $[x] = \{x^r : r \in R\}^*$ (closure in S).

2. Preliminary results.

THEOREM 2.1. *Let S be a compact UDS such that each subgroup of S is totally disconnected. Then, for each $x \in S \backslash E(S)$, $[x]$ is a U-semigroup.*

Proof. Let H denote the maximal subgroup of $[x]$ containing the identity (e) of $[x]$, and let K denote the kernel (minimal ideal) of $[x]$. Then H and K are connected subgroups of S. Hence $H = \{e\}$ and $K = \{f\}$, where f is the identity of K.

There exists a continuous one-to-one homomorphism σ from the additive nonnegative real numbers $\bar{R}$ into $[x]$ such that $[x] = H\sigma(\bar{R})^*$ (closure in $[x]$) [4, Theorem 3.1]. Since $H = \{e\}, [x] = \sigma(\bar{R})$. Note that $\sigma(\bar{R})^* \backslash \sigma(\bar{R}) = \{f\}$ [4, Theorem 3.1].

Let $I = [0, 1]$ under usual multiplication. Define $\psi: [x] \to I$ by $\psi(f) = 0$ and $\psi(p) = \exp(-\sigma^{-1}(p))$ if $p \neq f$. Then ψ is an iseomorphism of $[x]$ onto I.

COROLLARY 2.2. *Let S be a compact semigroup such that each subgroup of S is totally disconnected. Then S is an UDS if and only if each point of $S \backslash E(S)$ lies on an unique U-semigroup in S.*

COROLLARY 2.3. *Let S be a semigroup on the two-cell. Then S is an UDS if and only if each point of $S \backslash E(S)$ lies on an unique U-semigroup in S.*

3. Uniquely divisible semigroups on the two-cell. Throughout this section S denotes an *UDS* with identity (1) on the two-cell and B denotes the boundary of S. Note that $1 \in B$ [10]. If S has a zero (0) and $0 \in B$, then B_1 and B_2 denote the boundary arcs from 0 to 1 in S. Thus $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \{0, 1\}$.

LEMMA 3.1. *If S has a zero (0) and each point of $E(S)$ lies on a thread in S containing 1, then each point of S lies on a thread in S from 0 to 1.*

Proof. Since $0 \in E(S)$, there exists a thread T from 0 to 1 in S.

Let $e \in E(S)$. Then there exists a thread T_0 from e to 1 in S. Now eT is a thread from 0 to e in S. Thus $eT \cup T_0$ contains a thread $T(e)$ from 0 to 1 in S such that $e \in T(e)$. Hence, if $e \in E(S)$, then e lies on a thread $T(e)$ from 0 to 1 in S.

Let $x \in S \backslash E(S)$. Then, by Corollary 2.3, x lies on an unique U-semigroup I in S. Let z denote the zero of I and u the identity of I. Since $z, u \in E(S)$, there exists threads $T(z)$ and $T(u)$ from 0 to 1 in S such that $z \in T(z)$ and $u \in T(u)$. Thus $T(z) \cup I \cup T(u)$ contains a thread T^1 from 0 to 1 in S such that $x \in T^1$.

LEMMA 3.2. *If $E(S) = \{0, 1\}$, then $0 \in B$.*

Proof. Suppose $0 \notin B$. Let $x \in B \backslash E(S)$. Then $B \backslash [x] \neq \square$. Let $p \in B \backslash [x]$. Since $[x] \cap B$ is closed, there exists a point y in the arc from p to x on B which does not contain 1. Then $[y]$ must meet $[p]$ or $[x]$ in a point q not in $E(S)$. Thus q lies on two distinct U-semigroups in S. This is a contradiction to Corollary 2.3. Hence $0 \in B$.

LEMMA 3.3. *Suppose S has zero* (0) *and* $0 \in B$. *If each of B_1 and B_2 is a thread, then $S = B_1B_2 = B_2B_1$.*

Proof. Now $1 \in B_1 \cap B_2$. Hence $B \subset B_1B_2$. Define $\varphi: B_1B_2\ B_2 \to S$ by $\varphi((b_1b_2, b)) = b_1b_2 b$. Then φ is continuous, $\varphi((b_1b_2, 0)) = 0$, and $\varphi((b_1b_2, 1)) = b_1b_2$. Hence B_1B_2 is contractible, and thus $S = B_1B_2$. Similarly, $S = B_2B_1$.

LEMMA 3.4. *Suppose S has a zero* (0) *and* $0 \in B$. *If each point of S lies on a thread from* 0 *to* 1 *in S, then each of B_1 and B_2 is a thread.*

Proof. Let x and y be distinct points of $B_1\backslash\{0, 1\}$ such that y separates x from 1 on B_1. Suppose $[x] \neq [y]$. Let T_1 and T_2 denote threads from 0 to 1 in S containing x and y respectively. Then, since y separates x from 1 on B_1, $T_1 \cap T_2$ contains an idempotent f such that $xf = x$ and $fy = f$. Hence $xy = (xf)y = x(fy) = xf = x$. Thus, if y separates x from 1 on B_1 and $[x] \neq [y]$, then $xy = x$.

If $B_1\backslash E(S) = \square$, then the fact that B_1 is a thread follows from the preceding paragraph. Suppose $B_1\backslash E(S) \neq \square$. Let $z \in B_1\backslash E(S)$. Then there exists a U-semigroup I in S such that $z \in I$. Let a be the zero of I and b the identity of I. Let M be the component of $I \cap B_1$ containing z, $h = \inf M$, and $g = \sup M$ in the cut-point ordering ($\langle$) of B_1 from 0 to 1. Since $h = \inf M$, there exists a sequence $\{h_n\}$ of points of $B_1\backslash I$ such that $h_n < h$ for each $n \in N$ and $h_n \to h$. Thus, by the preceding paragraph, $h_nh = h_n$ for each $n \in N$. Since multiplication is continuous in S, $h_nh \to h^2$. Hence $h = h^2$. Since $h \in I$, $a = h$. Similarly, $g = b$, and hence $I \subset B_1$. Thus B_1 is a thread. Similarly, B_2 is a thread. This completes the proof of Lemma 3.4.

A commutative UDS S can be considered to be a generalization of a semilattice (a commutative idempotent semigroup). Indeed, if $S = E(S)$, then S is a semilattice. Consequently, Theorem 3.5 is a generalization of Theorem 3 in [1].

If S is commutative, then the kernel K (the minimal ideal) of S is a compact connected group. Hence K is either the circle group C or a point. It is not difficult to show that K is uniquely divisible. Thus, since C is not uniquely divisible, K is a point. Hence, if S is commutative, then S has a zero (0).

THEOREM 3.5. *If S is commutative and* $0 \in B$, *then these are equivalent*:

(i) *each point of $E(S)$ lies on a thread in S containing* 1;
(ii) *each point of S lies on a thread from* 0 *to* 1 *in S;*
(iii) *each of B_1 and B_2 is a thread;*

(iv) *S is the continuous homomorphic image of the cartesian product of two threads.*

Proof. (i) implies (ii). [Lemma 3.1].

(ii) implies (iii). [Lemma 3.4].

(iii) implies (iv). By Lemma 3.3, $S = B_1B_2$. Define $\psi: B_1 \times B_2 \to S$ by $\psi((b_1, b_2)) = b_1b_2$. Then ψ is a continuous homomorphism onto S.

(iv) implies (i). Let I_1 and I_2 be threads and φ a continuous homomorphism of $I_1 \times I_2$ onto S. Let $g \in E(S)$ and $p \in \varphi^{-1}(g)$. Then there exists a thread from $(0, 0)$ to $(1, 1)$ in $I_1 \times I_2$ containing p. Hence, by Theorem 2 of [3], $\varphi(T)$ is a thread in S containing g and 1.

COROLLARY 3.6. *If S is commutative and $E(S) = \{0, 1\}$, then S is iseomorphic to $(I \times I)/J$, where $I = [0, 1]$ is a U-semigroup and J is the ideal $\{(x, y): x = 0$ or $y = 0\}$.*

Proof. By Lemma 3.2, $0 \in B$. By Theorem 1 in [7], there exists a thread from 0 to 1 in S. Therefore, by Theorem 3.5, each of B_1 and B_2 is a thread, and thus are U-semigroups. The map $\psi: B_1 \times B_2 \to S$ defined by $\psi((b_1, b_2)) = b_1b_2$ is a continuous homomorphism of $B_1 \times B_2$ onto S.

Suppose $\psi((b_1, b_2)) = 0$. Then $b_1b_2 = 0$. Suppose $b_1 \neq 0 \neq b_2$. Then, for each $n \in N$, $b_1^{1/n}b_2^{1/n} = 0$. But $b_1^{1/n} \to 1$ and $b_2^{1/n} \to 1$. Thus $1 = 0$. This contradiction implies that either $b_1 = 0$ or $b_2 = 0$. Hence $\psi((b_1, b_2)) = 0$ if and only if $(b_1, b_2) \in J$.

Suppose $\psi((a, b)) = \psi((c, d))$, $(a, b), (c, d) \in (B_1 \times B_2)\backslash J$. Then $ab = cd$. Since B_1 and B_2 are U-semigroups, there exist $p \in B_1$ and $q \in B_2$ such that one of the following cases hold:

(i) $a = cp$ and $b = dq$;

(ii) $a = cp$ and $d = bq$;

(iii) $c = ap$ and $b = dq$:

(iv) $c = ap$ and $d = bq$.

We will assume that case (i) holds. The proof for the other cases is similar. Thus we have $cp \cdot dq = cd$. Hence $(pq)(cd) = cd$. Let $x = pq$ and $y = cd$. Then $xy = y$. Hence, for each $n \in N$, $x^ny = y$. If $x \neq 1$, then $x^n \to 0$. Thus, if $x \neq 1$, then $y = 0$, and hence $cd = 0$. By the preceding paragraph, either $c = 0$ or $d = 0$. But $c \neq 0 \neq d$. Hence $x = 1$ and $pq = 1$. Then for each $n \in N$, $p^nq^n = 1$. If $p \neq 1$, $p^n \to 0$, and hence $0 = 1$. Similarly, if $q \neq 1$, then $0 = 1$. This contradiction implies that $p = q = 1$. Thus $a = c$, $b = d$, and $(a, b) = (c, d)$. Hence ψ is one-to-one on $(B_1B_2)\backslash J$, thus completing the proof of the corollary.

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