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# CONGRUENCES ON REGULAR SEMIGROUPS

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**For any regular semigroup $S$ the relation $\theta$ is defined on the lattice, $\Lambda(S)$, of congruences on $S$ by: $(\rho, \tau) \in \theta$ if and only if $\rho$ and $\tau$ induce the same partition of the idempotents of $S$. Then $\theta$ is an equivalence relation on $\Lambda(S)$ such that each equivalence class is a complete modular sublattice of $\Lambda(S)$. If $S$ is an inverse semigroup then $\theta$ is a congruence on $\Lambda(S)$, $\Lambda(S)/\theta$ is complete and the natural homomorphism of $\Lambda(S)$ onto $\Lambda(S)/\theta$ is a complete lattice homomorphism.**

**Any congruence on an inverse semigroup $S$ can be characterized in terms of its kernel, namely, the set of congruence classes containing the idempotents of $S$. In particular, any congruence on $S$ induces a partition of the set $E_S$ of idempotents of $S$ satisfying certain normality conditions. In this note, those partitions of $E_S$ which are induced by congruences on $S$ and the largest and smallest congruences on $S$ correspond ing so such a partition of $E_S$ are characterized.**

**1. Preliminary results and definitions.** We adopt the notation and terminology of Clifford and Preston [2]. A semigroup $S$ is called *regular* if $a \in a\,Sa$, for all $a \in S$. If, for all $a \in S$, there exists an element $b \in S$ such that $aba = a$ and $bab = b$ then we say that $b$ is an *inverse* of $a$ and that $(a, b)$ is a *regular pair* [11]. In a regular semigroup, every element has an inverse. An *inverse semigroup* is a semigroup in which each element has a unique inverse. The elementary properties of regular and inverse semigroups can be found in [2]. In particular, a semigroup $S$ is an inverge semigroup if and only if it is regular and its idempotents commute ([2], Th. 1.17). The inverse of an element $a$ is then denoted by $a^{-1}$. For any idempotent $e$, $e^{-1} = e$, and, for any elements $a, b$ of $S$

$$(a^{-1})^{-1} = a, (ab)^{-1} = b^{-1}a^{-1} .$$

If $(a, a')$ is a regular pair, then $aa'$ and $a'a$ are both idempotents but are not always equal (even in an inverse semigroup).

A *regular* (*inverse*) *subsemigroup* $T$ of a semigroup $S$ is just a subsemigroup of $S$ which is a regular (inverse) semigroup in its own right.

For any semigroup $S$ we shall denote by $E_S$ the set of idempotents of $S$. The set $E_S$ can be partially ordered by defining $e \leqq f$ if and only if $ef = fe = e$. Of course, if $S$ is an inverse semigroup then this reduces to $ef = e$ and $E_S$ then becomes a semilattice, with $e \wedge f = ef$, called the *semilattice of idempotents* of $S$.

We shall call a subset $A$ of a partially ordered set $B$ *convex* if $x \leqq y \leqq z$, $x, z \in A$ implies that $y \in A$.

If $\rho$ is a congruence on a semigroup $S$ then we shall denote by $\rho|_{E_S}$ the partition of $E_S$ induced by $\rho$, that is $\rho|_{E_S} = \rho \cap (E_S \times E_S)$, and by $a\rho$ the $\rho$-class containing the element $a$. We shall also make use of the fact that, in an inverse semigroup, if $(a, b) \in \rho$ then $(a^{-1}, b^{-1}) \in \rho$ ([4] Corollary 2.3) and, consequently, $(aa^{-1}, bb^{-1}) \in \rho$.

Clearly a homomorphic image of a regular semigroup is regular and it was established in [12] that a homomorphic image of an inverse semigroup is an inverse semigroup.

Finally, two elements of a semigroup $S$ are said to be $\mathscr{L}$- ($\mathscr{R}$-) equivalent if they generate the same principal left (right) ideal of $S$. Clearly $\mathscr{L}$ and $\mathscr{R}$ are equivalence relations on $S$, as is the relation $\mathscr{H}$, defined by $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.

**2. Maximal regular subsemigroups.** In this section we shall generalize the well known result that for any idempotent $e$ of a semi-group $S$ there is a unique maximum subgroup of $S$ with identity $e$.

LEMMA 1.1. *Let $(a, a')$ and $(b, b')$ be regular pairs in a semigroup $S$. Then $a'a\,bb'$ and $bb'\,a'a$ are idempotents of $S$ if and only if $(ab, b'a')$ is a regular pair.*

*Proof.* Let $a'a\,bb'$ and $bb'\,a'a$ be idempotents. Then

$$(ab)(b'a')(ab) = aa'abb'a'abb'b = aa'abb'b = ab$$

and

$$(b'a')(ab)(b'a') = b'bb'a'abb'a'aa' = b'bb'a'aa' = b'a'$$

as required.

Conversely, if $(ab, b'a')$ is a regular pair, then

$$(a'abb')(a'abb') = a'(ab)(b'a')(ab)b' = a'abb'$$

and similarly $bb'a'a$ is an idempotent.

LEMMA 1.2. ([5] *Lemma* 1.1) *If $e$ and $f$ are idempotents in a regular semigroup $S$ then, for some idempotent $g$ in $S$, $(g, ef)$ is a regular pair.*

LEMMA 1.3. *For a regular semigroup $S$ the following are equivalent*:

(1) $E_SE_S \subseteqq E_S$;

(2) $e \in E_S$, $(e, x)$ *a regular pair implies that* $x \in E_S$;

(3) *$(a, a')$ and $(b, b')$ regular pairs implies that $(ab, b'a')$ is a regular pair.*

*Proof.* (1) imples (2). (Cf. [11] Th. 1.) Let $e \in E_S$ and $(e, x)$ be a regular pair. Then $x = xex = xeex \in E_S E_S \subseteqq E_S$. (2) implies (1). Let $e, f \in E_S$, then by Lemma 1.2, for some idempotent $g$, $(g, ef)$ is a regular pairs and so $ef \in E_S$.

(1) implies (3) by Lemma 1.1.

(3) implies (1). Let $e, f \in E_S$. Then, since $(e, e)$ and $(f, f)$ are regular pairs, $(ef, fe)$ is also a regular pair. Then $ef = (ef)(fe)(ef) = (ef)(ef)$ and $ef \in E_S$.

LEMMA 1.4. *Let $S$ be a regular semigroup such that $E_S$ is a subsemigroup. Then, for any regular pair $(a, a')$*

$$aE_S a' \subseteqq E_S .$$

*Proof.* Let $(a, a')$ be a regular pair and $e \in E_S$. Since

$$a'ae \in E_S E_S \subseteqq E_S ,$$

we have

$$(aea')^2 = aea'aea' = aa'aea'aea' = aa'aea' = aea' .$$

THEOREM 1.5. *Let $E$ be an idempotent subsemigroup of a semigroup $S$. Then $E^C = \{x \in S$: for some $x'$, $(x, x')$ is a regular pair, $xx', x'x \in E$, $xEx' \subseteqq E$ and $x'Ex \subseteqq E\}$ is the largest regular subsemigroup of with $E$ as its set of idempotents.*

*Proof.* Let $a, b \in E^C$ and let $a', b'$ be elements of $S$ such that $a, a'$ and $b, b'$ satisfy the conditions of membership for $a$ and $b$, respectively. Then clearly $a', b' \in E^C$. Moreover, since $a'a, bb' \in E$, we have $a'abb'$ and $bb'a'a$ contained in $E$ and so, by Lemma 1.1, $(ab, b'a')$ is a regular pair. Also

$$abb'a' \in aEa' \subseteqq E,\ b'a'ab \in b'Eb \subseteqq E$$

and

$$abEb'a' \subseteqq aEa' \subseteqq E,\ b'a'Eab \subseteqq b'Eb \subseteqq E .$$

Thus $ab \in E^C$ and $E^C$ is a subsemigroup of $S$. Since, as pointed out above, for each $a \in E^C$ some inverse of $a$ is also in $E^C$, $E^C$ is a regular subsemigroup of $S$.

To show that $E_{E^c} = E$, let $a$ be an idempotent in $E^C$ and let $(a, a')$ be a regular pair with $a, a' \in E^C$. Then

$$a' = a'aa' = a'aaa' \in EE \subseteqq E$$

and so

$$a = aa'a = aa'a'a \in EE \subseteqq E .$$

Hence $E_{E^c} = E$, since clearly $E \subseteqq E_{E^c}$.

Now suppose that $T$ is a regular subsemigroup of $S$ such that $E_T = E$. Then, for any $x \in T$, there exists an $x' \in T$ such that $(x, x')$ is a regular pair in $T$ and so $xx', x'x \in E_T = E$. Then, since $EE \subseteqq E$, we have, by Lemma 1.4,

$$xEx' \subseteqq E \text{ and } x'Ex \subseteqq E .$$

Thus $x \in E^C$.

COROLLARY 1.6. *Let $E$ be a subsemigroup of commuting idempotents of a semigroup $S$. Then $E^C$ (defined as in Theorem 1.5) is the largest inverse subsemigroup of $S$ with $E$ as its set of idempotents.*

*Proof.* By Theorem 1.5, $E^C$ is the largest regular subsemigroup with $E$ as its set of idempotents. Since the elements of $E$ commute, $E^C$ is, in fact, an inverse subsemigroup and so is clearly the largest such.

**3. The lattice of congruences on a regular semigroup.** For any semigroup $S$ we denote by $\Lambda(S)$ the lattice of congruences on $S$.

If $\sigma$ and $\rho$ are congruences on a semigroup $S$ such that $\sigma \subseteqq \rho$ then the relation $\rho/\sigma$ on $S/\sigma$ defined by

$$\rho/\sigma = \{(x\sigma, y\sigma) \in S/\sigma \times S/\sigma : (x, y) \in \rho\}$$

is a congruence on $S/\sigma$. Moreover, the mapping $\rho \to \rho/\sigma$ is a one-to-one order preserving mapping of the congruences $\rho$ on $S$ containing $\sigma$ onto the congruences on $S/\sigma$, as is easy to show.

It is straight forward to verify that if $\sigma, \rho, \tau \in \Lambda(S)$ and $\sigma \subseteqq \rho, \tau$ then

$$(\rho \cap \tau)/\sigma = \rho/\sigma \cap \tau/\sigma \text{ and } (\rho \vee \tau)\sigma = (\rho/\sigma) \vee (\tau/\sigma) .$$

It is convenient to point out here that, for any semigroup $S$, $\Lambda(S)$ is complete ([2] p. 24) and that if $C$ is a nonvoid subset of $\Lambda(S)$ then $\bigvee_{\rho \in C} \rho$ may be characterized as $\{(x, y) \in S \times S$: there exist $x_1, \cdots, x_n \in S$ and $\rho_1, \cdots, \rho_{n+1} \in C$ (not necessarily all distinct) such that $(x, x_1) \in \rho_1, (x_1, x_2) \in \rho_2, \cdots, (x_n, y) \in \rho_{n+1}\}$.

In order to show that a sublattice $R$ of $\Lambda(S)$ is a modular sublattice, it follows from ([1] Th. 3, p. 86) that it suffices to show that the congruences in $R$ commute; that is, that $\rho, \tau \in R$ implies that $\rho \circ \tau = \tau \circ \rho$.

We shall call a sublattice $R$ of $\Lambda(S)$ *a complete sublattice* if, for $C \subseteqq R$, $\bigvee_{\rho \in C} \rho$ and $\bigcap_{\rho \in C} \rho$ not only exist in $\Lambda(S)$, but also belong to $R$.

LEMMA 3.1. *Let $S$ be a semigroup and $R = \{\rho_i : i \in I\}$ be a subset of $\Lambda(S)$ such that $\sigma = \bigcap_{i \in I} \rho_i \in R$. If $R/\sigma = \{\rho_i/\sigma : i \in I\}$ is a sublattice (sublattice of commuting congruences, complete sublattice) of $\Lambda(S/\sigma)$ then $R$ is a sublattice (sublattice of commuting congruences, complete sublattice) of $\Lambda(S)$.*

*Proof.* Let $\rho_1, \rho_2 \in R$. Then $\rho_1/\sigma \cap \rho_2/\sigma$ and $\rho_1/\sigma \vee \rho_2/\sigma$ belong to $R/\sigma$, as $R/\sigma$ is a sublattice of $\Lambda(S/\sigma)$. Now,

$$\rho_1/\sigma \cap \rho_2/\sigma = (\rho_1 \cap \rho_2)/\sigma \quad \text{and} \quad \rho_1/\sigma \vee \rho_2/\sigma = (\rho_1 \vee \rho_2)/\sigma$$

and so, by the one-to-one nature of the mapping $\rho \longrightarrow \rho/\sigma$, it follows that $\rho_1 \cap \rho_2$ and $\rho_1 \vee \rho_2$ belong to $R$. Thus $R$ is a sublattice of $\Lambda(S)$.

Now let $R/\sigma$ be a sublattice of commuting congruences of $\Lambda(S/\sigma)$, let $\rho, \tau \in R$ and let $(a, b) \in \rho \circ \tau$. Then for some $c \in S$, $(a, c) \in \rho$ and $(c, b) \in \tau$. Hence $(a\sigma, c\sigma) \in \rho/\sigma$ and $(c\sigma, b\sigma) \in \tau/\sigma$. Since the elements of $R/\sigma$ commute it follows that $\rho/\sigma \circ \tau/\sigma = \tau/\sigma \circ \rho/\sigma$ and, consequently, that there exists a $d\sigma \in S/\sigma$ such that $(a\sigma, d\sigma) \in \tau/\sigma$ and $(d\sigma, b\sigma) \in \rho/\sigma$. Hence $(a, d) \in \tau$ and $(d, b) \in \rho$; that is, $(a, b) \in \tau \circ \rho$. Thus $\rho \circ \tau \subseteqq \tau \circ \rho$ and likewise $\rho \circ \tau \subseteqq \tau \circ \rho$. Hence $\rho \circ \tau = \tau \circ \rho$ and $R$ is a sublattice of commuting congruences of $\Lambda(S)$.

Finally, let $R/\sigma$ be a complete sublattice and let $C' \subseteqq R$. Then $\bigvee_{\rho \in C} \rho/\sigma$ exists and is contained in $R/\sigma$, say, $\tau/\sigma = \bigvee_{\rho \in C} \rho/\sigma$. Then $\rho/\sigma \subseteqq \tau/\sigma$, for all $\rho \in C$ and so $\rho \subseteqq \tau$. On the other hand, $\rho \subseteqq \tau'$, for all $\rho \in C$ implies that $\rho/\sigma \subseteqq \tau'/\sigma$, for all $\rho \in C$, and hence that

$$\tau/\sigma = \bigvee_{\rho \in C} \rho/\sigma \subseteqq \tau'/\sigma \,.$$

Thus $\tau \subseteqq \tau'$ and $\bigvee_{\rho \in C} = \tau \in R$.

The verification that $\bigcap_{\rho \in C} \rho \in R$ is even simpler. Thus $R$ is a complete sublattice of $\Lambda(S)$.

*Note.* It is almost immediate that if, in Lemma 3.1, $R$ is a sublattice (sublattice of commuting congruences, complete sublattice) of $\Lambda(S)$ then $R/\sigma$ is a sublattice (sublattice of commuting congruences, complete sublattice) of $\Lambda(S/\sigma)$.

LEMMA 3.2. ([6] *Lemma* 2.2) *Let* $\rho$ *be a congruence on a regular semigroup* $S$. *Then each idempotent* $\rho$-*class contains an idempotent of* $S$.

For any semigroup $S$, let $\Sigma(\mathscr{H}) = \{\rho \in \Lambda(S) : \rho \subseteqq \mathscr{H}\}$.
From [8] Lemmas 1 and 3, we have,

LEMMA 3.3. *Let* $S$ *be a regular semigroup. Then* $\Sigma(\mathscr{H})$ *is a sublattice of* $\Lambda(S)$ *of commuting congruences with a greatest and least element.*

We call a congruence $\rho$ on a semigroup $S$ *idempotent separating* if each $\rho$-class contains at most one idempotent.

That any congruence $\rho$ on a semigroup such that $\rho \subseteqq \mathscr{H}$ is idempotent separating follows from [2], Theorem 2.15, and the fact that every idempotent separating congruence on a regular semigroup is contained in $\mathscr{H}$ follows from Theorem 2.3 of [6]. Thus, for any regular semigroup $S, \Sigma(\mathscr{H})$ is the set of idempotent separating congruences on $S$.

Now any convex subset, with a largest and smallest member, of a complete lattice is clearly a complete sublattice. Hence, by Lemma 3.3, for any regular semigroup $S$, since $\Sigma(\mathscr{H})$ is clearly a convex subset of $\Lambda(S)$, it follows that $\Sigma(\mathscr{H})$ (the set of idempotent separating congruences on $S$) is a complete sublattice of $\Lambda(S)$.

THEOREM 3.4. *Let* $S$ *be a regular semigroup and let*

$$\begin{aligned}\theta &= \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S) : e\rho_1 \cap E_S = e\rho_2 \cap E_S,\ \textit{for each}\ e \in E_S\}\\ &= \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S) : \rho_1 \mid E_S = \rho_2 \mid E_S\}\ .\end{aligned}$$

*Then*

(i) $\theta$ *is a meet compatible equivalence on* $\Lambda(S)$;

(ii) *each* $\theta$-*class is a complete modular sublattice of* $\Lambda(S)$ (*with a greatest and least element*).

*Proof.* (i) Clearly $\theta$ is an equivalence relation on $\Lambda(S)$. Let $(\rho_1, \rho_2) \in \theta$ and $\rho_3 \in \Lambda(S)$. Then

$$\rho_1 \cap (E_S \times E_S) = \rho_2 \cap (E_S \times E_S)$$

and so

$$\rho_1 \cap \rho_3 \cap (E_S \times E_S) = \rho_2 \cap \rho_3 \cap (E_S \times E_S)$$

that is $(\rho_1 \cap \rho_3, \rho_2 \cap \rho_3) \in \theta$.

(ii) Let $A$ be a $\theta$-class, let $\sigma = \bigcap_{\tau \in A} \tau$ and let $\rho \in A$. For

$e, f \in E_S$, let $(e, f) \in \rho$. Then $(e, f) \in \tau$, for all $\tau \in A$ and so $(e, f) \in \sigma$. Conversely, as $\sigma \subseteqq \rho$, $(e, f) \in \sigma$ implies that $(e, f) \in \rho$. Thus $\rho|_{E_S} = \sigma|_{E_S}$ and $\sigma \in A$. Thus $A$ has a least member.

Now, for any $\rho \in A$, $\rho/\sigma$ is idempotent separating. For suppose that $f_1$ and $f_2$ are idempotents of $S/\sigma$ such that $(f_1, f_2) \in \rho/\sigma$. By Lemma 3.2, $f_1 = e_1\sigma$ and $f_2 = e_2\sigma$ for some idempotents $e_1, e_2$ of $S$. Thus $(e_1\sigma, e_2\sigma) \in \rho/\sigma$ and so $(e_1, e_2) \in \rho$. But $\rho|_{E_S} = \sigma|_{E_S}$ and so

$$f_1 = e_1\sigma = e_2\sigma = f_2 .$$

Hence $\rho/\sigma$ is idempotent separating.

On the other hand, for any congruence $\tau$ on $S/\sigma$,

$$\tau' = \{(a, b) \in S \times S : (a\sigma, b\sigma) \in \tau\}$$

is a congruence on $S$. Suppose that $\tau$ is idempotent separating. If, for $e, f \in E_S$, we have $(e, f) \in \tau'$, then $(e\sigma, f\sigma) \in \tau$ and so, as $\tau$ is idempotent separating, $e\sigma = f\sigma$. Thus $\tau'|_{E_S} = \sigma|_{E_S}$ and $\tau' \in A$. Now $\tau'/\sigma = \tau$ and so $\{\rho/\sigma : \rho \in A\}$ is just the sublattice of idempotent separating congruences on $S/\sigma$. Since this, by Lemma 3.3 and the remarks following it, is a complete sublattice of $\Lambda(S/\sigma)$ of commuting congruences, we conclude from Lemma 3.1 that $A$ is a complete sublattice of $\Lambda(S)$ of commuting congruence and so a complete modular sublattice of $\Lambda(S)$.

Finally, since $A$ is a complete sublattice of $\Lambda(S)$, $\bigvee_{\rho \in A} \rho \in A$ and $A$ has a greatest member.

**4. Congruences on inverse semigroups.** In this and the following sections we consider inverse semigroups, for which we are able to improve on the results of the previous sections.

DEFINITION 4.1. Let $S$ be an inverse semigroup and

$$P = \{E_\alpha : \alpha \in J\}$$

be a partition of $E_S$. Then $P$ is a *normal* partition of $E_S$ if

(i) $\alpha, \beta \in J$ implies that there exists a $\gamma \in J$ such that $E_\alpha E_\beta \subseteqq E_\gamma$

(ii) $\alpha \in J$ and $a \in S$ implies that there exists a $\beta \in J$ such that $aE_\alpha a^{-1} \subseteqq E_\beta$.

If, for an inverse semigroup $S$, $P = \{E_\alpha : \alpha \in J\}$ is a normal partition of $E_S$, then $E_\alpha$ is convex for each $\alpha \in J$. For if $e, g \in E_\alpha$ with $e \leqq f \leqq g$ and $f \in E_\beta$, then $ef = e$ implies that $E_\alpha E_\beta \subseteqq E_\alpha$ and so $f = gf \in E_\alpha E_\beta \subseteqq E_\alpha$. Moreover, we shall denote by $\pi_P$ the equivalence relation on $E_S$ induced by $P$ and show in the following theorem that there exists a congruence $\rho$ on $S$ such that $\rho|_{E_S} = \pi_P$. In fact, we give characterizations of the largest and smallest such congruences.

THEOREM 4.2. *Let $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of the semilattice of idempotents of an inverse semigroup $S$. Let $\sigma = \{(a, b) \in S \times S$: there exists an $\alpha \in J$ with $aa^{-1}, bb^{-1} \in E_\alpha$ and, for some $e \in E_\alpha$, $ea = eb\}$ and $\rho = \{(a, b) \in S \times S : \alpha \in J$ implies that, for some $\beta \in J$, $aE_\alpha a^{-1}, bE_\alpha b^{-1} \subseteqq E_\beta\}$. Then $\sigma$ and $\rho$ are, respectively, the smallest and largest congruences on $S$ such that $\sigma|_{E_S} = \rho|_{E_S} = \pi_P$.*

*Proof.* Clearly $\sigma$ is an equivalence relation. So let $(a, b) \in \sigma$ and $c \in S$, where $aa^{-1}, bb^{-1} \in E_\alpha$ and $ea = eb$, for some $e \in E_\alpha$. Now suppose that $(ac)(ac)^{-1} = acc^{-1}a^{-1} \in E_\gamma$ while $(bc)(bc)^{-1} = bcc^{-1}b^{-1} \in E_\delta$. Then, since $(aa^{-1})(acc^{-1}a^{-1}) = acc^{-1}a^{-1}$ and $(aa^{-1})(acc^{-1}a^{-1}) \in E_\alpha E_\gamma$, it follows that $E_\alpha E_\gamma \subseteqq E_\gamma$. Likewise $E_\alpha E_\delta \subseteqq E_\delta$. Now, $eacc^{-1}a^{-1} = eacc^{-1}a^{-1}e = ebcc^{-1}b^{-1}e = ebcc^{-1}b^{-1}$ where $eacc^{-1}a^{-1} \in E_\alpha E_\gamma \subseteqq E_\gamma$ and $ebcc^{-1}b^{-1} \in E_\alpha E_\delta \subseteqq E_\delta$. Hence $E_\gamma = E_\delta$. Now, for any $f \in E_\gamma$, $fe \in E_\alpha E_\gamma \subseteqq E_\gamma$ and

$$(fe)ac = f(ea)c = f(eb)c = (fe)bc\ .$$

Thus $(ac, bc) \in \sigma$.

For some $\gamma \in J$, $cE_\alpha c^{-1} \subseteqq E_\gamma$ and so $(ca)(ca)^{-1} = caa^{-1}c^{-1} \in cE_\alpha c^{-1} \subseteqq E_\gamma$ and $(cb)(cb)^{-1} = cbb^{-1}c^{-1} \in cE_\alpha c^{-1} \subseteqq E_\gamma$. Also, if $f = caa^{-1}ec^{-1}$ then $f \in cE_\alpha c^{-1} \subseteqq E_\gamma$ and

$$\begin{aligned} fca &= caa^{-1}ec^{-1}ca = caa^{-1}c^{-1}cea \\ &= caa^{-1}c^{-1}ceb = caa^{-1}ec^{-1}cb = fcb\ . \end{aligned}$$

Thus $(ca, cb) \in \sigma$ and $\sigma$ is a congruence on $S$. Moreover, it is evident that $\sigma|_{E_S} = \pi_P$.

Now suppose that $\tau$ is any congruence on $S$ such that

$$\tau|_{E_S} = \sigma|_{E_S} = \pi_P$$

and let $a, b$ be as above. Then $aa^{-1}\tau = bb^{-1}\tau = e\tau$ and so

$$\begin{aligned} a\tau &= (aa^{-1}a)\tau = (aa^{-1})\tau\, a\tau = e\tau\, a\tau = (ea)\tau = (eb)\tau \\ &= e\tau\, b\tau = bb^{-1}\tau\, b\tau = b\tau\ . \end{aligned}$$

Thus $\sigma \subseteqq \tau$ and $\sigma$ is the finest congruence on $S$ such that $\sigma|_{E_S} = \pi_P$.

The verification that $\rho$ is the largest such congruence is similar but simpler and so we omit it.

We devote the remainder of this section to obtaining an alternative characterization of the congruences $\sigma, \rho$ of Theorem 4.2 in terms of kernel normal systems.

DEFINITION 4.3 [9]. Let $S$ be an inverse semigroup. We call $\mathscr{N}$ a *kernel normal system* of $S$ if $\mathscr{N}$ is a collection of inverse subsemigroups of $S$, $\mathscr{N} = \{N_\alpha : \alpha \in J\}$ such that, if $E_\alpha = E_{N_\alpha}$ then

(1) $\{E_\alpha : \alpha \in J\}$ is a normal partition of $E_S$;
(2) $aa^{-1}, bb^{-1} \in E_\alpha$ and $a, ab^{-1} \in N_\alpha$ implies that $b \in N_\alpha$;
(3) $aa^{-1}, bb^{-1} \in E_\alpha, ab^{-1} \in N_\alpha$ and $aE_\beta a^{-1} \subseteqq E_\gamma$ implies that $aN_\beta b^{-1} \subseteqq N_\gamma$.

Then we have

THEOREM 4.4. (*Preston* [9], Th. 1). *Let $S$ be an inverse semigroup and let $\mathscr{N} = \{N_\alpha : \alpha \in J\}$ be a kernel normal system of $S$. Let $\rho_{\mathscr{N}} = \{(a, b) \in S \times S: aa^{-1}, bb^{-1} \in E_\alpha$ and $ab^{-1} \in N_\alpha$ for some $\alpha \in J\}$. Then $\rho_{\mathscr{N}}$ is a congruence on $S$ and $\{N_\alpha : \alpha \in J\}$ is the set of idempotents in $S/\rho_{\mathscr{N}}$.*

*Conversely, let $\rho$ be a congruence on $S$. Then $\mathscr{N} = \{e\rho : e \in E_S\}$ is a kernel normal system of $S$ and $\rho = \rho_{\mathscr{N}}$.*

Thus a congruence on an inverse semigroup is uniquely determined by the congruence classes which contain the idempotents.

THEOREM 4.5. *Let $S$ be an inverse semigroup and $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of $E_S$. For each $\alpha \in J$, let $T_\alpha$ be the largest inverse subsemigroup of $S$ such that $E_{T_\alpha} = E_\alpha$, let $M_\alpha = \{x \in T_\alpha : ex = e$ for some $e \in E_\alpha\}$ and let $N_\alpha = \{x \in T_\alpha : E_\alpha E_\beta \subseteqq E_\gamma$ implies that $xE_\beta x^{-1} \subseteqq E_\gamma\}$. Then $\mathscr{M} = \{M_\alpha : \alpha \in J\}$ and $\mathscr{N} = \{N_\alpha : \alpha \in J\}$ are kernel normal systems of $S$, $\rho_{\mathscr{M}} = \sigma$ and $\rho_{\mathscr{N}} = \rho$, where $\sigma$ and $\rho$ are defined as in Theorem 4.2.*

*Proof.* For each $\alpha \in J$, let $U_\alpha, V_\alpha$ be the $\sigma$ and $\rho$-classes, respectively, of $S$ containing $E_\alpha$.

Clearly $M_\alpha \subseteqq U_\alpha$. Also, since $E_{U_\alpha} = E_\alpha$, it follows that $U_\alpha \subseteqq T_\alpha$. Hence for $x \in U_\alpha$, we have first that $x \in T_\alpha$. Moreover, from the definition of $\sigma$, since $x, xx^{-1} \in U_\alpha$, that is, $(x, xx^{-1}) \in \sigma$, we have that, for some idempotent $e \in E_\alpha$, $ex = exx^{-1}$ and so $exe = exx^{-1}e = exx^{-1} \in E_\alpha$. Hence $(exe)x = exex = ex = exe$ and so $x \in M_\alpha$, $M_\alpha = U_\alpha$ and $\rho_{\mathscr{M}} = \sigma$.

For $x \in N_\alpha$, we have $xx^{-1} \in E_\alpha$. Now, for any $\beta \in J$, $xx^{-1}E_\beta xx^{-1} = xx^{-1}E_\beta \subseteqq E_\alpha E_\beta \subseteqq E_\gamma$, say. Then $xE_\beta x^{-1} \subseteqq E_\gamma$, by the definition of $N_\alpha$, and so $(x, xx^{-1}) \in \rho$. Consequently, $x \in V_\alpha$. On the other hand, since $E_{V_\alpha} = E_\alpha$, $V_\alpha \subseteqq T_\alpha$ and so $x \in V_\alpha$ implies that $x, x^{-1} \in T_\alpha$. For $\beta \in J$, suppose that $E_\alpha E_\beta \subseteqq E_\gamma$ and let $e \in E_\alpha$, $f \in E_\beta$ and $g \in E_\gamma$. Then $efe \in E_\alpha E_\beta E_\alpha \subseteqq E_\gamma$ and so $(e\rho)(f\rho)(e\rho) = (g\rho)$. Hence, since $e\rho = x\rho = x^{-1}\rho$, $(xfx^{-1})\rho = (x\rho)(f\rho)(x^{-1}\rho) = g\rho$; that is, $xfx^{-1} \in E_\gamma$. Since $P$ is a normal partition of $E_S$, it follows that $xE_\beta x^{-1} \subseteqq E_\gamma$ and that $x \in N_\alpha$. Thus $V_\alpha = N_\alpha$ and $\rho_{\mathscr{N}} = \rho$.

## 5. The lattice of congruences on an inverse semigroup.

THEOREM 5.1. *Let $S$ be an inverse semigroup and let*

$$\theta = \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S): \rho_1 |_{E_S} = \rho_2 |_{E_S}\} .$$

*Then*

(i) *$\theta$ is a congruence on $\Lambda(S)$;*

(ii) *each $\theta$-class is a complete modular sublattices of $\Lambda(S)$ (with a greatest and least element);*

(iii) *the quotient lattice $\Lambda(S)/\theta$ is complete and the natural homomorphism $\theta^\natural$ of $\Lambda(S)$ onto $\Lambda(S)/\theta$ is a complete lattice homomorphism.*

*Proof.* We already know from Theorem 3.4 that $\theta$ is a meet compatible equivalence on $\Lambda(S)$. To establish that $\theta$ is a congruence it only remains to be shown that for $(\rho_1, \rho_2) \in \theta$, $\rho_3 \in \Lambda(S)$ we have $(\rho_1 \vee \rho_3, \rho_2 \vee \rho_3) \in \theta$. Let $e \in E_S$ and $f \in e(\rho_1 \vee \rho_3) \cap E_S$. Then $f \in E_S$ and $(e, f) \in \rho_1 \vee \rho_3$. Hence there exist $x_1, x_2, \cdots, x_k \in S$ such that $(e, x_1) \in \rho_1, (x_1, x_2) \in \rho_3, \cdots, (x_k, f) \in \rho_3$. Thus $(e, x_1x_1^{-1}) \in \rho_1 (x_1x_1^{-1}, x_2x_2^{-1}) \in \rho_3$, $\cdots, (x_kx_k^{-1}, f) \in \rho_3$. But $(\rho_1, \rho_2) \in \theta$ and so $(e, x_1x_1^{-1}) \in \rho_2, (x_1x_1^{-1}, x_2x_2^{-1}) \in \rho_3$, $\cdots, (x_kx_k^{-1}, f) \in \rho_3$. Consequently, $(e, f) \in \rho_1 \vee \rho_3$ and $f \in e(\rho_2 \vee \rho_3) \cap E_S$. Similarly $e(\rho_2 \vee \rho_3) \cap E_S \subseteqq e(\rho_1 \vee \rho_3) \cap E_S$. Hence $\rho_1 \vee \rho_3 |_{E_S} = \rho_2 \vee \rho_3 |_{E_S}$ and $(\rho_1 \vee \rho_3, \rho_2 \vee \rho_3) \in \theta$.

Part (ii) follows immediately from Theorem 3.4.

(iii) To show that $\Lambda(S)/\theta$ is complete and the natural homomorphism $\theta^\natural$ of $\Lambda(S)$ onto $\Lambda(S)/\theta$ is complete, (i.e., $\theta^\natural$ preserves arbitrary joins and intersections as well as pairwise joins and intersections) it is sufficient to show that $\theta$ is a complete congruence in the following sense: if, for some index set $I$, $\rho_i, \rho'_i \in \Lambda(S)$ for all $i \in I$, and $(\rho_i, \rho'_i) \in \theta$, for all $i \in I$, then

(a) $$\left(\bigcap_{i\in I} \rho_i, \bigcap_{i\in I} \rho'_i\right) \in \theta$$

and

(b) $$\left(\bigvee_{i\in I} \rho_i, \bigvee_{i\in I} \rho'_i\right) \in \theta .$$

However, quite minor alterations to the proofs of (i) in Theorem 3.4 and (i) in Theorem 5.1 will establish (a) and (b), respectively. Hence we have (iii).

**6. Kernel normal systems.** Let $S$ be an inverse semigroup and define $\theta$ on $\Lambda(S)$ as in Theorem 5.1.

DEFINITION 6.1 [3]. Let $T$ be a semigroup, $\rho$ a congruence on $T$, and $B \subseteqq T$. Then $[B]\rho = \{x \mid (b, x) \in \rho$ for some $b \in B\}$.

The proofs of the following two lemmas are based on the methods of Goldie [3] and Preston [9].

LEMMA 6.2. *Let $T$ be a semigroup, $\rho_1, \rho_2$ congruences on $T$, and $B \subseteqq T$. Then $[[B]\rho_1]\rho_2 = [[B]\rho_2]\rho_1$ implies that $[[B]\rho_1]\rho_2 = B(\rho_1 \vee \rho_2)$.*

*Proof.* (i) It is immediate from the definition that $[[B]\rho_1]\rho_2 \subseteqq [B](\rho_1 \vee \rho_2)$.

(ii) Let $x \in [B](\rho_1 \vee \rho_2)$. Then $(b, x) \in \rho_1 \vee \rho_2$ for some $b \in B$ so that there exist $x_1, x_2, \cdots, x_k \in T$ such that $(b, x_1) \in \rho_1, (x_1, x_2) \in \rho_2, \cdots, (x_k, x) \in \rho_2$. Hence $x_1 \in [B]\rho_1$ and so $x_2 \in [[B]\rho_1]\rho_2 = [[B]\rho_2]\rho_1$. Thus $x_3 \in [[[B]\rho_2]\rho_1]\rho_1 = [[B]\rho_2]\rho_1 = [[B]\rho_1]\rho_2$. Proceeding by induction, it is easy to see that $x \in [[B]\rho_1]\rho_2$.

LEMMA 6.3. *Let $\rho, \sigma \in \Lambda(S)$ be such that $(\rho, \sigma) \in \theta$ and let $\{N_\alpha \mid \alpha \in J\}, \{M_\alpha \mid \alpha \in J\}$ be the kernel normal systems of $\rho$ and $\sigma$, respectively. Define $(N \vee M)_\alpha = \{k \mid kk^{-1} \in E_\alpha$ and $kn = m$ for some $n \in N_\alpha$ and $m \in M_\alpha\}$. Then $(N \vee M)_\alpha = [N_\alpha]\sigma = [M_\alpha]\rho$.*

*Proof.* ( i ) $(N \vee M)_\alpha \subseteqq [N_\alpha]\sigma \cap [M_\alpha]\rho$. Let $k \in (N \vee M)_\alpha$. Then $kk^{-1} \in E_\alpha$ and $kn = m$ for some $n \in N_\alpha$ and $m \in M_\alpha$. Thus $kk^{-1}$, $(n^{-1})(n^{-1})^{-1} \in E_\alpha$ and $k(n^{-1})^{-1} \in M_\alpha$ so that $(k, n^{-1}) \in \sigma$ and hence $k \in [N_\alpha]\sigma$.

Now, $k \in [N_\alpha]\sigma$ implies that $(k, a) \in \sigma$, for some $a \in N_\alpha$ and so $(k^{-1}k, a^{-1}a) \in \sigma$, where $a^{-1}a \in E_\alpha$. Hence $k^{-1}k \in E_\alpha$ and $(k^{-1}k, n) \in \rho$. Then $(k, m) = (kk^{-1}k, kn) \in \rho$ or $k \in [M_\alpha]\rho$.

(ii) $[N_\alpha]\sigma \subseteqq (N \vee M)_\alpha$. Let $k \in [N_\alpha]\sigma$. Then $(k, n) \in \sigma$ for some $n$ in $N_\alpha$. Thus $kk^{-1}, nn^{-1} \in E_\beta$ and $kn^{-1} \in M_\beta$ for some $\beta \in J$. But $n \in N_\alpha$ and so $nn^{-1} \in E_\alpha$ so that $E_\alpha = E_\beta$. Now $kn^{-1} \in M_\alpha$ implies that $k \in (N \vee M)_\alpha$.

(iii) $[M]_\alpha\rho \subseteqq [N \vee M]_\alpha$. Let $k \in [M_\alpha]\rho$, say $(k, m) \in \rho$ where $m \in M_\alpha$. Then, as $mm^{-1} \in E_\alpha$, we have $kk^{-1} \in E_\alpha$. Also, $(k^{-1}, m^{-1}) \in \rho$ and $m^{-1}m \in E_\alpha$ imply that $k^{-1}m \in N_\alpha$, say $k^{-1}m = n \in N_\alpha$. Then $kn = kk^{-1}m \in E_\alpha M_\alpha \subseteqq M_\alpha$ or $k \in (N \vee M)_\alpha$.

THEOREM 6.4. *Let $\{N_\alpha \mid \alpha \in J\}$ and $\{M_\alpha \mid \alpha \in J\}$ denote the kernel normal systems of $\rho$ and $\sigma$, respectively, where $(\rho, \sigma) \in \theta$. Let $(N \vee M)_\alpha = \{k \mid kk^{-1} \in E_\alpha$ and $kn = m$ for some $n \in N_\alpha$ and $m \in M_\alpha\}$ and $(N \wedge M)_\alpha = N_\alpha \cap M_\alpha$. Then $\{(N \vee M)_\alpha \mid \alpha \in J\}$ is the kernel normal system of $\rho \vee \sigma$ and $\{(N \wedge M)_\alpha \mid \alpha \in J\}$ is the kernel normal system of $\rho \cap \sigma$.*

*Proof.* It is immediate that $\{(N \wedge M)_\alpha \mid \alpha \in J\}$ is the kernel normal system of $\rho \cap \sigma$ since for each $e \in E_\alpha$, $e(\rho \cap \sigma) = e\rho \cap e\sigma = N_\alpha \cap M_\alpha$.

If $e \in E_\alpha$, then $e\rho = [E_\alpha]\rho = N_\alpha$ and $e\sigma = [E_\alpha]\sigma = M_\alpha$. Since the $\theta$-classes are sublattices of $\Lambda(S)$, $e \in E_\alpha$ implies that $e(\rho \vee \sigma) = [E_\alpha](\rho \vee \sigma)$. Lemma 6.3 shows that $[[E_\alpha]\rho]\sigma = [[E_\alpha]\sigma]\rho$ and hence Lemma 6.2 implies that $[E_\alpha](\rho \vee \sigma) = [[E_\alpha]\rho]\sigma = [N_\alpha]\sigma = (N \vee M)_\alpha$.

## REFERENCES

1. G. Birkhoff, *Lattice Theory*, American Math. Soc. Colloquium Publications, Vol. XXV, 1964.
2. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. 1, Math. Surveys of the American Math. Soc. **7** (Providence, R. I., 1961).
3. A. W. Goldie. *The Jordan-Hölder Theorems for general abstract algebras*, Proc, London Math. Soc. (2) **52** (1950), 107-131.
4. J. M. Howie, *The maximum idempotent separating congruence on an inverse semigroup*, Proc. Edinburgh Math. Soc. **14** (1964), 71-79.
5. J. M. Howie, and G. Lallement, *Certain fundamental congruences on a regular semigroup*, Proc. Glasgow Math. Assoc. (7). **3** (1966) 145-59.
6. G. Lallement, *Congruences et equivalences de Green sur un demi-groupe regulier* (to appear).
7. W. D. Munn, *A class of irreducible matrix representations of an arbitrary inverse semigroup*, Proc. Glasgow Math. Assoc. **5** (1961), 14-48.
8. ———, *A certain sublattice of the lattice of congruences on a regular semigroup*, Proc. Cambridge Philosophic Society **60** (1964), 385-391.
9. G. B. Preston, *Inverse semigroups*, J. London Math. Soc. **29** (1954), 396-403.
10. ———, *Congruences on completely o-simple semigroups*, Proc. Lond. Math, Soc. (3) **11** (1961), 557-576.
11. T. Saito, *Regular elements in an ordered semigroup*, Pacific J. Math. **13** (1963). 263-95.
12. V. V. Vagner, *The theory of generalized groups and heaps*, Mat. Sborn. (N. S.) **32** (1953), 545-632.

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