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HILBERT TRANSFORMS FOR THE p-ADIC AND p-SERIES FIELDS

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In this paper, a class of singular integral transforms of the Calderón-Zygmund type is constructed for the spaces $\mathfrak{L}_r(\Psi_p, \lambda)$; Ψ_p is the p-adic or p-series field, λ is additive Haar measure, $r > 1$. The transforms have the form

$$Lf(y) = \lim_{k\to\infty} \int_{\{m(x)\leq p^{-k}\}} [m(x)]^{-1}w(x)f(y-x)d\lambda(x) ,$$

where m is the modular function for the field and

$$\int_{\{m(x)=1\}} w(x)d\lambda(x) = 0 .$$

The fundamental result is the existence of the $\mathfrak{L}_r$-limit and the M. Riesz inequality $||Lf||_r \leq A_r ||f||_r$. Several examples of functions w defining transforms L are given. In particular, subsets Φ of Ψ_p such that $\Phi \cap -\Phi = \varnothing$ and $\Phi \cup -\Phi = \Psi_p \backslash \{0\}$ together with functions w satisfying $w(-x) = -w(x)$ yield transforms which are analogues of the classical Hilbert transform. Multipliers for L are also discussed. A preliminary theorem of independent interest states that the $\mathfrak{L}_2$-Fourier transform on certain 0-dimensional locally compact Abelian groups converges pointwise.

The construction of singular integrals is in § 3; the main result is (3.13). Section 2 contains preliminary results and § 4 gives examples and calculations. Section 3 begins with a notational review for the fields Ψ_p. Other notation is, generally, as in [5]. We also refer to [5] for the required background material from abstract harmonic analysis. For a locally compact Hausdorff space Y, $\mathfrak{C}(Y)$ is the complex-valued continuous functions on Y; $\mathfrak{C}_0(Y)$ and $\mathfrak{C}_{00}(Y)$ denote, respectively, continuous functions which are "small" outside of compact sets and continuous functions with compact support. The symbol Z denotes the integers, Z^+ the positive integers, and R the real numbers. The characteristic function of a set A is denoted by ξ_A; its complement by A'. The Fourier transform of a function f on a locally compact Abelian group G is denoted by $\hat{f}$; $\check{f}$ denotes the inverse Fourier transform, defined on the character group X of G. For a given Haar measure on G, we always assume that Haar measure on X is chosen so that $(\hat{f})^{\vee} = f$, if $\hat{f} \in \mathfrak{L}_1(X)$.

2. Three preliminary theorems. In this section, we single out three results ((2.1), (2.2), and (2.3)) which will be used frequently in the constructions of § 3. The results are of some interest in themselves.

Let F be a nondiscrete locally compact field with additive Haar measure λ and modular function m; hence,

$$\int_F f(xa)d\lambda(x) = [m(a)]^{-1}\int_F f(x)d\lambda(x)$$

for $f \in \mathfrak{L}_1(F, \lambda)$ and $a \in F\backslash\{0\}$. The measure $\mu = \lambda/m$ is a multiplicative Haar measure for $F\backslash\{0\}$. For $t > 0$, let $V_t = \{x \in F: m(x) \leqq t\}$. The family $\{V_t\}_{t>0}$ is a neighborhood base at 0 for the topology of F; see [1], pp. 32–34. The equalities below are easily verified:

$$xV_{m(y)} = V_{m(xy)} \text{ and } m(x) = \frac{1}{\lambda(V_1)}\lambda(V_{m(x)})\ .$$

(2.1) THEOREM. *The function* $m^{-1}\xi_{V_1'}$ *is in* $\mathfrak{L}_r(F, \lambda)$ *if and only if* $r > 1$; $m^{-1}\xi_{V_1}$ *is in* $\mathfrak{L}_r(F, \lambda)$ *if and only if* $r < 1$.

Proof. Since $\mu(F) = \infty$ (F is not compact), at least one of $m^{-1}\xi_{V_1}$ and $m^{-1}\xi_{V_1'}$ is not in $\mathfrak{L}_1(\lambda)$. By the inversion invariance of μ, both are not in $\mathfrak{L}_1(\lambda)$. The "only if" statements follow from the inequalities $m(x) \leqq 1$ and $m(x) > 1$ for $x \in V_1$ and $x \in V_1'$, respectively.

Since $V_t = \cup\{V_{m(x)}: m(x) \leqq t\}$, we have

$$\lambda(V_t) = \sup\{\lambda(V_{m(x)}): m(x) \leqq t\} = \lambda(V_1)\sup\{m(x); m(x) \leqq t\}:$$

thus, $\lambda(V_t) \leqq \lambda(V_1)t$. Using this inequality and supposing $r > 1$, we have

$$\begin{aligned}\int_{V_1'}\left[\frac{1}{m(x)}\right]^r d\lambda(x) &= r\int_F \xi_{V_1'}(x)\int_1^\infty \frac{1}{t^{r+1}}\xi_{[m(x),\infty[}(t)dtd\lambda(x)\\ &= r\int_1^\infty\int_F \frac{1}{t^{r+1}}\xi_{V_1'}(x)\xi_{V_t}(x)d\lambda(x)dt\\ &\leqq r\int_1^\infty \frac{1}{t^{r+1}}\lambda(V_t)dt \leqq r\lambda(V_1)\int_1^\infty \frac{1}{t^r}dt < \infty\ .\end{aligned}$$

(Fubini's theorem applies because $(x, t) \to \xi_{V_t}(x)$ is product-measurable.)

The result for $0 < r < 1$ follows from that for $r > 1$ by inversion invariance; if $r \leqq 0$, then m^{-r} is bounded on the compact set V_1.

We will use the following theorem of Edwards and Hewitt [3] on differentiation of indefinite integrals.

(2.2) THEOREM. *Let* G *be a locally compact group with left Haar measure* λ. *Suppose that there is a sequence* $(U_n)_{n=1}^\infty$ *of Borel subsets of* G *satisfying the following conditions*:

(i) *Every neighborhood of e contains some U_n, and $U_{n+1} \subset U_n$ for $n = 1, 2, \cdots$.*

(ii) *There is a constant C such that*

$$0 < \lambda(U_n U_n^{-1}) < C\lambda(U_n),\ n = 1, 2, \cdots .$$

Then the equality

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{1}{\lambda(U_n)} \int_{xU_n} f d\lambda = f(x)$$

holds l.a.e. for each $f \in \mathfrak{L}_{1,loc}(G)$ and a.e. for each $f \in \mathfrak{L}_1(G)$.

For the proof of (2.2), see [3]; we will apply it to certain 0-dimensional, locally compact, Abelian groups. If such a group is first countable, the conditions (2.2. i) and (2.2. ii) are met; in fact, the U_n's can be taken as subgroups.

If f is a function in $\mathfrak{L}_2(R)$, then the functions $f_n = f\xi_{]-n,n[}$ converge pointwise and in the $\mathfrak{L}_2$ norm to f. Thus the $\mathfrak{L}_2$ Fourier transforms $\hat{f}_n$ converge in the $\mathfrak{L}_2$ norm to $\hat{f}$. It is an open question whether $\hat{f}_n$ always converges pointwise a.e. [For a discussion see [9], p. 85. The analogous question for the circle group has recently been answered affirmatively by L. Carleson.] The following theorem asserts that for certain locally compact groups [not R!] the analogous question has an affirmative answer. We recall that every neighborhood of the unit e in a 0-dimensional, locally compact group contains a compact open subgroup; ([5] Th. (7.7), p. 62). For a subset Φ of the character group X of a locally compact group G, $A(G, \Phi)$ denotes the annihilator of Φ in G. Throughout this paper, convolution $(*)$ is taken with respect to Haar measure.

(2.3) THEOREM. *Let G be a locally compact Abelian group with first countable and 0-dimensional character group X, and let $\{\Phi_n\}_{n=1}^{\infty}$ be a basis at $e \in X$ consisting of compact open subgroups such that $\Phi_{n+1} \subset \Phi_n$. Suppose λ and μ are normalized Haar measures for G and X, respectively. If $f \in \mathfrak{L}_2(\lambda)$ and $f_n = f\xi_{A(G,\Phi_n)}$, then*

$$\hat{f}(\chi) = \lim_{n\to\infty} \hat{f}_n(\chi) \qquad a.e.$$

Proof. The subgroups $A(G, \Phi_n)$ of G are compact as the groups Φ_n are both compact and open (see [5], p. 369); we have

$$\hat{\xi}_{A(G,\Phi_n)}(\chi) = \int_G \xi_{A(G,\Phi_n)}(x)\overline{\chi(x)}d\lambda(x) .$$

If $\chi \in \Phi_n$, then $\chi(x) = 1$ for all $x \in A(G, \Phi_n)$; and so the value of the above integral is $\lambda(A(G, \Phi_n))$. If $\chi \notin \Phi_n$, then $\chi|_{A(G,\Phi_n)}$ is a non trivial character of the compact group $A(G, \Phi_n)$. A Haar measure on this group is simply λ restricted to it, and so in this case the integral is zero ([5], (23.19), p. 363). Thus we have proved that

$$\hat{\xi}_{A(G,\Phi_n)}(\chi) = \lambda(A(G, \Phi_n))\xi_{\Phi_n}(\chi) . \tag{1}$$

Using the identity $(\varphi*\psi)^{\vee} = \varphi^{\vee}\psi^{\vee}$ (valid for $\varphi \in \mathfrak{L}_1(X, \mu)$ and $\psi \in \mathfrak{L}_2(X,\mu)$) and the inversion formulas, it is easy to see that $(gf)^{\wedge} = \hat{g}*\hat{f}$ whenever $g \in \mathfrak{L}_1(G)$ is such that $\hat{g} \in \mathfrak{L}_1(X)$. Taking $g = \xi_{A(G,\Phi_n)}$ and using (1), we obtain

$$\hat{f}_n = (\xi_{A(G,\Phi_n)}f)^{\wedge} = \hat{\xi}_{A(G,\Phi_n)}*\hat{f} = \lambda(A(G, \Phi_n))\xi_{\Phi_n}*\hat{f} .$$

By Plancherel's theorem, we have $\| \xi_{A(G,\Phi_n)} \|_2^2 = \| \hat{\xi}_{A(G,\Phi_n)} \|_2^2$; thus, $\lambda(A(G, \Phi_n)) = [\lambda(A(G, \Phi_n))]^2\lambda(\Phi_n)$. Hence we have $\lambda(A(G, \Phi_n)) = 1/\mu(\Phi_n)$, and it follows that

$$\hat{f}_n(\chi) = \frac{1}{\mu(\Phi_n)} \int_{\chi\Phi_n} \hat{f}(\tau)d\mu(\tau) .$$

The sets $\{\Phi_n\}_{n=1}^{\infty}$ satisfy (2.2. i) and (2.2. ii) for X, and the function $\hat{f}$ is in $\mathfrak{L}_{1,\mathrm{loc}}(X)$ as it is in $\mathfrak{L}_2(X)$. Thus, by (2.2), we have $\lim_{n\to\infty}\hat{f}_n(\chi) = \hat{f}(\chi)$ for almost all $\chi \in \chi$.

We will apply (2.3) when $G = (\Psi_p, +)$.

3. Hilbert transforms for Ω_p (p-adic field) and Γ_p (p-series field). As a set, $\Psi_p(\Omega_p$ or $\Gamma_p)$ is all doubly infinite sequences $x = (x_n)_{n=-\infty}^{\infty}$ of integers such that $0 \leqq x_n \leqq p - 1$ for each n and such that $x_n = 0$ for almost all negative n. (The fields Ω_p and Γ_p differ in the definition of multiplication and addition; see [5], § 10 and [7], § 26.) The mapping

$$x \longrightarrow \chi_y(x) = \exp(i\sigma(xy)) \tag{3.1}$$

where

$$\sigma(x) = 2\pi \sum_{j=-\infty}^{0} x_j p^{j-1} \text{ on } \Omega_p \text{ and } \sigma(x) = 2\pi x_0 p^{-1} \text{ on } \Gamma_p \tag{3.2}$$

is a topological isomorphism of Ψ_p onto its character group. (The character group of Ω_p is computed in [5], pp. 400–402, but the function σ is not used there. Minor modifications show the role of σ as described above. With modifications in that computation, the result for Γ_p can also be obtained. The result is also in [4] and in [6].) We will usually identify the character group of Ψ_p with Ψ_p; thus, χ_y will be written as y. For a nonzero $x \in \Psi_p$, $s(x)$ denotes the unique integer such that $x_{s(x)} \neq 0$ and $x_n = 0$ if $n < s(x)$. We use the notations

$$A_k = \{x: s(x) \geqq k\};\ \Delta_k = \{x: s(x) = k\};\ u_k = (\delta_{kn})_{n=-\infty}^{\infty} , \tag{3.3}$$

$k \in Z$. The family $\{\Lambda_k\}_{k=-\infty}^{\infty}$ of compact open subgroups of Ψ_p forms a neighborhood base at 0. The multiplicative identity of Ψ_p is $u = u_0$. We also record the following identities:

$$m(x) = p^{-s(x)} \text{ and } V_{m(x)} = \Lambda_{s(x)} \tag{3.4}$$

$$\Delta_k^{-1} = \Delta_{-k};\ x\Delta_k = \Delta_{k+s(x)};\ x\Lambda_k = \Lambda_{k+s(x)};\ s(x^{-1}) = -s(x); \\ s(xy) = s(x) + s(y)\ . \tag{3.5}$$

$$A(\Psi_p, \Lambda_k) = \Lambda_{-k+1}\ . \tag{3.6}$$

(We note that the function m is a valuation for Ψ_p and that s is a logrithmic valuation. We will not use any valuation theory in this paper.)

Normalization of Haar measure λ on Ψ_p so that the companion Haar measure on the character group ($= \Psi_p$) is the same requires (by the proof of (2.3) and (3.6)) that $\lambda(\Lambda_0) = [\lambda(\Lambda_1)]^{-1}$. Since $\lambda(\Lambda_0) = p\lambda(\Lambda_1)$, we must have $\lambda(\Lambda_1) = p^{-1/2}$. It is then immediate that $\lambda(\Lambda_n) = p^{-n+(1/2)}$ for all integers n. With these notational preliminaries, we can give the definition of singular integrals.

First, w will denote a bounded λ-measurable function on Δ_0 satisfying

$$\int_{\Delta_0} w(x)d\lambda(x) = 0; \tag{3.7}$$

w is extended to all of Ψ_p by letting $w(x)(x^* = xu_{-s(x)} \in \Delta_0)$ if $x \neq 0$, and $w(0) = 1$. The kernels ψ which define the transforms are defined by

$$\psi(x) = \frac{w(x)}{m(x)}\ ; \tag{3.8}$$

we let $\psi_k = \psi\xi_{\Lambda_k'}$, $k \in Z$. (The function m on Ψ_p is a precise analogue of the function $x \to |x|$ on R. The real number analogue of Δ_0 is the two-element set $\{-1, 1\}$; and, the condition (3.7) for w is like demanding that $w(1) + w(-1) = 0$, if w were a function on $\{-1, 1\}$. Such a function defines the classical Hilbert transform.)

If $r > 1$, the convolution $f*\psi_k$ is in $\mathfrak{C}_0$ for all $f \in \mathfrak{L}_r$; and, if $f \in \mathfrak{L}_1$, it is defined a.e. and is in $\mathfrak{L}_r$ for all $r > 1$. ($\psi_k \in \mathfrak{L}_r$, $r > 1$, by (2.1).) In either case, we let

$$L_k f = f*\psi_k = \psi_k*f\ . \tag{3.9}$$

We will show that, under an additional restriction on w, the linear operators L_k carry $\mathfrak{L}_r(\Psi_p)$ boundedly into $\mathfrak{L}_r(\Psi_p)$ for every $r > 1$; and, furthermore, that the sequence of operators $(L_k)_{k=1}^{\infty}$ converges to a bounded operator L from $\mathfrak{L}_r$ into $\mathfrak{L}_r$. The properties of the $\mathfrak{L}_2$ Fourier transform (Plancherel's theorem; inversion) make the $\mathfrak{L}_2$ case easy, and we begin with it.

(3.10) THEOREM. *Suppose that $w(x) = w(x_0, x_1, \cdots, x_q)$; i.e., that $w(x)$ depends only on a finite number of the coordinates of x. Then for every $f \in \mathfrak{L}_2(\Psi_p)$, the functions $L_k f$, $k = 1, 2, \cdots$, are in $\mathfrak{L}_2(\Psi_p)$ and converge in the $\mathfrak{L}_2$ norm to a function Lf. The mapping L so defined is a bounded linear operator from $\mathfrak{L}_2(\Psi_p)$ to $\mathfrak{L}_2(\Psi_p)$. The linear operators L_k are uniformly bounded: there is a constant A_2, independent of f, such that*

(i) $$L_k f \,\|_2 \leqq A_2 \,\| f \|_2$$

for $k = 1, 2, \cdots$. We also have

(ii) $$\| Lf \|_2 \leqq A_2 \,\| f \|_2 \,.$$

Proof. Letting $\psi_{k,n} = \psi_k \xi_{\Lambda_n}$ for $n = -1, -2, -3, \cdots$ and $k = 1, 2, 3, \cdots$, we have $\lim_{n\to-\infty}\psi_{k,n} = \psi_k$ both pointwise and in the $\mathfrak{L}_2$ norm. The functions $\psi_{k,n}$ are in $\mathfrak{L}_1(\lambda)$, so that

$$\hat{\psi}_{k,n}(y) = \int_{\Lambda'_k \cap \Lambda_n} w(x)[m(x)]^{-1} \exp\,(-\,i\sigma(xy))d\lambda(x)\,.$$

By the invariance of the multiplicative Haar integral, we can write this, for $y \neq 0$, as

$$\hat{\psi}_{k,n}(y) = \int_S w(y^{-1}x)[m(x)]^{-1} \exp\,(-i\sigma(x))d\lambda(x) = \sum_{j=n+s(y)}^{k+s(y)-1} \int_{\Lambda_j} \cdot\;,$$

where $S = \Lambda'_{k+s(y)} \cap \Lambda_{n+s(y)}$ and the missing integrands are as in the previous expression. (When we use multiplicative invariance in this way, we make strong use of *field* properties of Ψ_p. We have used (3.5) in obtaining the set S.) Theorem (2.3) applied to the functions $\psi_{k,n}$ (take $\Phi_n = \Lambda_{n+1}$ and use (3.6)) gives the equality $\lim_{n\to-\infty}\hat{\psi}_{k,n}(y) = \hat{\psi}_k(y)$ a.e.; thus, $\hat{\psi}_k(y) = \sum_{j=-\infty}^{k+s(y)-1}\int_{\Lambda_j}$ a.e. The equality

$$\int_{\Lambda_j} w(y^{-1}x)[m(x)]^{-1} \exp\,(-i\sigma(x))d\lambda(x) = \int_{\Lambda_0} w(y^{*-1}x) \exp\,(-i\sigma(xu_j))d\lambda(x)$$

and (3.7) show that $\int_{\Lambda_j} = 0$ for $j > 0$. For any $y \neq 0$, there are k's such that $k + s(y) - 1 > 0$; hence,

(1) $$\varphi(y) = \lim_{k\to\infty} \hat{\psi}_k(y) = \sum_{j=-\infty}^{0} \int_{\Lambda_j} w(y^{-1}x)[m(x)]^{-1} \exp\,(-i\sigma(x))d\lambda(x)$$

exists a.e. A calculation like that given above shows that $\hat{\psi}_{k,n}(0) = 0$

for all n and k.

We will show that the convergence of $\hat{\psi}_{k,n}$ to $\hat{\psi}_k$ and of $\hat{\psi}_k$ to φ are actually everywhere and that the sequences $(\hat{\psi}_k)$ and $(\hat{\psi}_k - \hat{\psi}_{k,n})$ are uniformly bounded. The hypothesis on w implies that the range of w is finite; say $w(\psi_p) = \{a_1, a_2, \cdots, a_H\}$, with a's distinct. Each $y \neq 0$ defines a partition $\Pi(y) = \{\pi_h(y)\}_{h=1}^{H}$ of Δ_0, where

$$\pi_h(y) = \{x \in \Delta_0 \colon w(y^{*-1}x) = a_h\} .$$

Since $\Pi(y)$ is determined by the first $q + 1$ coordinates of y, there are at most $(p-1)p^q$ distinct partitions; call them $\{\Pi_i\}$. For a given $y \neq 0$, we have

$$(2) \qquad \hat{\psi}_{k,n}(y) = \sum_{j=n+s(y)}^{\min(0,k+s(y)-1)} \left[\sum_{h=1}^{H} a_h \int_{\pi_h(y)} \exp(-i\sigma(u_j x))dx\right] .$$

Each of the sets $\{y \neq 0; \Pi(y) = \Pi_i\}$ has infinite measure. Therefore each of these sets contains a point y for which $\hat{\psi}_{k,n}(y)$ converges in n for all $k > 0$. If $\Pi(z) = \Pi(y)$, then $\hat{\psi}_k(z) = \lim_{n\to-\infty}\hat{\psi}_{k,n}(z)$ must also exist; it differs from $\hat{\psi}_k(y)$ by the sum of a finite series. Thus, $\hat{\psi}_{k,n}(y)$ converges to $\hat{\psi}_k(y)$ for all y. It follows also that $\hat{\psi}_k(y)$ converges to $\varphi(y)$ for all y. Letting $n \to -\infty$ in (2), we see that $|\hat{\psi}_k(y)|$ has a bound depending only on $\Pi(y)$. Therefore, since there are only finitely many $\Pi(y)$'s, $(|\hat{\psi}_k(y)|)_{k=0}^{\infty}$ is uniformly bounded, say by M. The bound

$$(3) \quad |\hat{\psi}_k(y) - \hat{\psi}_{k,n}(y)| = \left|\sum_{j=-\infty}^{\min(0,n+s(y)-1)} \sum_{h=1}^{H} a_h \int_{\pi_h(y)} \exp(-i\sigma(u_j x))dx\right| \leqq M$$

also holds.

Let $f \in \mathfrak{L}_2(\lambda)$. By the bound (3) and the dominated convergence theorem, the sequence $(\hat{\psi}_{k,n}\hat{f})_{n=-1}^{-\infty}$ converges in $\mathfrak{L}_2$ to $\hat{\psi}_k\hat{f}$. Hence $\psi_{k,n}*f(= (\hat{\psi}_{k,n}\hat{f})^{\vee})$ converges in $\mathfrak{L}_2$ to $(\hat{\psi}_k\hat{f})^{\vee}$. But $\psi_{k,n}*f$ also converges uniformly to ψ_k*f; hence, $(\hat{\psi}_k\hat{f})^{\vee} = \psi_k*f$ a.e. In particular, we have proved that $L_k f \in \mathfrak{L}_2$ for all $k \in Z^+$. Applying dominated convergence and taking inverse Fourier transforms again, we see that

$$Lf = \lim_{k\to\infty} L_k f$$

exists in $\mathfrak{L}_2$ and that $(Lf)^{\wedge} = \varphi\hat{f}$. We have

$$\| Lf \|_2 = \| \varphi\hat{f} \|_2 \leqq \sup_{x \in \psi_p} |\varphi(x)| \, \|\hat{f}\|_2 .$$

Taking $A_2 = \sup_{x\in\psi_p} |\varphi(x)|$, we get (i) and (ii). The linearity of $\mathfrak{L}$ is an immediate consequence of the linearity of each L_k.[1]

To prove the analogue of (3.10) for $r \neq 2$, we require some pre-

[1] The equality $(Lf^{\wedge}) = \varphi\hat{f}$ means that φ is an $\mathfrak{L}_2$-multiplier for $\mathfrak{L}$. We will see later (3.14) that it is also an $\mathfrak{L}_r$-multiplier if $1 < r \leqq 2$.

liminaries on equimeasurable functions and a 0-dimensional covering lemma.

(3.11) PRELIMINARIES ON EQUIMEASURABILITY. For an extended real-valued μ-measurable function f on a measure space $(X, \mathscr{M}, \mu)$, we let $f^{\#}$ denote a decreasing function on $]0, \infty[$ that is equimeasurable with f. Such a function has the properties

$$\text{(ii)} \qquad \int_X f d\mu = \int_0^\infty f^{\#} d\lambda; \int_B f d\mu \leqq \int_0^{\mu(B)} f^{\#} d\lambda ,$$

for B μ-measurable and λ Lebesgue measure. For $f \in L_r(X)(r \geqq 1)$, we define, as in [3],

$$\text{(ii)} \qquad \beta_f(s) = \frac{1}{s}\int_0^s f^{\#}(t)dt, s > 0 .$$

The function β_f is continuous, is constant on $]0, s_0[$ for some $s_0 \geqq 0$, and is strictly decreasing on $[s_0, \infty[$. Let $y_0 = \lim_{s^+ \to s_0} \beta_f(s)$, and define β^f on $]0, \infty[$ as the inverse of β_f on $]0, y_0[$, as 0 on $]y_0, \infty[$, and s_0 at y_0. We have

$$\text{(iii)} \qquad \beta^f(\beta_f(s)) \geqq s \text{ for all } s > 0 \text{ and } \beta_f(\beta^f(y)) \leqq \text{ for all } y \leqq y_0;$$

$$\text{(iv)} \qquad \lim_{s\to\infty} \beta_f(s) = \lim_{y\to\infty} \beta^f(y) = 0 \text{ and } \lim_{y\to 0} \beta^f(y) = \infty .$$

The properties of β^f and β_f of course depend only on the properties of $f^{\#}$; *i.e.*, that it is a decreasing function in $\mathfrak{L}_r^+(]0, \infty[)$, and not on f. For $X =]0, \infty[$, the facts are contained in [3].

The following lemma is a 0-dimensional analogue of Lemma 1 of [3], p. 91.

(3.12) COVERING LEMMA. *Let G be a locally compact Abelian group having a neighborhood basis of the identity of the form $\{H_n\}_{n=-\infty}^{\infty}$, where the H_n's are compact open subgroups of G satisfying $H_{n+1} \subset H_n$ and $\bigcup_{n=-\infty}^{\infty} H_n = G$. Let $k_n = [H_{n-1}; H_n]$. For $f \in \mathfrak{L}_r^+(\lambda)(r \geqq 1, \lambda$ Haar measure) and $t > 0$, there are a subset P_t of $Z^+ \times Z$ and a mapping $(m, n) \to x_{m,n}$ of P_t into G such that $\{x_{m,n}H_n: (m, n) \in P_t\}$ is pairwise disjoint and the following inequalities hold:*

$$\text{(i)} \qquad t \leqq \frac{1}{\lambda(H_n)}\int_{x_{m,n}H_n} f d\lambda \leqq tk_n \; ((m, n) \in P_t)$$

$$\text{(ii)} \qquad \lambda(D_t) \leqq \beta^f(t) < \infty, \textit{ where } D_t = \bigcup_{P_t} x_{m,n}H_n$$

$$\text{(iii)} \qquad f(x) \leqq t \quad a.\text{e. in } D_t'$$

$$\text{(iv)} \qquad t\lambda(D_t) \leqq \int_{D_t} f d\lambda .$$

If $k_n \leqq C$ *(C constant,* $n \in Z$*), then*

$$\text{(v)} \qquad t\lambda(D_t) \leqq \int_{D_t} f d\lambda \leqq Ct\lambda(D_t) .$$

Proof. By (3.11. iv), we have $\beta_f(s) < t$ for all sufficiently large s. If there are n's such that $\beta_f(\lambda(H_n)) \geqq t$, let N be the largest integer such that $\beta_f(\lambda(H_N)) < t$. If $\beta_f(\lambda(H_n)) < t$ for all n, let $N = 0$. A countable number of disjoint cosets of H_N cover G, say $\bigcup_{m=1}^{\infty} v_{m,N} H_N$. For each of these cosets, we have

$$\frac{1}{\lambda(H_N)} \int_{v_{m,N} H_N} f d\lambda \leqq \frac{1}{\lambda(H_N)} \int_0^{\lambda(H_N)} f^{\#}(u) du = \beta_f(H_N) < t .$$

Write $H_N = \bigcup_{j=1}^{k_{N+1}} z_{j,N+1} H_{N+1}$, where the cosets in the union are disjoint. For each $v_{m,N}$, we have

$$v_{m,N} H_N = \bigcup_{j=1}^{k_{N+1}} v_{m,N} z_{j,N+1} H_{N+1} ,$$

and the family $\{v_{m,N} z_{j,N+1} H_{N+1}\}$ is pairwise disjoint. Relabel those sets in $\{v_{m,N} z_{j,N+1} H_{N+1}\}$ for which the average of f over the set is less than t as $\{v_{m,N+1} H_{N+1}\}_{m=1}^{\infty}$:

$$\frac{1}{\lambda(H_{N+1})} \int_{v_{m,N+1} H_{N+1}} f d\lambda < t .$$

There are finitely many remaining cosets $v_{m,N} z_{j,N+1} H_{N+1}$ (use Hölder's inequality to prove this, if $r > 1$), and these we label as

$$\{x_{m,N+1} H_{N+1}\}_{m=1}^{e_{N+1}} .$$

If $e_{N+1} = 0$, then the family $\{x_{m,N+1} H_{N+1}\}_{m=1}^{e_{N+1}}$ is void. If $e_{N+1} > 0$, suppose that $x_{m,N+1} H_{N+1} \subset v_{l,N} H_N$. We have

$$\int_{x_{m,N+1} H_{N+1}} f d\lambda \leqq \int_{v_{l,N} H_N} f d\lambda < \lambda(H_N) t = k_{N+1} \lambda(H_{N+1}) t ,$$

so that

$$t \leqq \lambda(H_{N+1})^{-1} \int_{x_{m,N+1} H_{N+1}} f d\lambda < k_{N+1} t .$$

We inductively define nonnegative integers e_n and sets $\{v_{m,n}\}_{m=1}^{\infty}$ and $\{x_{m,n}\}_{m=1}^{e_n}$ $(n = N+1, N+2, \cdots)$ such that the families $\{x_{m,n} H_n\}_{m=1}^{e_n}$ and $\{v_{m,n} H_n\}_{m=1}^{\infty}$ are disjoint for each n, each of these families is pairwise disjoint for each n, and the following relations hold:

$$(1) \quad \bigcup_{m=1}^{\infty} v_{m,n} H_n = \left(\bigcup_{m=1}^{\infty} v_{m,n+1} H_{n+1}\right) \cup \left(\bigcup_{m=1}^{e_{n+1}} x_{m,n+1} H_{n+1}\right), \ n = N, N+1, \cdots ;$$

(2) $\dfrac{1}{\lambda(H_n)}\displaystyle\int_{v_{m,n}H_n} fd\lambda < t$ for $m = 1, 2, \cdots$ and $n = N, N+1, \cdots$;

(3) $t \leqq \dfrac{1}{\lambda(H_n)}\displaystyle\int_{x_{m,n}H_n} fd\lambda < k_n t$ for $m = 1,2, \cdots$

and $n = N+1, N+2, \cdots$.

(If $e_n = 0$, (3) holds vacuously.) The inductive step differs only in notation from the construction above giving the sets $\{x_{m,N+1}\}$ and $\{v_{m,N+1}\}$ satisfying (1)—(3) for $n = N+1$.

Denote by P_t the subset of $Z^+ \times Z$ defined by the condition that $(m, n) \in P_t$ if an element $x_{m,n}$ appears in the above construction. Note that it is possible that $P_t = \varnothing$, i.e., $e_{N+1} = e_{N+2} = \cdots = 0$. If this is the case, let $D_t = \varnothing$; otherwise, let $D_t = \bigcup_{P_t} x_{m,n}H_n$. We have seen that $\{x_{m,n}H_n\}_{m=1}^{e_n}$ is pairwise disjoint for every n, and it is also clear that $(x_{m', n'}H_{n'}) \cap (x_{m,n}H_n) = \varnothing$ if $n' \neq n$. Thus the family

$$\{x_{m,n}H_n\colon (m, n) \in P_t\}$$

is pairwise disjoint.

The function f is in $\mathfrak{L}_r(G)$, and therefore also in $\mathfrak{L}_{1,loc}$. Clearly G is σ-compact, so that "l.a.e." and "a.e." coincide. Hence, by (2.2) we have $\lim_{n\to\infty}\lambda(H_n)^{-1}\int_{xH_n} fd\lambda = f(x)$ for almost all x. If $x \in D_t'$, then for every $n \leqq N$ x is in some $v_{m,n}H_n$; thus, $xH_n = v_{m,n}H_n$. By (2), the inequality $\lambda(H_n)^{-1}\int_{xH_n} fd\lambda < t$ holds for $n = N, N+1, \cdots$; and (iii) is established.

If $P_t = D_t = \varnothing$, the remaining assertions of the lemma are trivial. For $P_t \neq \varnothing$ and F a finite subset of P_t, we have

$$\begin{aligned} t &\leqq \Big[\lambda\Big(\bigcup_F x_{m,n}H_n\Big)\Big]^{-1}\int_{\bigcup_F x_{m,n}H_n} fd\lambda \leqq \lambda\Big(\bigcup_F \cdot\Big)^{-1}\int_0^{\lambda\left(\bigcup_F \cdot\right)} f^{\#}(t)dt \\ &= \beta_f\Big(\sum_F \lambda(x_{m,n}H_n)\Big) . \end{aligned}$$

Taking the inverse β^f in this inequality (see (3.11. iii)) gives $\sum_F \lambda(x_{m,n}H_n) \leqq \beta^f(t)$ for all finite subsets F of P_t; (ii) follows.

The inequalities in (i), (iv), and (v) follow from those in (3).

We now prove the main theorem of the paper.

(3.13) THEOREM. *Let w be as in (3.10); i.e., $\int_{\Delta_0} wd\lambda = 0$, and $w(x)$ depends on only finitely many of the coordinates of x. Suppose that $r > 1$. For every $f \in \mathfrak{L}_r(\Psi_p)$, the functions $L_k f$ $(k \in Z^+)$ are in $\mathfrak{L}_r(\Psi_p)$. The linear operators L_k from $\mathfrak{L}_r(\Psi_p)$ to $\mathfrak{L}_r(\Psi_p)$ are uniformly bounded:*

there is a constant A_r, independent of f and of k, such that

(i) $$\|L_k f\|_r \leqq A_r \|f\|_r$$

for $k = 1,2,\cdots$. For every $f \in \mathfrak{L}_r(\Psi_p)$, the sequence $(L_k f)_{k=1}^{\infty}$ converges in the $\mathfrak{L}_r$ norm to a function Lf. The inequality

(ii) $$\|Lf\|_r \leqq A_r \|f\|_r$$

holds for all $f \in \mathfrak{L}_r \Psi_p)$.

Proof. The function w is bounded and $m^{-1}\xi_{\Lambda_k'}$ $(k \in Z)$ is in all $\mathfrak{L}_s$ spaces $(s > 1)$. It follows that $L_k f \in \mathfrak{C}_0$ if $f \in \mathfrak{L}_r$, $r > 1$; and that $L_k f \in \mathfrak{L}_s$ for all $s > 1$, if $f \in \mathfrak{L}_1$.

We give the proof in three steps. In some portions of the proof we will include the case $r = 1$.

Step I. Suppose for now that $1 \leqq r \leqq 2$, and let $k \in Z^+$ and $f \in \mathfrak{L}_r^+$ be fixed. For $t > 0$, define $\Phi_t = \{x \in \Psi_p : |L_k f(x)| > t\}$.

The heart of the proof lies in estimating the measure of Φ_t. Following Calderón and Zygmund [3], we will prove that there are constants c_1 and c_2, independent of k and t, such that

(1) $$\lambda(\Phi_t) \leqq \frac{c_1}{t^2}\int_{\Psi_p}([f]_t)^2 d\lambda + c_2 \beta f(t)$$

where

$$[f]_t(x) = \begin{cases} f(x) & \text{if } f(x) \leqq t \\ t & \text{if } f(x) > t \,. \end{cases}$$

The subgroups Λ_n $(n \in Z)$ of Ψ_p satisfy the conditions of the H_n's in (3.12). Since $k_n = [\Lambda_{n-1} : \Lambda_n] = p$ for all $n \in Z$, we may take $C = p$ in (3.12. v).

Let

$$h(x) = \begin{cases} \dfrac{1}{\lambda(\Lambda_n)}\displaystyle\int_{x_{m,n}+\Lambda_n} f d\lambda & \text{if } x \in x_{m,n} + \Lambda_n \subset D_t \\ f(x) & \text{if } x \in D_t' \,, \end{cases}$$

and set $g(x) = f(x) - h(x)$; thus, $f(x) = h(x) + g(x)$ for all $x \in \Psi_p$ and $g(x) = 0$ for $x \in D_t'$. Define

$$\Phi_{t,1} = \left\{x : |L_k h(x)| > \frac{t}{2}\right\} \text{ and } \Phi_{t,2} = \left\{x : |L_k g(x)| > \frac{t}{2}\right\}.$$

We obtain (1) by estimating $\lambda(\Phi_{t,1})$ and $\lambda(\Phi_{t,2})$. The function h

is bounded on D_t by tp (3.12. i) and on D'_t by t(3.12.iii). Since the set D_t has finite measure and $h = f$ on D'_t, it follows that h is in $\mathfrak{L}_r^+$. Hence g is in $\mathfrak{L}_r$. Furthermore, we have

$$\int_{\Psi_p} h^2 d\lambda \leqq \sup_{x \in \Psi_p} | h(x) |^{2-r} \int_{\Psi_p} h^r d\lambda .$$

Thus $h \in \mathfrak{L}_2$ and the inequalities

$$\lambda(\Phi_{t,1}) \frac{t^2}{4} \leqq \int_{\Phi_{t,1}} | L_k h |^2 d\lambda \leqq \int_{\Psi_p} | L_k h |^2 d\lambda \leqq A_2^2 \int_{\Psi_p} h^2 d\lambda$$

yield the estimate $\lambda(\Phi_{t,1}) \leqq 4A_2^2 t^{-2} \int_{\Psi_p} h^2 d\lambda$. Finally, we have

$$\int_{\Psi_p} h^2 d\lambda = \int_{D_t} h^2 d\lambda + \int_{D'_t} f^2 d\lambda \leqq p^2 t^2 \lambda(D_t) + \int_{\Psi_p} ([f]_t)^2 d\lambda .$$

Thus we get our estimate for $\lambda(\Phi_{t,1})$:

$$\lambda(\Phi_{t,1}) \leqq \frac{c_1}{t^2} \int_{\Psi_p} ([f]_t)^2 d\lambda + b\lambda(D_t) , \tag{2}$$

where c_1 and b are constants independent of k, t, and f.

To estimate $\lambda(\Phi_{t,2})$, we write

$$\lambda(\Phi_{t,2}) \leqq \lambda(D_t) + \lambda(\Phi_{t,2} \cap D'_t) , \tag{3}$$

and consider $\lambda(\Phi_{t,2} \cap D'_t)$. For each $x \in \Psi_p$, the functions

$$y \longrightarrow g(y)\psi_k(x - y)_{\bigcup_F (x_{m,n}+\Lambda_n)}(y)$$

converge dominately to the $\mathfrak{L}_1$-function $y \longrightarrow g(y)\psi_k(x - y)\xi_{D_t}(y)$ as the finite set F expands to P_t. We thus obtain

$$L_k g(x) = \sum_{P_t} \int_{x_{m,n}+\Lambda_n} g(y)\psi_k(x - y) d\lambda(y) \tag{4}$$

for all $x \in \Psi_p$. Consider a term of this series for $x \in D'_t$. If (m, n) is such that $(x_{m,n} + \Delta_n) \cap (x + \Lambda_k) = \varnothing$, then $x - y$ is in Λ'_k for all $y \in x_{m,n} + \Lambda_n$. For these (m, n), we can replace ψ_k by ψ in (4). Thus, using the equality $\int_{x_{m,n}+\Lambda_n} g d\lambda = 0$ (valid for all (m, n) by the definition of g), we can write

$$\begin{aligned} &\int_{x_{m,n}+\Lambda_n} g(y)\psi_k(x - y) d\lambda(y) \\ &= \int_{x_{m,n}+\Lambda_n} g(y)[\psi(x - y) - \psi(x - x_{m,n})] dy \end{aligned} \tag{5}$$

Since $x - x_{m,n} \notin \varDelta_n (x \in D'_t)$, we have $s(x-y) = s(x - x_{m,n})$ for $y \in x_{m,n} + \varDelta_n$; hence,

$$(6) \qquad |\psi(x-y) - \psi(x - x_{m,n})| = \left| \frac{w(x-y) - w(x - x_{m,n})}{m(x - x_{m,n})} \right| .$$

If $w(x) = w(x_0, x_1, \cdots, x_q)(x \in \varDelta_0)$ and M is a bound for w, let $\gamma = 2M\xi_{\varDelta'_{q+1}}$. The function γ satisfies

$$\text{(A)} \qquad \int_{\varDelta_0} \frac{\gamma(x)}{m(x)} d\lambda(x) < \infty ;$$

$$\text{(B)} \qquad |w(x) - w(y)| \leqq \gamma(x-y), \text{ all } x, y \in \varDelta_0 ;$$

$$\text{(C)} \qquad 0 \leqq \sup \gamma(\varDelta_j) \leqq \inf \gamma(\varDelta_{j-1}), \ j \in Z.^2$$

Still supposing that $y \in x_{m,n} + \varDelta_n$, we use (B) and (C) to write

$$\begin{aligned} |w(x-y) - w(x - x_{m,n})| &\leqq \gamma((x-y)^* - (x - x_{m,n})^*) \\ &= \gamma(u_{-s(x-x_{m,n})}(x_{m,n} - y)) \\ &\leqq \gamma((x - x_{m,n})^{-1} u_{n-1}) . \end{aligned}$$

Using this estimate in (6) and the resulting inequality in (5), we obtain

$$(7) \qquad \begin{aligned} &\left| \int_{x_{m,n}+\varDelta_n} g(y) \psi_k(x-y) d\lambda(y) \right| \\ &\leqq \frac{\gamma((x - x_{m,n})^{-1} u_{n-1})}{m(x - x_{m,n})} \int_{x_{m,n}+\varDelta_n} |g(y)| d\lambda(y) \end{aligned}$$

for all $x \in D'_t$ and (m, n) such that $(x + \varDelta_k) \cap (x_{m,n} + \varDelta_n) = \varnothing$. If $x \in D'_t$ and $(x + \varDelta_k) \cap (x_{m,n} + \varDelta_n) \neq \varnothing$, then n is larger than k and $x - x_{m,n}$ is in $\varDelta_k$. This implies that $x - y \in \varDelta_k$ and $\psi_k(x-y) = 0$ for $y \in x_{m,n} + \varDelta_n$; therefore, (7) is trivial in this case. Using (7) in (4) and integrating over D'_t gives

$$\begin{aligned} \int_{D'_t} |L_k g(x)| d\lambda(x) &\leqq \sum_{P_t} \int_{D'_t} \frac{\gamma((x - x_{m,n})^{-1} u_{n-1})}{m(x - x_{m,n})} \left[\int_{x_{m,n}+\varDelta_n} |g(y)| dy \right] dx \\ &\leqq \sum_{P_t} \int_{(x_{m,n}+\varDelta_n)'} = \sum_{P_t} \left[\int_{\varDelta'_n} \frac{\gamma(x^{-1} u_{n-1})}{m(x)} dx \right] \left[\int_{x_{m,n}+\varDelta_n} |g(y)| dy \right] \\ &= \left[\int_{\varDelta_0} \gamma(x) [m(x)]^{-1} dx \right] \left[\int_{D_t} |g(y)| dy \right] = a \int_{D_t} |g| d\lambda , \end{aligned}$$

2) We single out these properties of the trivial function γ because they are all that is needed in the subsequent analysis. The hypothesis $w(x) = w(x_0, \cdots, x_q)$ is not used in any portion of the proof except to guarantee the results of (3.10) for the $\mathfrak{L}_2$ function h and to establish the conditions (A), (B) and (C) for the γ defined here. Hence the results of this theorem can be proved by starting with any bounded w for which the essential condition (3.7) is satisfied, for which the results of (3.10) can be proved, and for which there is a function γ satisfying (A), (B), and (C).

where a is a constant depending only on w. Since

$$\int_{D_t} h d\lambda = \int_{D_t} f d\lambda$$

and $|g| \leqq f + h$, the inequalities

$$\int_{D_t} |g| \, d\lambda \leqq 2\int_{D_t} f d\lambda \leqq 2tp\lambda(D_t)$$

hold; hence, we have

$$\int_{D'_t \cap \Phi_{t,2}} \frac{t}{2} d\lambda(x) \leqq \int_{D'_t \cap \Phi_{t,2}} |L_k g(x)| \, d\lambda(x) \leqq 2apt\lambda(D_t) .$$

The inequality $\lambda(D'_t \cap \Phi_{t,2}) \leqq 4ap\lambda(D_t)$ follows; and this combined with (3) yields the estimate $\lambda(\Phi_{t,2}) \leqq a_1\lambda(D_t)$, a_1 independent of t, k, and f. This final estimate and (2) yield

$$\lambda(\Phi_t) \leqq \lambda(\Phi_{t,1}) + \lambda(\Phi_{t,2}) \leqq \frac{c_1}{t^2}\int_{\Psi_p} ([f]_t)^2 d\lambda + c_2\lambda(D_t) ;$$

(1) follows from (3. 12. ii).

Step II. Using the measure estimate (1) and the equality

$$\int_{\Psi_p} |L_k f|^r \, d\lambda = r\int_0^\infty \lambda(\Phi_t) t^{r-1} dt \qquad (r > 1) ,$$

the proof of (i) for $1 < r < 2$ is essentially as in [3], pp. 97–99. The case $r > 2$ is obtained by a duality argument from the result for $r < 2$; this, too, is in [3]. We omit these details.

Step III. It remains to show that the sequence $(L_k f)_{k=1}^\infty$ converges in the $\mathfrak{L}_r$-norm, for every $f \in \mathfrak{L}_r$. We begin by showing that the family $\mathfrak{T} = \{\tau\colon \tau(x) = \sum_{j=1}^J a_j \xi_{\Theta_j}(x)\}$, where the a_j's are complex numbers and the Θ_j's are compact and open is dense in $\mathfrak{L}_r$ and that each $L\tau$ converges. The family $\mathfrak{S}$ obtained by demanding that the Θ_j's be measurable of finite measure is dense in $\mathfrak{L}_r$, so it suffices to show that $\mathfrak{T}$ is dense in $\mathfrak{S}$. If Φ is λ-measurable of finite measure and $\delta > 0$, then there is a compact open set Θ satisfying

$$(8) \qquad \lambda(\Theta' \cap \Phi) < \delta \quad \text{and} \quad \lambda(\Theta \cap \Phi') < \delta .$$

To prove (8), let Υ and $\mathscr{T}$ be compact and open sets, respectively, such that $\Upsilon \subset \Phi \subset \mathscr{T}$, $\lambda(\Phi' \cap \mathscr{T}) < \delta$, and $\lambda(\Upsilon' \cap \Phi) < \delta$. For each $x \in \Upsilon$, there is an n_x such that $x + \Lambda_{n_x} \subset \mathscr{T}$. A finite union, say Θ, of the sets $x + \Lambda_{n_x}$ covers $\Upsilon\colon \Upsilon \subset \Theta \subset \mathscr{T}$. The set Θ is clearly compact and open. We have $\lambda(\Theta' \cap \Phi) \leqq \lambda(\Upsilon' \cap \Phi) < \delta$ and $\lambda(\Theta \cap \Phi') \leqq$

$\lambda(\mathscr{T} \cap \Phi') < \delta$; thus (8) holds. For a given λ-measurable set Φ of finite measure and a positive ε, select a compact open set such that (8) holds with $\delta = (1/2)\varepsilon^r$. Then we have $\| \xi_\Phi - \xi_\Theta \|_r < \varepsilon$. For a function $\zeta = \sum_{j=1}^J a_j \xi_{\Phi_j}$ in $\mathfrak{S}$, select compact open sets Θ_j $(j = 1, 2, \cdots, J)$ such that $\| \xi_{\Phi_j} - \xi_{\Theta_j} \|_r < \varepsilon/J | a_j |$. Letting $\tau = \sum_{j=1}^J a_j \xi_{\Theta_j}$, we have $\| \zeta - \tau \|_r < \varepsilon$. Hence, $\mathfrak{T}$ is dense in $\mathfrak{L}_r$.

If Θ is compact and open, dominated convergence shows that

$$\lim_{j \to -\infty} \int_{\Lambda_j \cap \Theta} \psi_k(x - y) d\lambda(y) = \int_\Theta \psi_k(x - y) d\lambda(y)$$

for every k. Using the equality

$$\lim_{j \to -\infty} \int_{\Lambda_j} \psi_k(x - y) d\lambda(y) = 0 ,$$

and translating, we can thus write

$$L_k \xi_\Theta(x) = \lim_{j \to -\infty} \int_{\Lambda_j} \psi(-y) \xi_{\Lambda_k'}(-y)[\xi_\Theta(x + y) - \xi_\Theta(x)] dy . \tag{9}$$

There is an integer n_0 and finitely many disjoint cosets $\{x_i + \Lambda_{n_0}\}_{i=1}^M$ with union Θ. If $y \in \Lambda_{n_0}$, then $y + x \in \Theta$ if and only if $x \in \Theta$. Thus, (9) shows that $L_k \xi_\Theta(x) = L_{n_0} \xi_\Theta(x)$, for $k \geqq n_0$. It follows easily that for every $\tau \in \mathfrak{T}$ there is an integer $n_0(\tau)$ such that $L_k \tau = L_{n_0} \tau$, whenever $k \geqq n_0$. Thus $L_k \tau$ converges to a function $L\tau$ both pointwise and in the $\mathfrak{L}_r$ norm. Finally, let $f \in \mathfrak{L}_r$ and suppose $\varepsilon > 0$. Select $\tau \in \mathfrak{T}$ such that $\| f - \tau \|_r < (2A_r)^{-1}\varepsilon$. We have

$$\| L_k f - L_n f \|_r < \| L_k \tau - L_n \tau \|_r + \varepsilon ,$$

and so $\| L_k f - L_n f \|_r < \varepsilon$ for $k, n \geqq n_0(\tau)$. Let $Lf = \lim_{k \to \infty} {}_r\, L_k f$. The inequality (ii) is immediate.

(3.14) THEOREM. *The function* $\varphi = \lim_{k \to \infty} \hat{\psi}_k$ *is an* $\mathfrak{L}_r$*-multiplier for* L $(1 < r \leqq 2)$.

Proof. We first show that $\mathfrak{L}_2$ and $\mathfrak{L}_r$ $(1 < r < 2)$ Fourier transforms agree on $\mathfrak{L}_2 \cap \mathfrak{L}_r$. Thus, suppose that $h \in \mathfrak{L}_r \cap \mathfrak{L}_2$ and let $\hat{h}^r$ and $\hat{h}$ denote its $\mathfrak{L}_r$ and $\mathfrak{L}_2$ transforms, respectively. The functions $h_n = h\xi_{\Lambda_n}$ $(n = -1, -2, \cdots)$ are in $\mathfrak{L}_1$ and $\lim_{n \to -\infty} \hat{h}_n = \hat{h}$ a.e. (2.3). Thus, we have

$$\int_{\Psi_p} | \hat{h}^r - \hat{h} |^r d\lambda = \int_{\Psi_p} \lim_{n \to -\infty} | \hat{h}^r - \hat{h}_n |^r d\lambda \leqq \lim_{n \to -\infty} \int_{\Psi_p} | \hat{h}^r - \hat{h}_n |^r d\lambda = 0 ;$$

and so, $\hat{h}^r = \hat{h}$ a.e. We can now drop the r on $\wedge_r$ without fear of ambiguity.

Let $f \in \mathfrak{L}_r$. Since the functions $\hat{\psi}_{k,n}$ (see the proof of (3.10)) converge boundedly to $\hat{\psi}_k$, we obtain

$$(1) \qquad \lim_{n\to-\infty} \| \hat{\psi}_{k,n}\hat{f} - \hat{\psi}_k\hat{f} \|_{r'} = 0 ,$$

as in the $\mathfrak{L}_2$ case. We now know (3.13) that $\psi_k * f$ is in $\mathfrak{L}_r$, so the Hausdorff-Young inequality implies

$$(2) \qquad \lim_{n\to-\infty} \| (\psi_{k,n} * f)^\wedge - (\psi_k * f)^\wedge \|_{r'} = 0 .$$

Since $(\psi_{k,n} * f)^\wedge = \hat{\psi}_{k,n}\hat{f}$ $(\psi_{k,n} \in \mathfrak{L}_1)$, (1) and (2) give the equality $(\psi_k * f)^\wedge = \hat{\psi}_k\hat{f}$. The functions $(\psi_k * f)^\wedge$ converge in $\mathfrak{L}_{r'}$ to (Lf) by the Hausdorff-Young inequality; and, the functions $\hat{\psi}_k\hat{f}$ converge in the $\mathfrak{L}_{r'}$ norm to $\varphi\hat{f}$ because $\hat{\psi}_k$ converges boundedly to φ.

4. Examples.

We give some examples of w's defining L's.

(4.1) *Φ-Kernels.* Suppose $\Phi \subset \Psi_p$ satisfies $\Phi \cup -\Phi = \Psi_p\backslash\{0\}$ and $\Phi \cap -\Phi = \emptyset$. Let w be a bounded λ-measurable function on Δ_0 such that $w(-x) = -w(x)$ for all $x \in \Delta_0$; the condition (3.7) is immediate for such a w. By additive inversion invariance, we can write

$$(\mathrm{i}) \qquad L_k f(y) = -\int_{\Phi_k} \psi(x)[f(y+x) - f(y-x)]d\lambda(x) ,$$

where $\Phi_k = \Phi \cap \Lambda'_k$. If there is a $q \in Z^+$ such that every $w(x)$ depends on only the first $q+1$ coordinates of x, then the hypothesis of Theorem (3.13) is satisfied for w. We call the corresponding kernel ψ a *Φ-kernel.* If L_k is generated by a Φ-kernel, the functions $L_k f$ as given in (i) converge in the $\mathfrak{L}_r$ norm to a function Lf $(f \in \mathfrak{L}_r$ $(r > 1))$. In particular, we can let $w(x) = \mathrm{sgn}_\Phi(x)$ $(=1$ if $x \in \Phi$ and -1 if $x \in -\Phi)$. If there is a q such that a knowledge of $x_0, \cdots, x_q$ determines whether x is in Φ or $-\Phi$, then $(\mathrm{sgn}_\Phi)m^{-1}$ is a Φ kernel and

$$(\mathrm{ii}) \qquad Lf(y) = \lim_{k\to\infty}{}_r - \int_{\Phi_k} \frac{f(y+x) - f(y-x)}{m(x)} d\lambda(x).$$

The limit in (ii) giving the transform L is a precise analogue of the limit defining the classical Hilbert transform for R; and, Theorem (3.13) is an analogue of corresponding results for R. The set Φ corresponds to the positive real numbers and the sets Φ_k to the sets $](1/k), \infty[$ $(k \in Z^+)$.

For a Φ-kernel ψ, the multiplier φ can be written as

$$(\mathrm{iii}) \qquad \varphi(y) = \lim_{k\to\infty}\lim_{n\to-\infty}(-2i)\int_{\Phi_k \cap \Lambda_n} \psi(x)\sin(-\sigma(xy))dx.$$

If $w = \mathrm{sgn}_\Phi$, then

$$\text{(iv)} \qquad \varphi(y) = \lim_{k\to\infty} \lim_{n\to-\infty} (-2i)\int_{\Phi_k \cap \Delta_n} \frac{\sin(-\sigma(xy))}{m(x)}\, dx .$$

The limit in (iv) is like the limit

$$\lim_{k\to\infty} \lim_{m\to\infty} (-2i)\pi^{-1}\int_{1/k}^{m} \frac{\sin(xy)}{x}\, dx = -i\operatorname{sgn} y ,$$

which is a multiplier for the classical Hilbert transform. See, *e.g.*, [9], pp. 119–120.

(4.2) *A calculation.* There are many sets Φ satisfying the conditions in (4.1). We now consider one of these in more detail. Suppose that p is odd and define

$$\Phi = \left\{x\colon 1 \leqq x_{s(x)} \leqq \frac{p-1}{2}\right\} ;$$

then

$$-\Phi = \left\{x\colon \frac{p+1}{2} \leqq x_{s(x)} \leqq p-1\right\} .$$

(Another natural Φ is $\{x\colon x_{s(x)}$ is odd$\}$.) Letting $w = \mathrm{sgn}_\Phi$, we know that the limit (4.1.ii) exists for $f \in \mathfrak{L}_r$. We compute Lf for $f = \xi_{\Delta_0'} m^{-1}$; the result is given in (4), *infra*. Let

$$\mathscr{T}_j = \Phi \cap \Delta_j = \left\{x \in \Delta_j\colon 1 \leqq x_j \leqq \frac{p-1}{2}\right\} .$$

Using (4.1.i) then translating, we can write

$$\text{(1)} \qquad \begin{aligned} L_k f(y) &= -\sum_{j=-\infty}^{k-1} p^j \int_{\mathscr{T}_j} \left[\frac{\xi_{\Delta_0'}(y+x)}{m(y+x)} - \frac{\xi_{\Delta_0'}(y-x)}{m(y-x)}\right] dx \\ &= -\sum_{j=-\infty}^{k-1} p^j \left[2\int_{\mathscr{T}_j} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)} - \int_{\Delta_j} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}\, dx\right] . \end{aligned}$$

If $s(y) > j$, then we have $s(x+y) = s(x-y) = s(x) = j$ for all $x \in \Delta_j$. In particular, $x+y$ is in Δ_0' if and only if $x - y \in \Delta_0'$; and hence the first expression for $L_k f(y)$ in (1) shows that the j^{th} term of the sum is zero. If $s(y) < j$, we have $s(y+x) = s(y-x) = s(y)$ for all $x \in \Delta_j$, and again the j^{th} term of the sum is zero. Thus, if $k > s(y)$, we have

$$\text{(2)} \qquad \begin{aligned} Lf(y) &= L_k f(y) \\ &= -p^{s(y)}\left[2\int_{\mathscr{T}_{s(y)}} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}\, dx - \int_{\Delta_{s(y)}} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}\, dx\right] . \end{aligned}$$

If $s(y) \geqq 0$, then the equality $\xi_{\Delta_0'}(y+x) = \xi_{\Delta_0'}(y-x) = 0$ holds for all $x \in \Delta_{s(y)}$, and hence $Lf(y) = 0$ for $y \in \Delta_0$. For the following calculations, we suppose that y is fixed, put $s(y) = S$, and define

$$Z_{S+j} = \{x \in \Delta_S : s(x+y) = S+j\} ;$$
$$E_{S+j} = \{x \in \Delta_S : s(x+y) > S+j\} \qquad (j = 0, 1, 2, \cdots) .$$

We have

$$\begin{aligned}\int_{\Delta_S} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}dx &= \int_{Z_S} + \int_{E_S} = p^S\lambda(Z_S) + \int_{E_S} \\ &= p^S\lambda(\Delta_S)\cdot\frac{p-2}{p-1} + \int_{Z_{S+1}} + \int_{E_{S+1}} \\ &= p^{1/2}(p-2) + p^{-1/2}(p-1) + \left(\int_{Z_{S+2}} + \cdots + \int_{Z_{-1}} + \int_{E_{-1}}\right).\end{aligned}$$

If $s(x+y) > -1 (x + y \in \Delta_0)$, then $\int_{E_{-1}} = 0$; the other integrals in the last line above are $p^{-1/2}(p-1)$. We thus obtain

$$(3) \qquad \int_{\Delta_S} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}dx = p^{-1/2}[(p-2) + (-S-1)(p-1)] .$$

It remains to calculate the first integral on the right side of (2). If $y \in \mathscr{T}_S$, then $y + x$ is Δ_S for all $x \in \mathscr{T}_S$. In this case, we have

$$\int_{\mathscr{T}_S} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}dx = p^S\lambda(\mathscr{T}_S) = \frac{1}{2}p^{-1/2}(p-1) .$$

If $y \in -\mathscr{T}_S$, then there exists an $x \in \mathscr{T}_S$ such that $s(x+y) > S$. Computing as above, we have

$$\begin{aligned}\int_{\mathscr{T}_S} \frac{\xi_{\Delta_0'}(y+x)}{m(y+x)}dx &= \int_{Z_S} + \int_{E_S} = p^S\lambda(\mathscr{T}_S)\left[\frac{\frac{p-1}{2}-1}{\frac{p-1}{2}}\right] + \sum_{j=1}^{-S-1}\int_{Z_{S+j}} \\ &= p^{-1/2}\left(\frac{1}{2}(p-3) + (-S-1)(p-1)\right) .\end{aligned}$$

Using the two above equalities and (3) in (2) gives

$$Lf(y) = \begin{cases} -p^{S-(1/2)}((p-1)-(p-2)+(S+1)(p-1))\xi_{\Delta_0'}(y) & \text{if } y \in \Phi \\ -p^{S-(1/2)}((p-3)-(p-2)+(-S-1)(p-1))\xi_{\Delta_0'}(y) & \text{if } y \in -\Phi . \end{cases}$$

This in turn can be written

$$(4) \qquad Lf(y) = f(y)p^{-1/2}(-1 - s(y)(p-1))\,\mathrm{sgn}_\Phi(y) .$$

(4.3) *Kernels from additive characters.* For a character χ_y of Ψ_p, we have

$$(\mathrm{i}) \quad \int_{\Delta_0}\chi_y(x) = \int_{\Lambda_0}\chi_y(x)dx - \int_{\Lambda_1}\chi_y(x)dx.$$

If $y \in \Delta_0$, then $\chi_y|_{\Delta_0}$ is a nontrivial character of the group Δ_0 and $\chi_y|_{\Delta_1} \equiv 1$. Hence $\int_{\Delta_0} \chi_y(x)dx = 0$, $\int_{\Delta_1} x_y(x)dx = \lambda(\Delta_1)$, and $\int_{\Delta_0'} \chi_y(x)dx = -\lambda(\Delta_1)$. Thus $\int_{\Delta_0} \mathrm{Img}\, \chi_y(x)dx = 0$; and setting

$$w_y(x) = \mathrm{Img}\, \chi_y(x)\,, \qquad y \in \Delta_0\,, \quad \chi \in \Delta_0\,,$$

we obtain a w satisfying the hypothesis of (3.13). Note that $w_y(x) = \sin(-\sigma(xy))$.

If $y \in \Delta_0'$, then $\chi_y|_{\Delta_0}$ and $\chi_y|_{\Delta_1}$ are nontrivial character of Δ_0 and Δ_1, respectively. It follows from (i) that the function $\chi_y|_{\Delta_0}$ defines a generating function w. In this case, each of the functions $x \longrightarrow \mathrm{Re}\, \chi_y(x)$ and $x \longrightarrow \mathrm{Img}\, \chi_y(x)$ $(x \in \Delta_0)$ is also a generating function.

(4.4) *Kernel's from characters of Δ_0.* If τ is a nontrivial character of the multiplicative group Δ_0, then $\int_{\Delta_0} \tau(x)d\lambda(x) = 0$. Continuity requires that $\tau(u + \Delta_q) = 1$ for some $q > 0$. We can write any $x \in \Delta_0$ in the form $x = x'(u + x'')$, where $x' \in \Delta_0$, $x_k' = 0$ if $k \geqq q$, and $x'' \in \Delta_q$. Hence, $\tau(x)$ is determined by the 1st q coordinates of x, and so τ generates a singular integral for which the results of (3.13) are valid.

Both the author and Professor Mitchell Taibleson have extended the results of this paper in several directions. These extensions will be published in due time.

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