PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 1, 1968

THE SILOV BOUNDARY FOR A LATTICE-ORDERED SEMIGROUP

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Let X be a compact Hausdorff space and let S be a point separating collection of $\bar{\mathrm{R}}^+$-valued lower semicontinuous functions on X which is closed under addition. Assume that S is a lower semi-lattice with respect to the partial order $\lesssim$ (where $f \lesssim g$ if $g = f + h$, for some $h \in S$). Further, assume S contains all the nonnegative constant functions λ and is such that $\lambda \leqq f$ implies $\lambda \lesssim f$ (where $\lambda \leqq f$ if $\lambda \leqq f(x)$ for all $x \in X$). Then, the Šilov boundary of S is precisely $\{x \mid (f \wedge g)(x) = \min \{f(x), g(x)\} \; \forall f, g \in S\}$ if, in addition, for all f, g, and $h \in S$ we have $f + (g \wedge h) = (f + g) \wedge (f + h)$.

This theorem[1] extends a result due to H. Bauer [1]. He showed that if $H \subseteqq C(X, \mathrm{R})$ is a linear subspace which contains the constant functions, separates the points of X, and is a lattice with respect to the partial order $\leqq$, then the Šilov boundary of H is precisely $\{x \mid (f \wedge g)(x) = \min \{f(x), g(x)\} \; \forall f, g \in H\}$. This result can be obtained from the above theorem by applying it to the semigroup H^+ of nonnegative functions in H.

The analogous theorem for upper semi-lattices is false. If, however, S is a lattice with respect to $\lesssim$ which satisfies the hypotheses of the theorem, then $\partial S \subseteqq \{x \mid (f \vee g)(x) = \max \{f(x), g(x)\} \; \forall f, g \in S\}$ and this inclusion can be proper.

2. Basic assumptions.

Let X be a compact Hausdorff space and let S be a set of $\bar{\boldsymbol{R}}^+$ – valued lower semicontinuous functions on X which is closed under addition and separates the points of X. A closed set $B \subseteqq X$ is called a *boundary* for S if each function in S attains its minimum on B. Bauer [1] has shown that there exists a boundary, denoted by ∂S and called the Šilov boundary of S, which is a subset of every boundary.

Two partial orders $\leqq$ and $\lesssim$ can be defined on S which are compatible with addition. Set $f \leqq g$ if $f(x) \leqq g(x)$ for all $x \in X$ and set $f \lesssim g$ if $g = f + h$, for some $h \in S$. Then, clearly, $f \lesssim g$ implies $f \leqq g$.

Assume that S is a lower semi-lattice with respect to $\lesssim$ (i.e. if f and g are in S their meet $f \wedge g$ exists in S). Let $A = \{x \in X \mid \forall f,$

[1] The author wishes to thank the referee for extending this theorem from continuous functions to lower semicontinuous functions, the crucial step being the arguement for Proposition 1, and for shortening the arguement of Proposition 2.

$g \in S$, $(f \wedge g)(x) = \min \{f(x), g(x)\}\}$. Then, as the following, examples show, not much can be said about the relationship of A and ∂S.

EXAMPLE 1 (*due to E.J. Barbeau*). Let $X_1 = \{1, 2, 3\}$ and let S_1 be the set of positive real-valued functions f on X with $f(1) \leqq 1/2 (f(2) + f(3))$. Then S is a point separating subcone of $C(X, \mathrm{R})$ which contains the constants and is a lattice with respect to the partial order $\leqq$. In fact, for $f, g \in S_1$, $(f \vee g)(i) = \max \{f(i), g(i)\}$, $(f \wedge g)(j) = \min \{f(j), g(j)\}$ if $j = 2$ or 3, and $(f \wedge g)(1) = 1/2[(f \wedge g)(2) + (f \wedge g)(3)]$. The set $A = \{2, 3\}$ and $\partial S_1 = X_1$.

EXAMPLE 2. Let $X_2 = \{4, 5\}$ and let S_2 be the set of positive real-valued functions for X with $f(4) \leqq f(5)$. Then $\partial S_2 = \{4\}$ and $A = X_2$.

EXAMPLE 3. Let $X = X_1 \cup X_2$ and let S be the set of functions f with $f \mid X_i \in S_i$. Then $A = \{2, 3, 4, 5\}$ and $\partial S = \{1, 2, 3, 4\}$.

Assume now that S is a lower semi-lattice with respect to $\leqq$, Define A, as before, to be $\{x \mid (f \wedge g)(x) = \min \{f(x), g(x)\} \ \forall f, g \in S\}$, where $f \wedge g$ now denotes meet with respect to $\leqq$.

LEMMA. *A is closed.*

Proof. If $x_0 \notin A$ then, for some f and $g \in S$, there is $\lambda > 0$ with $\min (f, g)(x_0) > \lambda > (f \wedge g)(x_0)$. Set $f_1 = f \wedge \lambda$ and $g_1 = g \wedge \lambda$ Then, since $f_1 \wedge g_1 = (f \wedge g) \wedge \lambda$, one of the inequalities $\min (f, \lambda)(x_0) \geqq f_1(x_0)$, $\min (g, \lambda)(x_0) \geqq g_1(x_0)$, and $\min (f_1, g_1)(x_0) \geqq (f_1 \wedge g_1)(x_0)$ is strict.

Consequently, if $x_0 \notin A$ there exist $f, g \in S$ with g real-valued on X and $\min (f, g)(x_0) > (f \wedge g)(x_0)$. Now $f = f \wedge g + h$ and $g = f \wedge g + k$, so $\min (f, g) = f \wedge g + \min (h, k)$. Since $\min (h, k)(x_0) > 0$, $\min (h, k)$ is lower semi-continuous, and g is real-valued, a neighbourhood of x_0 is disjoint from A.

PROPOSITION 1. Let $x_0 \in A$ and let U be an open neighbourhood of x_0. If S contains all the nonnegative constant functions there exist $\varepsilon > 0$ and $f \in S$ with (1) $f(x_0) = 0$ and (2) $\{x \mid f(x) < \varepsilon\} \subseteqq U$.

Proof[2]. Assume the proposition to be false. Then as $f \in S$, with $f(x_0) = 0$, and $\varepsilon > 0$ vary, the closed sets of the form $\{x \notin U \mid f(x) \geqq \varepsilon\}$ define a filter base. Hence there is a point $x_1 \notin U$ such that if $f \in S$ and $f(x_0) = 0$, then $f(x_1) = 0$.

For $g \in S$ if $\lambda = g(x_0) < +\infty$, then $(g \wedge \lambda)(x_0) = \lambda$. Now there

[2] This arguement, based on a technique of Loeb and Walsh in [3] is due to the referee.

exist functions h and $k \in S$ with $g \wedge \lambda + h = g$ and $g \wedge \lambda + k = \lambda$. Since $h(x_0) = k(x_0) = 0$ it follows that $g(x_1) = (g \wedge \lambda)(x_1) = \lambda = g(x_0)$. If $g(x_0) = +\infty$, then $n = (g \wedge n)(x_0) = (g \wedge n)(x_1) \leqq g(x_1)$ and so $g(x_1) = +\infty$. This contradicts the fact that S separates points.

COROLLARY. *If S is a lower semi-lattice with respect to $\leqq$ which contains all the nonnegative constant functions, then $A \subseteqq \partial S$.*

3. **The main theorem.** From now on S is assumed to have the following properties:

(a) S contains all the nonnegative constant functions λ.

(b) $\lambda \leqq f$ implies $\lambda \leqq f$; and

(c) for all $f, g, h \in S$, $f + (g \wedge h) = (f + g) \wedge (f + h)$.

PROPOSITION 2. Let $I \subseteqq S$ be maximal with respect to the following properties:

(1) $f, g \in I \Rightarrow f + g \in I$

(2) $g \leqq f$ and $f \in I \Rightarrow g \in I$

(3) $1 \notin I$.

Then there is a point $x_0 \in A$ with $I = \{f \in S \mid f(x_0) = 0\}$.

Assuming this proposition the main theorem of this note is quickly proved.

THEOREM. *Let X be compact Hausdorff and let S be a point-separating collection of lower semi-continuous functions $f: X \to [0, +\infty]$, Assume that S is closed under addition and is a lower semi-lattice with respect to the partial order $\leqq$ (where $f \leqq g$ if $g = f + h$, for some $h \in S$.)*

The Šilov boundary of S coincides with $A = \{x \mid \forall f, g \in S, (f \wedge g)(x) = \min\{f(x), g(x)\}\}$ if S satisfies properties (a), (b) *and* (c).

In particular this is the case if S satisfies (a), (b) *and the cancellation law.*

Proof. Since A is a closed subset of ∂S it suffices to show that each $f \in S$ with finite minimum α attains α on A. Let $M = \{x(f(x) = \alpha\}$. Then, since $f \wedge \alpha = \alpha$ there exists a function $h \in S$ with $f = \alpha + h$ which vanishes on M. Let I_0 be the set of functions in S which vanish on M. Then I_0 satisfies conditions (1), (2) and (3) of Proposition 2. Since I_0 can be embeded in a set $I \subseteqq S$ maximal with respect to these properties, it follows from Proposition 2 that $M \cap A \neq \phi$.

EXAMPLE 4. Let $X = \{1, 2, 3\}$ and let S be the semigroup of positive real-valued functions f with $f(1) \leqq f(2)$ and $3/4\, f(2) - 1/4\, f(1) \geqq f(3)$. Then S satisfies all the hypotheses of the theorem except (b). Here $A = \{1\}$ and $\partial S = \{1, 3\}$.

4. Proof of proposition 2. Let $B = \{x \mid f \in I \Rightarrow f(x) = 0\}$. Then, since I satisfies (1), (2) and (3), $B \neq \varnothing$ and furthermore, if $\varnothing \neq K \subseteqq \complement B$ is compact there exists $h \in I$ with $\min_K h > 0$. The maximality of I implies that if $h \in S$ vanishes at a point of B, then $h \in I$.

Define ψ: $S \rightarrow \mathbf{R}^+$ by setting $\psi(f) = \min_B f$. Then ψ has the following properties:

(i) $\psi(f) = \sup \{\lambda \mid \text{for some } h \in I,\ \lambda \leqq f + h\}$

(ii) $\psi(f) = 0 \Leftrightarrow f \in I$

(iii) $\psi(f \wedge \psi(f)) = \psi(f)$

(iv) $f \in S \Rightarrow f \mid B = \psi(f) \mid B$.

(i) Let $\lambda < \psi(f)$. Then $K = \{x \mid f(x) \leqq \lambda\}$ is compact and disjoint from B. If $K \neq \varnothing$, there exists $h \in I$ with $\min_K h \geqq \lambda$ and so $\lambda \leqq f + h$. If $\lambda \leqq f + h$ then $\lambda \leqq \psi(f)$ as h vanishes on B.

(ii) Clear.

(iii) $\psi(f \wedge \psi(f)) \leqq \psi(f)$ anyway. Let $\lambda < \psi(f)$ and let $h \in I$ with $\lambda \leqq f + h$. Now $\lambda \leqq \psi(f) + h$ implies by (c) that $\lambda \leqq f \wedge \psi(f) + h$ and so $\lambda \leqq \psi(f \wedge \psi(f))$.

(iv) Assume $\psi(f) < +\infty$ as it is trivial if $\psi(f) = +\infty$. Then $f = f \wedge \psi(f) + g$ for some $g \in S$. Now $\psi(f) \geqq \psi(f \wedge \psi(f)) + \psi(g) = \psi(f) + \psi(g)$ and so $\psi(g) = 0$. Consequently, $f \mid B = f \wedge \psi(f) \mid B$ and hence $f \mid B = \psi(f) \mid B$.

Property (iv) implies that $B = \{x_0\}$ for some $x_0 \in X$. It remains to show that $x_0 \in A$. Let $f, g \in S$ and let $\lambda = f(x_0) \wedge g(x_0)$. If $\lambda \leqq (f \wedge g)(x_0)$ then $\lambda = (f \wedge g)(x_0)$ and so $x_0 \in A$.

Let $\alpha < \lambda$. Then there exist functions h and k in I with $\alpha \leqq f + h$ and $\alpha \leqq g + k$. Property (c) implies that $\alpha \leqq \alpha + (h \wedge k) = (\alpha + h) \wedge (\alpha + k) \leqq (f + h + k) \wedge (g + h + k) = (f \wedge g) + (h + k)$. Hence, $\alpha \leqq \psi(f \wedge g) = (f \wedge g)(x_0)$.

Remarks. This arguement, due to the referee, is a shortened version of an argument of the author which showed that A could be identified with the additive functions ψ: $S \rightarrow \bar{\boldsymbol{R}}^+$ that preserve finite meets and for which $\psi(\lambda) = \lambda$ if λ is a constant. (c.f. Bauer [1]).

5. Upper semi-lattices. Examples 1, 2 and 3 show that when S is an upper semi-lattice with respect to $\leqq$ there is no particular relationship between ∂S and $B = \{x \mid (f \vee g)(x) = \max \{f(x), g(x)\}\ \forall f, g \in S\}$.

Assume that S is an upper semi-lattice with respect to $\leqq$. Then, if S is also a lower semi-lattice with respect to $\preceq$ and if S satisfies the hypotheses of the theorem, $\partial S \subseteqq B$ whenever the identity $f + g = f \vee g + f \wedge g$ holds for all $f, g \in S$ (for example, this is the case if $f + (g \vee h) = (f + g) \vee (f + h)$ for all $f, g\, h \in S$). However, as the following example shows, ∂S can be distinct from B.

EXAMPLE 5. Let $X = \{1, 2\}$ and let S be the set of nonnegative functions f with $f(1)$ finite and $f(1) = f(2)$ or $f(2) = +\infty$. Then S is a lower semi-lattice. Here $\partial S = A = \{1\}$. However, $B = X$ since for $f, g \in S$ max $(f, g) = f \vee g$.

If S is an upper semi-lattice but not a lattice with respect to $\leqq$ then B can be a proper subset of ∂S as shown by the next example.

EXAMPLE 6. Let $X = \{1, 2, 3\}$ and let S be the set of nonnegative real-valued functions f with $f(3) \geqq 4/3\, f(2) - 1/3\, f(1)$. Then S contains the nonnegative constant functions λ and $\lambda \leqq f$ implies $\lambda \preceq f$. S is an upper semi-lattice.

Here, $B = \{1, 2\}$ and $\partial S = X$. Consequently, S is not a lower semi-lattice with respect to $\leqq$. For example, there is no function in S which is the meet of $f = (1, 1, 1)$ and $g = (1, 1/4, 1/2)$.

Putting Examples 5 and 6 together, as was done before to obtain Example 3 from Examples 1 and 2, we see that for upper semi-lattices S, even those which satisfy hypotheses analogous to those of the theorem on lower semi-lattices, there is no particular relationship between ∂S and B.

6. The case of a vector space. Let X be compact and let $H \subseteqq C(X, \boldsymbol{R})$ be a point-separating set of continuous functions $f: X \to \bar{\boldsymbol{R}}$ which is a lattice with respect to the partial order $f \leqq g$ if $f(x) \leqq g(x)$ for all $x \in X$. Assume that H has the following properties:

(1) if $f \in H$, then $R(f) = \{x \mid |f(x)| < +\infty\}$ is dense;

(2) if $f, g \in H$ there is a function $h \in H$ with $h(x) = f(x) + g(x)$, for $x \in R(f) \cap R(g)$;

(3) if $f \in H$ and $\lambda \in \boldsymbol{R}$ there is a function $k \in H$ with $k(x) = \lambda f(x)$ for $x \in R(f)$; and

(4) H contains the constant functions.

These properties imply that H is a vector lattice. Denote by S the positive cone H^+ of H. Then S is an additive semigroup with cancellation which is a lattice with respect to the partial order $f \leqq g$ if $g = f + h$, for some $h \in S$. Clearly, the hypotheses of the theorem are satisfied by S, and so the Šilov boundary of S is the set of points in X such that $(f \wedge g)(x) = \min\{f(x), g(x)\}$ for all $f, g \in S$. Since $f + g = f \wedge g + f \wedge g$, this is the set of points at which the lattice operations hold pointwise.

Denote by H^* the set $\{f \in H \mid \text{for some } n, n \geqq |f|\}$. Then H^* is the subvector lattice of H consisting of bounded functions.

If $B \subseteqq X$ is closed, it will be called a *boundary* for H^* if (1) each function in H^* attains its maximum on B, and (2) if $x \in B$ and $f(x) = f(y)$, for all $f \in H^*$, then $y \in B$. It is well known that H^* has a unique minimal boundary if H^* separates the points of X (c.f. [1]). By passing to an identification space Y of X and taking inverse images of sets in Y, it then follows that H^* has a unique minimal boundary in X. This set will be called the *Šilov boundary* of H^*.

PROPOSITION 3. The Šilov boundary of H^+ is the Šilov boundary of H^*.

Proof. Let A denote the Šilov boundary of H^+. Then, A is a boundary for H^*.

It is clear that each function in H^* attains its maximum on A. It remains to show that if $x \in A$ and $f(x) = f(y)$, for all $f \in H^*$, then $y \in A$.

Assume $x \in A$ and $y \neq x$. Then, since $f = f^+ - f^-$ there exists $f \in H^+$ with either $\lambda = f(x) > f(y) \geqq 0$ or $\lambda = f(x) < f(y) \leqq 0$. In the first case let $g = f \wedge \lambda$ and in the second case let $g = f \vee \lambda$. Then, since $x \in A$, $g(x) = \lambda$ and $g(y) \leqq f(y)$ or $g(y) \geqq f(y)$. Hence, in either case, $g(x) \neq g(y)$. Since $g \in H^*$, it follows that A is a boundary for H^*.

The Šilov boundary A of H^+ is the set of points in X at which the lattice operations hold pointwise. Since $H^* \mid A$ separates the points of A it is dense in $C(A, \mathbf{R})$. Hence, A is the Šilov boundary for H^*.

EXAMPLE 1. Let H be the vector lattice of differences of positive harmonic functions on some open set $\Omega \subseteqq \mathbf{R}^n$. Let X be the compactification of Ω determined by H ([2] p. 97). Viewing H^+ as a cone of functions on X, it follows that the Šilov boundary of H^+ coincides with the Šilov boundary of H^*, which in this case consists of the bounded harmonic functions on Ω. Further, if $H_1{}^+$ denotes the functions on Ω. Further, if $H_1{}^+$ denotes the functions f in H^+ with $f \wedge 1 \neq 0$, the Šilov boundary of H^+ coincides with that of H^+.

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Received September 15, 1966.

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