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GENERALIZED SEMIGROUP KERNELS

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This paper is concerned with the problem of generalizing thc notion of a kernel of a semigroup. Various kernels are introduced and their mutual relationships are investigated. Conditions are found on a semigroup which are necessary and sufficient in order that certain of its kernels be trivial.

The "generalized" kernels we introduce here have properties which are reminiscent of the notion of a radical. Our results, however, are quite different from certain of the investigations along these lines (see, for example, [3] and [13]). Our work is more closely related to that of Schwartz [10], [11], and [12]. We refer to [2] for definitions not explicitly given.

1. Mutually annihilating sums and kernels. The following definition seems to be due to Ljapin [6]. If S is a semigroup, then S is said to be a mutually annihilating sum of semigroups $\{S_\lambda\}_{\lambda \in \Lambda}$ if and only if S is (isomorphic to) a semigroup with zero such that if 0 is the zero of S, then

(i) for λ in Λ, S_λ is a subsemigroup of S with contains 0,

(ii) each member of S is in S_λ for some λ in Λ, and

(iii) for λ and γ in Λ, $\lambda \neq \gamma$, $S_\lambda \cap S_\lambda = \{0\} = S_\lambda S_\gamma$.

We shall be concerned with semigroups S which are mutually annihilating sums of semigroups each of which has some one fixed semigroup property P (to say that P is a semigroup property means that P is a property such that if one of two isomorphic semigroups has property P, then so does the other). There is a rather obvious connection between mutually annihilating sums and subdirect sums which we make explicit in the lemma below.

We use the concept of a subdirect sum as in the theory of rings, *i.e.*, to say that S is a subdirect sum of semigroups $\{T_\mu\}_{\mu \in \Omega}$ means that S is (isomorphic to) a subsemigroup of the direct product $\Pi_{\mu \in \Omega} T_\mu$ such that if for some $\nu \in \Omega, \pi_\nu$ is the projection of $\Pi_{\mu \in \Omega} T_\mu$ onto T_ν, then the homomorphism $\pi_\nu | S$ is onto T_ν. The following lemma is not difficult to prove.

LEMMA 1.1 *If S is a semigroup with zero, then S is a mutually annihilating sum of semigroups each having property P if and only if there is a collection $\{T_\mu\}_{\mu \in \Omega}$ of semigroups such that*

(1) *for each $\mu \in \Omega$, T_μ is a semigroup with zero which has property P, and*

(2) *S is a subdirect sum of the collection $\{T_\mu\}_{\mu \in \Omega}$ such that each member of S, when viewed as a member of $\Pi_{\mu \in \Omega} T_\mu$, has at most one nonzero component.*

Let K_P denote the set $\{I \mid I$ is an ideal and S/I has property $P\}$.

THEOREM 1.2. *Suppose P is a semigroup property, S is a semigroup, and J is an ideal of S. Then S/J is a mutually annihilating sum of semigroups each having property P if and only if there is a subset K of K_P such that* (i) *J is the K-kernel of S, and* (ii) *if I and I' are distinct members of K, then $S = I \cup I'$.*

Proof. Assume K is a subset of K_P such that (i) and (ii) of the theorem are true. It is clear from (i) that S/J is, in a natural way, a subdirect sum of the collection $\{S/I \mid I \in K\}$. Property (ii) implies that each member of the subdirect sum has no more than one nonzero component. It then follows from (i) and the lemma that S/J is a mutually annihilating sum of semigroups each having property P.

Now assume J is an ideal of S and S/J is a mutually annihilating sum of a collection $\{S_\lambda\}_{\lambda \in \Lambda}$, where, for each λ in Λ, S_λ is a semigroup having property P. Let φ denote the natural homomorphism from S onto S/J. For each $\lambda \in \Lambda$, let I_λ denote the set of all x in S such that either $\varphi(x)$ is zero or $\varphi(x)$ is not in S. If $K = \{I_\lambda \mid \lambda \in \Lambda\}$, then K satisfies (i) and (ii) of the theorem.

REMARK 1.3. In case S has a zero and J is zero, the theorem asserts that S is a mutually annihilating sum of semigroups each of which has property P if and only if there is a subset K of K_P such that (i) $\cap K = 0$ and (ii) if I and I' are in K, $I \neq I'$, then $S = I \cup I'$.

For each semigroup S, let $\mathscr{M} = \mathscr{M}_S$ denote the set of all maximal ideals of S. The following corollaries are immediate applications of Remark 1.3.

COROLLARY 1.4. *Assume S is a semigroup with zero. Then $\cap \mathscr{M} = 0$ if and only if S is a mutually annihilating sum of semigroups each of which either is a null semigroup of order two or is a 0-simple semigroup.*

COROLLARY 1.5. *If S is a semigroup and J is the $\mathscr{M}$-kernel of S, then the $\mathscr{M}$-kernel of S/J is zero.*

The $\mathscr{M}$-kernel of a semigroup determines, to some extent, which maximal ideals are prime (an ideal J of a semigroup S is said to be prime if and only if either $J = S$ or the complement of J is a subsemigroup of S).

THEOREM 1.6. *Suppose S is a semigroup which has a maximal ideal. If J denotes the $\mathscr{M}$-kernel of S, then each maximal ideal of S is prime if and only if there is a collection $\{S_\alpha\}_{\alpha\in\Lambda}$ of simple subsemigroups of S such that*

(1) $S = J \cup \bigcup_{\alpha\in\Lambda} S_\alpha$,

(2) *for* $\alpha\in\Lambda$, $\beta\in\Lambda$, $\alpha\neq\beta$, $S_\alpha\cap S_\beta$ *is void and* $S_\alpha S_\beta\subseteqq J$, and

(3) *for each* $\alpha\in\Lambda$, $J\cap S_\alpha$ is void.

Proof. First assume that each maximal ideal of S is prime and that J is the $\mathscr{M}$-kernel of S. From previous arguments, it is known that S/J is isomorphic to a mutually annihilating sum of semigroups each of which is isomorphic to S/M for some $M\in\mathscr{M}$. Since S/M is a simple semigroup with zero for each $M\in\mathscr{M}$ (recall that M is prime), it follows that there is a collection $\{S_\alpha\}_{\alpha\in\Lambda}$ of simple semigroups such that S/J is a mutually annihilating sum of $\{S_\alpha^0 \mid \alpha\in\Lambda\}$. For each $\alpha\in\Lambda$, we identify S_α with the subsemigroup T_α of S such that

$$(T_\alpha \cup J)/J = S_\alpha$$

and $T_\alpha\cap J$ is void. Then the collection $\{S_\alpha\}_{\alpha\in\Lambda}$ satisfies (1), (2), and (3) of the theorem.

Assume, on the other hand, that $\{S_\alpha\}_{\alpha\in\Lambda}$ is a collection of simple subsemigroups of S such that (1), (2), and (3) hold where J denotes the $\mathscr{M}$-kernel of S. Then each maximal ideal of S is of the form

$$J \cup \bigcup_{\alpha\in\Lambda\setminus\{\beta\}} S_\alpha$$

for some $\beta\in\Lambda$. Thus each maximal ideal of S is prime.

2. The $\mathscr{P}$-kernel of a semigroup. We now turn our attention to a different kind of kernel of a semigroup. Let $\mathscr{P}$ denote the set of all prime ideals of S. We now characterize the $\mathscr{P}$-kernel of S. First we need some notation and definitions.

To say that S is a band means that S is an idempotent semigroup. S is said to be a rectangular band if and only if S is a band and $a\ b\ a = a$ for all a and b in S. Rectangular bands may be characterized as semigroups of the form $X\times Y$ where X and Y are arbitrary sets and where the operation on $X\times Y$ is defined by

$$(x, y)(x', y') = (x, y')$$

for x, x' in X and y, y' in Y (see, for example, [4] or [7]).

We assume, from this point on, that S is any semigroup, that E is the maximal semilattice homomorphic image of S, and that η is the natual homomorphism from S onto E. Define a relation φ on S by $(a, b) \in \varphi$ if and only if there exists $x \in S^1, y \in S^1, c \in S, d \in S$, and positive integers m and n such that $a = x c y, b = x d y$, and $c^m = d^n$. Clifford has observed, [1], that if φ^t is the transitive closure of φ, then S/φ^t is the maximal band homomorphic image of S. He also noted that the maximal semilattice homomorphic image of S/φ^t is the maximal semilattice homomorphic image of S. Each φ^t-congruence class of S will be called an archimedean component of S. This definition, which agrees with the usual one in case S is commutative, has not been used before in case S is not commutative. Clifford's observation may be rephrased, "any semigroup is a semilattice union of semigroups each of which is a rectangular band of archimedean components of S".

The following theorem is due to Petrich (see [8] and [9]).

THEOREM 2.1. *(Petrich) In order that P be a prime ideal of the semigroup S it is necessary and sufficient that there exists a prime ideal Q of E such that* $P = \bigcup_{e \in Q} \eta^{-1}(e)$.

The following corollary is immediate.

COROLLARY 2.2. *The $\mathscr{P}$-kernel of the semigroup S is precisely the inverse image of the $\mathscr{P}$-kernel of E under η (even in case either is void).*

LEMMA 2.3. *If E is a semilattice, then the $\mathscr{P}$-kernel of E is void in case E contains no zero element and otherwise is the zero of E.*

Proof. Suppose z is in the $\mathscr{P}$-kernel of E. If z were not a zero of E, then $\{x \in E \mid x \ngeq z\}$ would be a prime ideal of E which does not contain z.

The next theorem follows immediately from previous results.

THEOREM 2.4. *The $\mathscr{P}$-kernel of the semigroup S is void in case E does not contain a zero and otherwise is the inverse image of the zero of E under η.*

COROLLARY 2.5. *If the semigroup S contains a zero, then the $\mathscr{P}$-kernel of S is zero if and only if the equations $a c b = 0$ and $c^n = d^m$ imply $a d b = 0$ for $a \in S^1, b \in S^1, c \in S, d \in S$, and positive integers m and n. Note that in case S is commutative the latter* *condition merely asserts that* 0 *is the only nilpotent member of* S.

PROOF. By Theorem 2.4 the $\mathscr{P}$-kernel of S is $\eta^{-1}(z)$ where z is the zero of E. Since $\eta^{-1}(z)$ contains the zero of S, it must contain only one archimedean component of S. Thus the $\mathscr{P}$-kernel of any semigroup with zero is precisely the archimedean component containing the zero. The corollary now follows from the way φ was defined.

The following corollaries are evident.

COROLLARY 2.6. *The following statements are equivalent*:
(1) *the maximal semilattice homomorphic image of* S *is trivial*,
(2) *the* $\mathscr{P}$*-kernel of* S *is* S, *and*
(3) S *is a rectangular band of its archimedean components.*

COROLLARY 2.7. *The maximal band image of a semigroup is a rectangular band if and only if the maximal semilattice image is trivial.*

Finally we consider an application to semilattice theory. To say that F is a face of a semilattice E means that F is a (nonvoid) subsemigroup of E such that either F is E or the complement of F in E is a prime ideal of E. A prime ideal P of E is principal if and only if it is of the form $\{x \in E \mid e \not\leqq x\}$ for some $e \in E$ (in this case e is called the generator of P).

THEOREM 2.7. *If* E *is a semilattice, then each proper face of* E *is finite if and only if*
(1) *each proper prime ideal of* E *is principal*,
(2) *each ascending chain in* E *is finite, and*
(3) *each nonzero element of* E *is covered by at most a finite number of elements of* E.

Proof. First assume each proper face of E is finite. If P is a proper prime ideal of E, then P is principal and has as generator the product of all elements of E not in P. It is equally clear that (2) and (3) follow.

Now assume (1), (2), and (3) are true. Let F denote any proper face of E. Then $E \backslash F$ is a proper prime ideal and thus is principal. Let e denote the generator of $E \backslash F$. Then $x \in F$ if and only if $x \geqq e$. Define a sequence A of subsets of E inductively by

(i) $x \in A_1$ if and only if $x \in E$ and x covers e, and (ii) if k is a positive integer, $x \in A_{k+1}$ if and only if $x \in E$ and x covers some member of A_k.

For each positive integer i, A_i is finite. One can show that there is a positive integer n such that A_n is void by assuming otherwise and by constructing an infinite ascending chain in E. Thus

$$F = \bigcup_{i=1}^{n} A_i$$

and F is finite. The theorem follows.

If $\mathscr{J}$ is any collection of ideals of a semigroup S, $S/(\cap \mathscr{J})$ is always a subdirect sum of the collection $\{S/T \mid T \in \mathscr{J}\}$. In case $S = E$ is a semilattice the intersection of the collection of all prime ideals of E is void or is a zero of E. Thus one obtains the following corollary of Theorem 2.7.

COROLLARY 2.8. *Assume E is a semilattice such that*

(1) *each proper prime ideal of E is principal,*

(2) *each ascending chain in E is finite, and*

(3) *each nonzero element of E is covered by at most a finite number of elements of E.*

Then E is a subdirect sum of the collection $\{F^\circ \mid F$ is a finite face of $E\}$.

3. Relationships among various kernels. As in the previous section S denotes any semigroup, E its maximal semilattice homomorphic image, and η the natural homomorphism from S onto E. Throughout this section K_T will denote the intersection of all ideals of the semigroup T and will be called the kernel of T. If N denotes the void set, we define $K_N = N$. Likewise P_T and M_T will denote the $\mathscr{P}$ and $\mathscr{M}$ kernels of T respectively.

THEOREM 3.1. *If A is an ideal of the semigroup S, then $K_A = K_S$. Thus we have*

$$K_S = K_{P_S} = K_{M_S} .$$

Proof. Let A denote any ideal of S. If K_S is not void, then for each ideal J of A

$$K_S = K_S J K_S K_S \subseteqq K_S J K_S .$$

Thus $K_S J K_S$ is an ideal of K_S. Since K_S is simple,

$$K_S = K_S J K_S \subseteqq AJA \subseteqq J .$$

Thus $K_S \subseteqq K_A$.

Conversely, if K_A is not void, then K_S is equal to the intersection of the collection $\mathscr{H}$ where $J \in \mathscr{H}$ if and only if $J = I \cap A$ for some ideal J of S. But each such J is an ideal of A, thus $K_S \subseteqq K_A$.

It follows that $K_S = K_A$ for each ideal A of S. Clearly if P_S or M_S is void so is K_S. The theorem follows.

COROLLARY 3.2. *If S is a semigroup, then the kernel of S is the same as the kernel of P_S and thus is the kernel of a rectangular band of archimedean components of S.*

In order to obtain the relationship between the $\mathscr{M}$-kernel, M_S, and the $\mathscr{P}$-kernel, P_S, we need more information about the maximal ideals of S. The next theorem provides such information and has some interest in its own right. First we need another definition. An ideal I of $\eta^{-1}(e)$, for $e \in E$, is said to be induced by S if and only if $I \cup (S \backslash \eta^{-1}(e))$ is an ideal of S. It is easy to see that an ideal I of $\eta^{-1}(e)$ is induced by S if and only if

(1) $f_1 \in E \backslash \{e\}, f_2 \in E \backslash \{e\}$, and $f_1 f_2 = e$ imply $\eta^{-1}(f_1)\eta^{-1}(f_2) \subseteqq I$ and

(2) $f \in E$ and $f > e$ imply $\eta^{-1}(f)I \subseteqq I$ and $I\, \eta^{-1}(f) \subseteqq I$.

THEOREM 3.3. *If M is a subset of the semigroup S, then M is a maximal ideal of S if and only if there exists $e \in E$ such that either*

(1) *e is a maximal element of E such that $\eta^{-1}(e)$ is simple and* $M = \bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)$, or

(1) *there is a maximal ideal M_e* of $\eta^{-1}(e)$ *such that M_e is induced by S and $M = M_e \cup \bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)$.*

Proof. Suppose M is a maximal ideal of S and that $a \in S \backslash M$. Let e denote $\eta(a)$. First we show that $\eta^{-1}(f) \subseteqq M$ for all $f \in E \backslash \{e\}$. Consider $F = \{x \in E \mid x \ngeqq e\}$. The set $M \cup \eta^{-1}(F)$ is a proper ideal of S. Since M is maximal, $\eta^{-1}(F) \subseteqq M$. Thus if $f \in E$ such that $\eta^{-1}(f) \nsubseteqq M$, then $e \leqq f$. Suppose there exists $f_0 \in E$ such that

$$\eta^{-1}(f_0) \nsubseteqq M$$

such that $f_0 > e$. If $F_0 = \{x \in E \mid x \ngeqq f_0\}$, then $M \cup \eta^{-1}(F_0)$ is a proper ideal of S. Thus $\eta^{-1}(F_0) \subseteqq M$. But $f_0 > e$ implies that $e \in F_0$ and that $\eta^{-1}(e) \supseteqq M$, contrary to the choice of $e \in E$. It follows that $\eta^{-1}(f) \subseteqq M$ for each $f \neq e$.

We now show that if $M \cap \eta^{-1}(e)$ is not void, then $M_e = M \cap \eta^{-1}(e)$ is a maximal ideal of $\eta^{-1}(e)$ which is induced by S. Suppose there is an ideal J of $\eta^{-1}(e)$ such that $M_e \subset J \subset \eta^{-1}(e)$. Then $M \cup J$ is an ideal of S such that $M \subset M \cup J \subset S$. Thus no such J exists and M_e is a maximal ideal of $\eta^{-1}(e)$. Clearly M_e is induced by S.

Similar reasoning shows that if $M \cap \eta^{-1}(e)$ is void, then e is maximal in E and that $\eta^{-1}(e)$ is simple.

The proof of the other half of the theorem is easy and is omitted.

LEMMA 3.4. The $\mathscr{M}$-kernel of a semigroup is never void.

Proof. Assume the $\mathscr{M}$-kernel of some semigroup S is void. Then the $\mathscr{M}$-kernel of S° is zero and thus S° is a mutually annihilating sum of semigroups each of which either is a null semigroup of order two or is a simple semigroup with zero. Since 0 is a prime ideal of S°, S° must be a simple semigroup with zero. Thus S is simple and the $\mathscr{M}$-kernel of S is S contrary to the assumption that the $\mathscr{M}$-kernel of S is void.

THEOREM 3.5. *In order that the $\mathscr{M}$-kernel of S be a subset of the $\mathscr{P}$-kernel of S it is necessary and sufficient that E contain a zero z and that for each $e \in E \backslash \{z\}$, e is maximal in E and $\eta^{-1}(e)$ is simple.*

REMARK. We do *not* require in the previous theorem that E contain elements other than z.

Proof. Assume $M_S \subseteqq P_S$. Since M_S is not void, neither is P_S; thus there is a zero z in E and $P_S = \eta^{-1}(z)$. Assume $e \in E \backslash \{z\}$. We show that e is maximal in E and that $\eta^{-1}(e)$ is simple. To do this it suffices, by Theorem 3.3, to show that $\bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)$ is a maximal ideal of S. Assume $\bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)$ is not a maximal ideal of S. Since $M_S \subseteqq P_S$, $x \in \eta^{-1}(e)$ implies that there exists a maximal ideal M_x of S such that $x \notin M_x$. Since $\bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)$ is not a maximal ideal of S, Theorem 3.3 implies that there exists a maximal ideal N_x of $\eta^{-1}(e)$ such that $M_x = N_x \cup \bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)$. Since for each $x \in \eta^{-1}(e)$, $x \notin N_x$, we have that $\bigcap_{x \in \eta^{-1}(e)} N_x$ is void. But $\bigcap_{x \in \eta^{-1}(e)} N_x$ contains the $\mathscr{M}$-kernel of $\eta^{-1}(e)$ which, by the lemma, is *not* void. We have established the necessity of the condition.

Now assume E has a zero and that if $e \in E \backslash \{z\}$, then e is maximal in E and $\eta^{-1}(e)$ is simple. If E contains no element other than z, then $M_S \subseteqq S = \eta^{-1}(z) = P_S$. Assume E contains elements other than z. For each $e \in E \backslash \{z\}$ it is easy to see that $\bigcup_{f \; E \backslash \{e\}} \eta^{-1}(f)$ is a maximal ideal of S. Thus $\bigcap_{e \in E \backslash \{z\}} [\bigcup_{f \in E \backslash \{e\}} \eta^{-1}(f)] = \eta^{-1}(z)$ contains $\bigcap \mathscr{M}$ and $M_S \subseteqq \eta^{-1}(z) = P_S$. The theorem follows.

COROLLARY 3.6. *If E has a zero z and $\eta^{-1}(z)$ is simple, then the following statements are equivalent*:

(1) *$M_S = P_S$, and*

(2) *whenever $e \in E \backslash \{z\}$, e* is maximal in E and $\eta^{-1}(e)$ is simple.

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