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AN L^1 ALGEBRA FOR LINEARLY QUASI-ORDERED COMPACT SEMIGROUPS

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This paper is concerned with obtaining an L^1 algebra for compact commutative linearly quasi-ordered topological semigroups. It will be shown that there is a suitable measure on such a semigroup so that the maximal ideal space of the L^1 algebra with respect to this measure and the bounded measurable semicharacters modulo equal almost everywhere can be identified. It is also proved that the condition $x^2 = y^2 = xy$ implies $x = y$ is necessary and sufficient that the L^1 algebra be semisimple.

Before we begin the study of the semigroups in the title, we wish to point out the differences in the measures discussed here and the L^1 algebra constructed here with those found in such papers as [3], [4], [7], [9], [10] and [11]. It is shown in [11] that "r^*-invariant" measures on compact semigroups are supported on the minimal ideal of such semigroups. The measures used here will have support outside the minimal ideal. The work in [4] on compact simple semigroups reduces to the compact group situation in commutative semigroups and thus there is no relation to our work. In [3], the use of idempotent measures insures that the support is an ideal, again distinct from the situation here. The measures in [10] and [11] are again supported on the minimal ideal of the semigroup and for compact commutative semigroups would reduce to Haar measure on the minimal ideal, which is a compact topological group. The development in [7] of the L^1 algebra of an interval of idempotent elements is more interesting, since it involves part of the structure of $S/\mathscr{J}$ (see below). In this paper, if such an interval is in the semigroup we have required it to have measure 0. In the special case of algebraically irreducible compact connected semigroups [14], an L^1 algebra, which combines the measure in [7] and the techniques to follow, could provide interesting results.

Let S be a compact commutative topological semigroup with identity satisfying the condition (*) $x, y \in S$ and $x^2 = y^2 = xy$ implies $x = y$. It is known [8, VIII 1.3] that such an S is the union of a family of nonintersecting subsemigroups each of which is a cancellation semigroup and that there is a finest such decomposition. This finest decomposition may be too fine for the proper introduction of a measure on S. In fact, the maximal subgroups of S need not be contained in elements of the decomposition. We will consider a class of semigroups which allow a decomposition into cancellation subsemigroups each element of which will be either a compact topological group or a cancellation semigroup, which is embeddable in a locally compact topological group as a subset with interior. In this way, we will be able to introduce a measure on such an S, construct $L^1(S)$ as in [12] and use some of the theory developed there.

Let S now also satisfy the condition that the principal ideals of S are linearly ordered by inclusion. Such S were called linearly quasi-ordered in [13] and their structure was considered there. We will use the results concerning this order without further reference here.

Let E denote the idempotent elements of S and for each $e \in E$, let $H(e)$ be the maximal subgroup of S containing e. Since S is commutative with identity, Se is the principal ideal generated by e. If e, $f \in E$ and $Se \subset Sf$, then we write $e \leqq f$ and note that $e \leqq f$ if and only if $ef = e$. It is also clear that E, with $\leqq$, is a naturally totally ordered set in the sense of Clifford [1]. Let $e, f \in E$ with $e \leqq f$, then $[e, f] = H(e) \cup (Sf - Se)$, $(e, f] = Sf - Se$ and $(e, f) = Sf - (H(f) \cup Se)$.

Let $\mathfrak{D}(S)$ be that decomposition of S such that $A \in \mathfrak{D}(S)$ if and only if either $A = H(e)$ for some $e \in E$ or $A = (e, f) \neq \varnothing$ where $e, f \in E$ and $(e, f) \cap E = \varnothing$. We will show that $\mathfrak{D}(S)$ is a decomposition of S into cancellation subsemigroups.

LEMMA 1. *Let S be a compact commutative linearly quasi-ordered topological semigroup with identity and satisfying* (*). *If* $(e, f) \in D(S)$ *then* (1) (e, f) *contains a subsemigroup* $N(f)$ *and* $N(f)$ *is either isomorphic to the usual open unit interval* $(0, 1)$ *or isomorphic to* $\{2^{-n}\}_{n=1}^{\infty}$ *as a subset of the usual unit interval* (2) $(e, f) = N(f)H(f)$ *and* (3) (e, f) *is a cancellation semigroup.*

Proof. Let $e, f \in E$ with $(e, f) \neq \varnothing$ and $(e, f) \cap E = \varnothing$.

(1) Let us consider (e, f) as a subset of $S_0 = Sf/Se$ $(Sf - Se \approx Sf/Se - \{0\})$ and let $\mathscr{J}$ be the equivalence relation of Green [2]; i.e., $x \mathscr{J} y$ if and only if $S_0 x = S_0 y$. Set $T = S_0/\mathscr{J}$ and let ϕ be the natural map. Then T is a naturally totally ordered semigroup in the sense of Clifford [1] and is a compact topological semigroup with 0 and 1 and no other idempotents in the order topology (which agrees with the original topology). If $0 \leqq a, b, c \leqq 1$ in T and $ac = bc$ then (since $a \leqq b$ or $b \leqq a$, we assume $b \leqq a$) there is a $d \in T$ such that $b = ad$. From $ac = bc = adc = ad^n c$ for all integers $n > 0$ and the compactness of T, $d^n \to 0$ or $d^n \to 1$ and thus $ac = 0$ or $a = b$, since $bd^n = ad^{n+1}$. If $x, y \in T$ with $x^2 = y^2 = xy$ and $xy \neq 0$ then $x = y$ by the above and if $xy = 0$, $x^2 = 0$ and if $x \neq 0$ there is an $s \in (e, f)$

with $s^2 \in H(e)$. Then $s^2 = es^2 = e^2s^2 = (es)(s)$ in S and $es = s$ since S satisfies (*), a contradiction. Hence $x = 0 = y$ and T satisfies (*).

If $ac = 0$ in T, we assume without any loss of generality that $a = tc$ for some $t \in T$. Then $Ta^2 = Tatc = Ttac = 0$. Thus $a^2 = 0 = 0^2 = 0a$ and $a = 0$ since T satisfies (*). This is a contradiction, hence $b = a$ and $T - \{0\}$ is a cancellation semigroup which is easily seen to be archimedean. By a result of Hölder [1], T is isomorphic to a subsemigroup of the usual unit interval $[0, 1]$.

It is clear that $T = [0, 1]$ if $1 \in (T - \{1\})^-$ (the bar denotes closure) and $T = \{1\} \cup \{0\} \cup \{x_0, x_0^2, \cdots, x_0^n, \cdots\}$ if $1 \notin (T - \{1\})^-$. When $T = [0, 1]$, $N(f)$ exists by [13]; in the other case, for any $s \in (e, f)$ which maps to x_0 under the $\mathscr{J}$ equivalence $N(f) = \{s^n\}_{n=1}^\infty$ will work.

(2) Let $x \in N(f)$, then for any $g \in H(f)$, $Sgx = Sfx = Sx$ and $gx \mathscr{J} x$. On the other hand, if $y \mathscr{J} x$ then there exists s_1 and $s_2 \in [e, f]$ so that $x = s_1 y$ and $y = s_2 x$. Now

$$\phi(x) = \phi(s_1)\phi(y) = \phi(s_1)\phi(s_2)\phi(x)$$

so that $\phi(s_1)\phi(s_2) = 1$ and s_1 and $s_2 \in H(f)$; thus $y \in xH(f)$. It follows that $N(f)H(f) = (e, f)$.

(3) It is clear that (e, f) is a subsemigroup of S. In order to see that (e, f) is a cancellation semigroup, let a, x and $y \in (e, f)$ with $ax = ay$. Let us choose the representations $a = t_1 g_1$, $x = t_2 g_2$ and $y = t_3 g_3$ where $t_1, t_2, t_3 \in N(f)$ and $g_1, g_2, g_3 \in H(f)$. From $ax = ay$, we have $t_1 t_2 g_1 g_2 = t_1 t_3 g_1 g_3$, hence $t_1 t_2 = t_1 t_3$ and then $t_2 = t_3$. Thus, for some $t \in N(f)$, $g_3^{-1} g_2 t = t$. If $N(f) = \{s^n\}_{n=1}^\infty$ and $gs^k = s^k$ for some $k > 1$ and $g \in H(f)$, then $gs = s$. For if not, and if $gs^2 = s^2$ then $g^2 s^2 \neq s^2$ by (*), and if $gs^{2^l} = s^{2^l}$ then $g^2 s^{2^l} \neq s^{2^l}$; but $gs^k = s^k$ implies $gs^m = s^m$ for all $m \geqq k$ and $g^2 s^{2^l} = s^{2^l}$ for $2^l \geqq 2k$ and $gs = s$. It follows that $H(f)$ has a subgroup G leaving $N(f)$ fixed. Now $x = t_2 g_2$ and $y = t_2 g_3$ and $g_2 g_3^{-1} \in G$ implies $x = y$, but $t_1 t_2 g_1 g_2 = t_1 t_2 g_1 g_3$, which implies $g_2 g_3^{-1} \in G$ and thus $x = y$. If $N(f) = (0, 1)$ the open unit interval, and if $gx = x$ where $g \in H(f)$ and $x \in N(f)$, then $g\sqrt{x} = \sqrt{x}$ since

$$(g\sqrt{x})\sqrt{x} = g(x) = x = \sqrt{x}\,\sqrt{x} = (g\sqrt{x})^2 = g(gx) = gx\ .$$

Thus g leaves $N(f)$ fixed. Let $x_n \to f$, $\{x_n\} \subset N(f)$. Then $gx_n \to gf = g$ but $gx_n = x_n \to f$ so $g = f$ and it is clear that $N(f)$ is a cancellation semigroup.

It should be noted that the proof above shows that if $(e, f) \in \mathfrak{D}(S)$ and $(e, f)/\mathscr{J} \approx (0, 1)$, then $H(f)$ operates in a fixed point free manner on (e, f). Thus, $(e, f]$ is a cancellation semigroup. We also note that if $(e, f)/\mathscr{J} \neq (0, 1)$ then $(e, f]$ need not be a cancellation semigroup, for let $[e, f] = \{-1, 1\} \cup \{0\} \cup \{2^{-n}\}_1^\infty$ where -1 and 1 both act as the identity on their complement.

It was shown in [5] that the condition (*) was necessary and sufficient that the l_1 algebra of a discrete topological semigroup be semisimple. We have used the condition here first to obtain the preceding lemma so that the choice of a decomposition for a linearly quasi-ordered semigroup into subsemigroups on which measures can be introduced would be as natural as possible. We now will drop the condition (*) until the time to prove it is a necessary and sufficient condition for the semisimplicity of the algebra we shall construct. However, in order to be able to introduce a measure on S satisfying conditions such as in [12, 2.5], we assume that the sets (e, f), with $(e, f) \cap E = \varnothing$, are subsemigroups of S.

Let $\mathfrak{D}(S)$ be such that $D \in \mathfrak{D}(S)$ implies D is a subsemigroup of S.

Let $E^1 = [e \in E: e \in \overline{Se \backslash H(e)}]$. Let $\mathfrak{A}(S)$ be that decomposition of S into subsemigroups such that $A \in \mathfrak{A}(S)$ if and only if

either (1) $A = H(e)$ for some $e \notin E^1$;

or (2) $A = (e, f]$ where $(e, f) \in \mathfrak{D}(S)$ and $f \in E^1$;

or (3) $A = (e, f)$ where $(e, f) \in \mathfrak{D}(S)$ and $f \notin E^1$;

or (4) $A = H(e)$ where $(f, e) \notin \mathfrak{D}(S)$ for all $f \leqq e$.

A measure can be introduced on each $A \in \mathfrak{A}(S)$ in the following manner. If $A = H(e)$ and $e \notin E^1$, let m_e be normalized Haar measure on the compact group $H(e)$. If $A = H(e)$ and $e \in E^1$, let $m_e = 0$. In the remaining cases, $A = (e, f]$ or $A = (e, f)$. Let $\tau: N(f) \times H(f) \rightarrow (e, f)$ be given by $\tau(t, h) = th$; then τ is a continuous function. Let μ_f denote Haar measure on $H(f)$. Let λ_f denote Lebesgue measure on $N(f)$ if $N(f) \approx (0, 1)$ and counting measure on $N(f)$ if $N(f) \approx \{2^{-n}\}_{n=1}^{\infty}$. Giving $N(f) \times H(f)$ the product measure $\lambda_f \times \mu_f$, we decree $B \subset (e, f)$ measurable if $\tau^{-1}(B)$ is measurable in $N(f) \times H(f)$, and define $m_f(B) = \lambda_f \times \mu_f(\tau^{-1}(B))$.

In this way each $A \in \mathfrak{A}(S)$ has a measure m_A assigned such that $E \subset A$ and $x \in A$ imply $m_A(Ex) \geqq m_A(E)$. A set $B \subset S$ is measurable if and only if $B \cap A$ is measurable for all $A \in \mathfrak{A}(S)$ and $m(B) = \sum_{A \in \mathfrak{A}(S)} m_A(B \cap A)$ is a measure on S.

THEOREM 1. *Let S be a compact commutative linearly quasi-ordered topological semigroup with identity. If the decomposition $\mathfrak{D}(S)$ is such that each element of the decomposition is a subsemigroup of S, then the decomposition $\mathfrak{A}(S)$ consists of semigroups and there is a measure m on S such that $A \in \mathfrak{A}(S)$, $E \subset A$, and $x \in A$ imply $m_A(Ex) \geqq m_A(E)$, where $m_A = m \mid A$.*

DEFINITION. For S a locally compact topological semigroup and m a regular Borel measure on S, $L^1(S, m) = [\mu: \mu \in M(S)$ and $\mu << m]$.

The definition of $L^1(S, m)$ above is the same as we used in [12]. The measure m has been chosen so that the conditions of [12, Th. 2.5] are satisfied. We also note that for each $A \in \mathfrak{A}(S)$ the hypotheses of [12, Th. 2.3] and [12, Th. 3.3] are satisfied. The Theorem 3 and 4 to follow arise because the condition, $\bar{x} * L^1(S, m) \subset L^1(S, m)$ for all $x \in S$ of [12, Th. 3.4] need not hold. However, that condition was sufficient but not necessary. We have here a weaker condition that is sufficient to obtain the same conclusions.

THEOREM 2. *Let S be as in the previous theorem. Each measurable semicharacter θ on an $A \in \mathfrak{A}(S)$ can be uniquely extended to a measurable semicharacter $\hat{\theta}$ on S which is zero below A; i.e.,*

$$\begin{aligned} A = H(e),\ x \in Se\backslash H(e) &\Rightarrow \hat{\theta}(x) = 0 . \\ A = (e, f],\ x \in Se \qquad &\Rightarrow \hat{\theta}(x) = 0 . \end{aligned}$$

Proof. If $A = H(e)$, then θ measurable on $H(e)$ implies θ is continuous on $H(e)$.

Define $\hat{\theta}$ on S by

$$\hat{\theta}(x) = \begin{cases} \theta(xe) & \text{if } x \in (S\backslash Se) \cup H(e) \\ 0 & \text{if } x \in Se\backslash H(e) . \end{cases}$$

Since $\hat{\theta}$ is continuous on $S\backslash Se \cup H(e)$ and $Se\backslash H(e)$ is open, hence measurable, $\hat{\theta}$ is measurable on S.

If $A = (e, f]$ then let

$$\hat{\theta}(x) = \begin{cases} \theta(xf) & x \in S\backslash Se \\ 0 & x \in Se . \end{cases}$$

Now $\hat{\theta}$ is continuous on $S\backslash Se$. Since θ can be extended to the group generated by $N(f) \times H(f)$ to be measurable, and therefore continuous, and Se is closed, $\hat{\theta}$ is measurable on S.

COROLLARY. *If in addition S is connected, then each such extension of a measurable semicharacter on an $A \in \mathfrak{A}(S)$, is continuous* a.e. (m).

This makes for the great distinction between groups and semigroups. Let $S = [0, 1]$, $xy = \min(x, y)$. Then, every set is measurable and for each $A = H(e)$, $m_A = 0$ unless $A = \{0\}$, where m_A is point mass at 0 and so is m. Thus each semi-character is measurable and continuous a.e.

However, in the above example, all nonzero and not identically 1 semicharacters are equal almost everywhere to 0. Hence, they all generate the multiplicative linear functional h on $L^1(S, m)$,

$$h(\mu) = \int \tau \, d\mu = \int 0 \, d\mu = 0 .$$

It should also be noted that the above semigroup satisfies condition (*).

DEFINITION. Let Δ denote the m measurable semicharacters on S modulo the relation equal almost everywhere with respect to m.

Let τ be a representative of an element of Δ. For $\mu \in L^1(S, m)$, define $h(\mu) = \int \tau \, d\mu$. Then, for $\mu, \gamma \in L^1(S, m)$,

$$\begin{aligned} h(\mu * \gamma) &= \int \tau d(\mu * \gamma) = \iint \tau(xy)\mu(dx)\gamma(dy) = \iint \tau(x)\tau(y)\mu(dx)\gamma(dy) . \\ &= h(\mu)h(\gamma) \end{aligned}$$

and h is a multiplicative linear functional on $L^1(S, m)$. Further, if $\lambda = \tau$ a.e. (m), then

$$\int \lambda d\mu = \int \lambda - \tau \, d\mu + \int \tau \, d\mu = \int \tau \, d\mu$$

since $\lambda - \tau = 0$ a.e.(m) and hence $\lambda - \tau = 0$ a.e.(μ). Thus, there is

THEOREM 3. *Let S be a compact commutative linearly quasi-ordered topological semigroup with identity. Let S be such that the decomposition $\mathfrak{A}(S)$ consists of semigroups and m the measure on S defined as above. Then each measurable semicharacter τ on S induces a multiplicative linear functional h on $L^1(S, m)$ such that $h(\mu) = \int \tau \, d\mu$. Further two measurable semicharacters induce the same h if and only if they are equal* a.e.(m).

Let h be any multiplicative linear functional on $L^1(S, m)$. It was shown in [12] that if $x \in S$ and $\mu \in L^1(S, m)$ imply $\bar{x} * \mu \in L^1(S, m)$ then $\tau(x) = h(\bar{x} * \mu)/h(\mu)$ was a measurable semicharacter on $S(h(\mu) \neq 0)$. It is not always true that $\bar{x} * \mu \in L^1(S, m)$, but we will still be able to construct the measurable semicharacter from a given multiplicative linear functional.

EXAMPLE. Let $S = [0, 2]$ where $[0, 1]$ is a usual unit interval, $[1, 2]$ is a usual unit interval and each element of $[1, 2]$ acts as an identity element on $[0, 1]$. Let μ be Lebesgue measure of the interval $[3/2, 2]$ and $x = 1/2$. Then $\mu << m$ but

$$(\bar{x} * \mu)(\{x\}) = \mu[y; xy = x] = \mu([1, 2]) \neq 0$$

while $m(\{x\}) = 0$. Thus $\bar{x} * \mu \notin L^1(S, m)$.

Let h be a nonzero multiplicative linear functional on $L^1(S, m)$. Now, for some $A \in \mathfrak{A}(S)$, $h \mid L^1(A, m_A) \neq 0$, for if not then since $L^1(S, m)$ is the least closed subalgebra of $C(S)^*$ containing all $L^1(A, m_A)$, $A \in \mathfrak{A}(S)$, h woule be identically zero.

Consider then $\mathfrak{A}_0 = [A: A \in \mathfrak{A}(S)$ and $h \mid L^1(A, m_A) \neq 0]$ and partially order by $A_1 \leqq A_2$ if and only if $SA_2 \subset SA_1$. Now if $A = H(e)$, $e \in E$, then $SA = Se$ and if $A = (e, f)$, $e, f \in E$, $SA = Sf \backslash H(f)$ and if $A = (e, f]$, $e, f \in E$, $SA = Sf$. We now have a family of closed sets $\{SA: A \in \mathfrak{A}_0\}$, and these are nested; thus $\bigcap_{A \in \mathfrak{A}_0} SA \neq \varnothing$. If $\mathfrak{A}_0$ has a maximal element A_0, let $\mu_0 \in L^1(A_0, m_{A_0})$ such that $h(\mu_0) \neq 0$. If $x \in (S \backslash SA_0) \cup A_0$, then $\bar{x} * \mu_0 \in L^1(S, m)$. If $x \in A_0$ then trivially

$$\bar{x} * \mu_0 \in L^1(A_0, m_{A_0}) .$$

If $x \in S \backslash SA_0$, then, letting e denote the maximal idempotent element of A_0, $xe \in A_0$ is such that $\overline{xe} * \mu_0 = \bar{x} * \mu_0$ and hence $\bar{x} * \mu_0 \in L^1(S, m)$.

Let us now define

$$\tau(x) = \begin{cases} h(\bar{x} * \mu_0)/h(\mu_0) & x \in (S \backslash SA_0) \cup A_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then τ is a semicharacter on S and its restriction to A_0 is a measurable semicharacter on A_0. Thus τ is measurable on S.

If $\mathfrak{A}_0$ has no maximal element, let $T = \bigcap [SA: A \in \mathfrak{A}_0]$ and $x \in S \backslash T$. Now $x \in A_0$ for some $A_0 \in \mathfrak{A}(S)$ and there is an $A \in \mathfrak{A}_0$ such that $SA \subset SA_0$. Hence, there is a $\mu_0 \in L^1(S, m)$ whose support is contained in A and $h(\mu_0) \neq 0$. Let e denote the maximal idempotent in A, then $\bar{x} * \mu_0 = \overline{xe} * \mu_0 \in L^1(A, m_A)$ and $\bar{x} * \mu_0 \in L^1(S, m)$. Note that for any $\nu \in L^1(S, m)$ for which $\bar{x} * \nu \in L^1(S, m)$ and $h(\nu) \neq 0$,

$$h(\mu * (\bar{x} * \nu)) = h(\mu) h(\bar{x} * \nu) = h(\bar{x} * \mu) h(\nu)$$

and thus

$$h(\bar{x} * \mu_0)/h(\mu_0) = h(\bar{x} * \nu)/h(\nu) .$$

We define

$$\tau(x) = \begin{cases} h(\bar{x} * \mu_0)/h(\mu_0) & \text{if } x \in S \backslash T \text{ and } h(\mu_0) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that τ is a well defined semicharacter on S and that $\tau \mid A$ is measurable for each $A \in \mathfrak{A}(S)$. Hence τ is measurable on S.

Let $\mu \in L^1(S, m)$ and let $\phi \in L^\infty(S, m)$ such that $h(\mu) = \int \phi d\mu$ is a multiplicative linear functional on $L^1(S, m)$. If $\mathfrak{A}_0$ has a maximal element and τ is the measurable semicharacter constructed above and

$\mu_0 \in L^1(S, m)$ such that $h(\mu_0) \neq 0$, $\tau(x) = h(\bar{x} * \mu_0)/h(\mu_0)$ (if $\tau(x) \neq 0$) then

$$\begin{aligned} h(\mu_0)\int \tau(x)\mu(dx) &= \int h(\bar{x} * \mu_0)\mu(dx) = \iiint \phi(zw)\bar{x}(dz)\mu_0(dw)\mu(dx) \\ &= \iint \phi(xw)\mu_0(dw)\mu(dx) = \int \phi(y)(\mu_0 * \mu)(dy) = h(\mu_0 * \mu) \\ &= h(\mu_0)h(\mu) \text{ and hence } h(\mu) = \int \tau d\mu \,. \end{aligned}$$

If, on the other hand, $\mathfrak{A}_0$ has no maximal element, and τ is constructed as before this case, then we can still obtain $h(\mu) = \int \tau d\mu$. Now

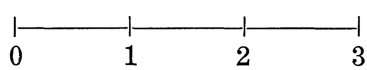
$$\begin{aligned} \int_S \tau d\mu &= \sum \int_A \tau d\mu \text{ (the sum over all } A \in \mathfrak{A}(S)) \\ &= \sum \int \tau d(\mu \mid A) \,. \end{aligned}$$

Now, for each $A \in \mathfrak{A}(S)$ where $\tau \neq 0$ on A there is a $\mu \in L^1(S, m)$ such that $h(\mu \mid A) \neq 0$ and $\bar{x} * \mu_A \in L^1(S, m)$ for all $x \in A$. Thus by repeating the earlier proof $h(\mu \mid A) = \int_A \tau d\mu$ and hence $h(\mu) = \int_S \tau d\mu$. There follows

THEOREM 4. *Let S be a compact commutative linearly quasi-ordered topological semigroup with identity. Let S be such that $\mathfrak{A}(S)$ consists of subsemigroups and m is the measure on S defined therefrom. Then each nonzero multiplicative linear functional h on $L^1(S, m)$ is such that there is a measurable semicharacter τ on S such that $h(\mu) = \int \tau d\mu$ for all $\mu \in L^1(S, m)$.*

The above two theorems establish a one to one correspondence between the maximal ideal space of $L^1(S, m)$ and the space Δ of equivalent measurable semicharacters on S.

EXAMPLE. Let $S = [0, 3]$ where $[0, 1]$ is a usual unit interval and $[2, 3]$ is a usual unit interval but $[1, 2]$ is a continuum of idempotent elements and each interval acts as identity for the ones below



The measurable semicharacters on S separate points, but the semicharacters $\chi_{[x,3]}$ where $1 \leq x \leq 2$ agree (a.e.) with $\chi_{[2,3]}$ which is also a measurable semicharacter.

THEOREM 5. *Let S be a compact connected commutative linearly quasi-ordered topological semigroup with identity and $L^1(S, m)$ the associated L^1 algebra as in the proceeding. Then $L^1(S, m)$ is semisimple if and only if* (*) $x, y \in S$ *with* $x^2 = y^2 = xy$ *implies* $x = y$.

Before proving the theorem, let us note that we do not exclude there being a continuum of idempotent elements in $S/\mathscr{J}$. It is also clear that condition (*) is equivalent to the separation of points by measurable semicharacters, since as was shown in [6], each $A \in \mathfrak{A}(S)$, $A \neq H(e)$, is a cancellation semigroup.

Proof. If $L^1(S, m)$ is semisimple, then $L^1(A, m \mid A)$ is semisimple for each $A \in A(S)$. It follows that each $A \in A(S)$ is a cancellation semigroup and hence (*) is satisfied.

On the other hand, if (*) is satisfied each $A \in \mathfrak{A}(S)$ is a cancellation semigroup and $L^1(A, m \mid A)$ is semisimple. If μ is in the radical of $L^1(S, m)$, then $\int \tau d\mu = 0$ for all measurable semicharacters τ. Let $A \in \mathfrak{A}(S)$ with $m(A) \neq 0$. Now A is either a compact group $H(f)$ with $f \notin E'$, or $A = (e, f)$ or $A = (e, f]$.

Let τ be any semicharacter which is the extension of a semicharacter on A and 0 below A. Let

$$\theta = \begin{cases} \tau & \text{on } S \backslash A \\ 0 & \text{elsewhere} . \end{cases}$$

Then θ is a measurable semicharacter on S and not equal a.e.(m) to τ. Thus $\int \tau - \theta d\mu = 0$. But, $\int \tau - \theta d\mu = \int_A \tau d(\mu \mid A)$ and $\mu \mid A$ is in the radical of $L^1(A, m \mid A)$, that is $\mu \mid A = 0$ and hence $\mu \equiv 0$ and $L^1(S, m)$ is semisimple.

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