PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 1, 1968

ON SUPPORTS OF REGULAR BOREL MEASURES

D. J. HEBERT AND H. ELTON LACEY

The existence of a regular Borel measure whose support is a given compact Hausdorff space X imposes definite structures on $X, C(X)$, and $C(X)^*$. In this paper a necessary and sufficient condition is given to insure that X is the support of a regular Borel measure. This involves the intersection number of a collection of open sets in X. Measures which vanish on a sigma ideal of a sigma field of subsets of X which contains a basis for the topology of X are also considered. In particular, for a certain class of compact Hausdorff spacs X, necessary and sufficient conditions are given to insure the existence of a nonatomic regular Borel measure whose support is X. The final section of the paper is devoted to a study of normal measures; i.e., measures which vanish on meager Borel sets. Normal measures on X are shown to be related to normal measures on the projective resolution of X.

NOTATION AND TERMINOLOGY. Set theoretical and topological terminology is that of [12], the terminology of linear topological spaces is that of [14], and measure theory terminology follows [11]. All spaces considered are taken to be nonempty and all measures considered are finite. If X is a compact Hausdorff space, $C(X)$ denotes the space of continuous real-valued functions on X in the supremum norm, $C(X)^*$ denotes the space of all continuous linear functionals on $C(X)$, or, equivalently, the space of all signed regular Borel measures, and $B(X)$ denotes the space of all bounded real-valued functions on X in the supremum norm.

1. Intersection numbers.

The following definitions are motivated by the concept of an intersection number as given in [13]. Let X be a compact Hausdorff space and B be a Boolean algebra.

1.1. If $S = (f_1, \cdots, f_n)$ is a finite sequence in $B(X), i(S) = (1/n) \| \sum_{i=1}^{n} f_i \|$. If $A \subseteqq C(X)$, then $I(A) = \inf \{i(S): S$ is a finite sequence in $A\}$.

1.2. If $S = (A_1, \cdots, A_n)$ is a finite sequence of subsets of X, $i(S) = \max \{(k/n)$: there is a subsequence $(A_{i_1}, \cdots, A_{i_k})$ of S such that $\bigcap_{j=1}^{k} A_{i_j} \neq \emptyset\}$. If H is a collection of subsets of X, then $I(H) = \inf \{i(S): S$ is a finite sequence in $H\}$.

1.3. If $S = (E_1, \cdots, E_n)$ is a finite sequence in B, then $i(S) =$

$\max\{(k/n)$: there is a subsequence $(E_{i_1}, \cdots, E_{i_n})$ of S such that $\bigwedge_{j=1}^{k} E_{i_j} \neq 0\}$ and $I(H) = \inf\{i(S)$: S is a finite sequence in $H\}$.

The relationship between the above concepts is the following: if Y is the Stone space of B (see [10] or [18]), then for $H \subseteqq B$, $I(H) = I\{h(E): E \in H\} = I\{C_{h(E)}: E \in H\}$, where h is the isomorphism of B onto the clopen (i.e., closed and open) sets of Y and $C_{h(E)}$ is the characteristic function of $h(E)$ (this notation for the characteristic function is used throughout).

The numbers $I(A)$ and $I(H)$ above are called the intersection numbers of the collections A and H respectively.

LEMMA 1.4. *Let X be a compact Hausdorff space and let F and G be nonempty subsets of the positive cone of $C(X)$. If $a > 0$ is such that, for each $f \in F$, there is a $g \in G$ with $a\, g \leqq f$, then $a\, I(G) \leqq I(F)$.*

Proof. Let $S = (f_1, \cdots, f_n)$ be a finite sequence in F. For each i, $1 \leqq i \leqq n$, there is a g_i in G such that $ag_i \leqq f_i$. Thus

$$0 \leqq \sum_{i=1}^{n} ag_i \leqq \sum_{i=1}^{n} f_i \quad \text{and} \quad \left\| \sum_{i=1}^{n} ag_i \right\| \leqq \left\| \sum_{i=1}^{n} f_i \right\| .$$

If $T = (g_1, \cdots, g_n)$, then

$$aI(G) \leqq ai(T) = a\left(\frac{1}{n}\right)\left\| \sum_{i=1}^{n} g_i \right\| = \left(\frac{1}{n}\right)\left\| \sum_{i=1}^{n} ag_i \right\| \leqq \left(\frac{1}{n}\right)\left\| \sum_{i=1}^{n} f_i \right\| = i(S) .$$

Hence $aI(G) \leqq I(F)$.

DEFINITION 1.5. Let B be a Boolean algebra and let H be a nonempty subset of B. Then H is said to be *positive* if and only if $I(H) > 0$. Similarly, if H is a nonempty collection of nonempty subsets of a given set, then H is said to be a *positive collection* whenever $I(H) > 0$.

THEOREM 1.6. *If X is a compact Hausdorff space with topology G, then there is a regular Borel measure whose support is X if and only if $G \backslash \{\phi\}$ is the union of a countable family of positive collections.* (See [13], Th. 4)

Proof. Suppose $G \backslash \{\phi\}$ is the union of a countable family $\{G_n\}$ of positive collections. For each f in $C(X)$ and each u, let

$$A_f = \left\{x \in X: f(x) > \frac{1}{2}\right\}$$

and $F_n = \{f \in C(X): f \geqq 0, ||f|| = 1$, and A_f is in $G_n\}$, and denote the convex hull of F_n by Q_n. It is a matter of computation to show that $||f|| \geqq I(F_n)$ for each $f \in Q_n$ and that for each $f \in F_n$, $(1/2)C_{A_f} \leqq f$. By Lemma 1.4, $0 < (1/2)I(G_n) \leqq I(F_n)$, and by the above, $||f|| \geqq I(F_n) > 0$ for all $f \in Q_n$. There is a positive linear functional ϕ_n on $C(X)$ such that $\phi_n(f) \geqq I(F_n)$ for all $f \in Q_n$. For, if $U = \{g \in C(X): ||g|| < I(F_n)\}$, then $Q_n + U$ is open and convex and $(Q_n + U) \cap -P = \varnothing$, where $P = f \in C(X) \mid f \geqq 0\}$. By [14], p. 118, there is a continuous linear functional ϕ_n on $C(X)$ such that $\phi_n(-f) \leqq \phi_n(g + h)$ for each $f \in P$, $g \in Q_n$, and $h \in U$. Since 0 is in $-P$, $0 \leqq \phi_n(g + h)$ for all $g \in Q_n$, and $h \in U$. Thus ϕ_n is a positive linear functional and, without loss of generality, it is assumed that $||\phi_n|| = 1$. Suppose $g \in Q_n$, $I(F_n) > \varepsilon > 0$, and let 1 denote the constant function 1 on X. Then $(\varepsilon - I(F_n))1$ is in U, and $0 \leqq \phi_n[g + (\varepsilon - I(F_n))1] = \phi_n(g) - I(F_n)\phi_n(1) + \varepsilon\phi_n(1) = \phi_n(g) - I(F_n) + \varepsilon$ and hence, $\phi_n(g) \geqq I(F_n) - \varepsilon$. Let $\phi = \sum_{n=1}^{\infty} (1/2_n)\phi_n$. Then ϕ is a positive linear functional on $C(X)$ and $||\phi|| = 1$. Now, suppose $F \in P$ and $f \neq 0$. Then $g = f/||f||$ has norm 1 and thus, $g \in F_n$ for some n. Hence, $\phi(g) \geqq \phi_n(g) \geqq I(F_n) > 0$. It follows immediately that the regular Borel measure that corresponds to ϕ has support all of X.

Conversely, suppose X has a regular Borel measure whose support is X. Then there is a positive normalized linear functional ϕ on X such that $\phi(f) > 0$ if $f \geqq 0, f \neq 0$. Let $F = \{f \in C(X) \mid f \geqq 0, ||f|| = 1\}$ and $F_n = \{f \in F \mid \phi(f) > (1/n)\}$. By computation if follows that

$$\frac{1}{n} \leqq \inf \{\phi(f): f \in F_n\} \leqq I(F_n) .$$

For each $f \in F$ and each n, let $B_f = \{x \in X: f(x) > 0\}$ and $H_n = \{B_f: f \in F_n\}$. From Urysohn's lemma it follows that for each open subset U of X there is an $f \in F$ such that $f(x) = 0$ for $x \notin U$ and $f(x) = 1$ for some $x \in U$ and thus, $f \leqq C_{B_f} \leqq C_U$. If $G_n = \{T \in G: U$ contains a member of $H_n\}$, then for each U in G_n there is an $f \in F_n$ such that $f \leqq C_{B_f} \leqq C_U$. By Lemma 1.4, $0 < I(F_n) \leqq I(H_n) \leqq I(G_n)$. Clearly $G\backslash\{\phi\} = \bigcup_{n=1}^{\infty} G_n$.

COROLLARY 1.7. *If X is a compact Hausdorff space and B is a basis for the topology G of X, then there is a regular Borel measure whose support is X if and only if $B\backslash\{\phi\}$ is the union of A countable family of positive collections.*

2. Nonatomic measures. In this section, certain conditions are shown to be sufficient for the existence of a nonatomic regular Borel measure whose support is a given compact Hausdorff space. The study of supports of nonatomic regular Borel measures is shown to be related to the study of perfect separable compact Hausdorff spaces.

If m is a regular Borel measure on a compact Hausdorff space X, then m is called *nonatomic* if for each x in X, $m(\{x\}) = 0$.

PROPOSITION 2.1. [17] If X is a compact Hausdorff space, then there is a nonzero, nonatomic regular Borel measure on X if and only if X has a nonempty perfect subset.

A well-known topological lemma is also needed.

LEMMA 2.2. *If C is a closed subset of a topological space X, then* $\operatorname{int} C = \operatorname{int} \operatorname{cl} \operatorname{int} C$.

THEOREM 2.3. *If X is a perfect compact Hausdorff space such that there exists a regular Borel measure m on X whose support is X, then there are perfect subsets X_1 and X_2 of X such that*

(i) *either X_1 is empty or there is a nonatomic regular Borel measure whose support is X_1,*

(ii) *either X_2 is empty or X_2 is separable,*

(iii) $X = X_1 \cup X_2$,

(iv) $\operatorname{int} X_1$ *contains* $X \backslash X_2$ *and* $\operatorname{int} X_2$ *contains* $X \backslash X_1$.

Proof. Suppose $C = \{x_1, x_2, \cdots\} = \{x \text{ in } X: m(\{x\}) > 0\}$ and $m = m_1 + m_2$ where m_1 is nonatomic and $m_2 = \sum_{n=1}^{\infty} m(\{x_n\}) e_{x_n}$. Let $X_1 = \operatorname{Supp}(m_1)$ and let $X_2 = \operatorname{cl} \operatorname{int} \operatorname{Supp}(m_2)$. Since $m_1(X \backslash \operatorname{Supp}(m_1)) = 0$ and $m_2[X \backslash \operatorname{Supp}(m_2)] = 0$, if $A = X \backslash \operatorname{Supp}(m_1) \cap X \backslash \operatorname{Supp}(m_2)$, then

$$0 = m_1(A) = m_2(A) = m(A) .$$

Since the complement of the support of a regular Borel measure is open and since the m-measure of a nonempty open set is positive, A is open; therefore empty. It follows that $\operatorname{Supp}(m_2)$ contains the open set $X \backslash \operatorname{Supp}(m_1)$: hence, $\operatorname{int} \operatorname{Supp}(m_2)$ contains $X \backslash X_1$, and X_2 contains $X \backslash X_1$. Therefore, $X_1 \cup X_2 = X$. Also, $\operatorname{cl} X_2$ contains $\operatorname{cl} X \backslash X_1$, and hence $\operatorname{int} X_1 = X \backslash \operatorname{cl} X \backslash X_1$ contains $X \backslash \operatorname{cl} X_2 = X \backslash X_2$. Since $\operatorname{int} X_2$ is $\operatorname{int} \operatorname{cl} \operatorname{int} \operatorname{Supp}(m_2)$, by 2.2 $\operatorname{int} X_2 = \operatorname{int} \operatorname{Supp}(m_2)$. Hence, $\operatorname{int} X_2$ contains $X \backslash X_1$. Suppose that x is in X_2 and U is an open set subset of X containing x. Since $X_2 = \operatorname{cl} \operatorname{int} \operatorname{Supp}(m_2)$, $U \cap \operatorname{int} \operatorname{Supp}(m_2)$ is nonempty. It is easy to see that $\operatorname{Supp}(m_2) = \operatorname{cl} C$; hence, there is a member y of C such that y is also an element of the open set $U \cap \operatorname{int} \operatorname{Supp}(m_2)$. This implies that X_2 is separable, ($C \cap X_2$ is a countable dense subset). If x is not in $\operatorname{int} \operatorname{Supp}(m_2)$, then $y \neq x$. If x is in $\operatorname{int} \operatorname{Supp}(m_2)$, then since X is perfect, there is a member of

$U \cap \operatorname{int} \operatorname{Supp}(m_2)$ which is distinct from x. Hence, X_2 is perfect.

COROLLARY 2.4. *If X is a compact Hausdorff space on which there is a regular Borel measure whose support is X, and such that each countable subset of X is nowhere dense, then there is a nonatomic regular Borel measure on X whose support is X.*

Proof. First note that since each singleton is nowhere dense, X is perfect. Continuing the notation of the preceding theorem, since $\operatorname{Supp}(m_2) = \operatorname{cl} C$, and C is nowhere dense, $\operatorname{int} \operatorname{Supp}(m_2)$ is empty, and hence X_2 is empty. This implies that $X = X_1$, and the proof is finished.

According to Theorem 2.3, the problem of the existence of a nonatomic regular Borel measure whose support is a given compact Hausdorff space X, which has a measure whose support is X, may be reduced to the same problem for a separable perfect compact Hausdorff space. The theorem also makes possible the development of other sufficient conditions on such spaces via the following corollary.

COROLLARY 2.5. *If X is a perfect compact Hausdorff space with a regular Borel measure whose support is X, and such that each point of X is contained in the support of a nonatomic regular Borel measure, then there is a nonatomic regular Borel measure whose support is X.*

Proof. Suppose that X_1, X_2, and m_1 are chosen as in the theorem. If X_2 is nonempty, let $\{x_1, x_2, \cdots\}$ be a countable dense subset of X_2. For each k, let ν_k be a nonatomic regular Borel measure whose support contains x_k. Then $\nu = \sum_{n=1}^{\infty} (1/2^k)\nu_k$ is a nonatomic regular Borel measure whose support is X_2, and $m_1 + \nu$ is a nonatomic regular regular Borel measure whose support is X.

LEMMA 2.6. *If X is a perfect compact Hausdorff space, and x is an element of X such that there is a countable base of open neighborhoods of x, then there is a nonatomic regular Borel measure whose support contains x.*

Proof. Let $\{U_1, U_2, \cdots\}$ be a countable base of open neighborhoods of x. For each k, U_k contains a perfect subset P_k, which by (2.1) contains the support of a nonatomic regular Borel measure ν_k. Let $\nu = \sum_{n=1}^{\infty} (1/2^k)\nu_k$. If U is an open set containing x, then U contains U_k for some k; hence, U contains P_k, and $\nu_k(U) > 0$.

THEOREM 2.7. *If X is a perfect compact Hausdorff space which satisfies the first axiom of countability, and there is a regular Borel measure whose support is X, then there is a nonatomic regular Borel measure whose support is X.*

Proof. It follows immediately from the preceding results.

COROLLARY 2.8. *If X is a perfect compact metrizable Hausdorff space, then there is a nonatomic regular Borel measure whose support is X.*

3. J-null measures.

Let B be a Boolean sigma-algebra, J be a sigma-ideal in B, and G be a subset of B. Using some of the methods in [13], conditions may be obtained for the existence of a measure on B which vanishes on J and which is bounded away from zero on G. A set G which satisfies such conditions is called *J-sigma-positive*. Let B be a sigma-field of subsets of a topological space, and let J be a sigma-ideal in B containing no open sets. If B contains a basis G of open sets, then there is a measure on B, which vanishes on J and is positive on nonempty open sets if and only if $G\backslash\{\phi\}$ is contained in a countable union of J-sigma-positive collections. This result is used along with results of § 2 to give necessary and sufficient conditions for the existence of a nonatomic regular Borel measure on a class of compact Hausdorff spaces X whose support is all of X.

Let B be a Boolean algebra and let h be the natural isomorphism of B onto the Boolean algebra of clopen subsets of the Stone space Y of B. The following lemma are basic results in [13].

LEMMA 3.1. *If G is a positive subset of B, then there is a finitely additive measure m on B such that $m(E) \geqq I(G)$ for each E in G.*

LEMMA 3.2. *If m is a finitely additive measure on B and G is a subset of B, then* $\inf\{m(E): E$ *is in* $G\} \leqq I(G)$.

DEFINITION 3.3. If B is a Boolean sigma-algebra and C is a subset of B, then C is said to be a *monotone class* whenever for each monotone sequence $E_1 \leqq E_2 \leqq \cdots$ of elements of C; $\bigvee_{n=1}^{\infty} E_n$ is an element of C.

DEFINITION 3.4. Let B be a Boolean sigma-algebra, G be a subset of B, and $\hat{G} = \{E$ in B: $D \leqq E$ for some D in $G\}$. The collection G is said to be *sigma-positive* if and only if G is positive and $B\backslash\hat{G}$ is a monotone class.

The following construction of a measure on a sigma-algebra is based on an idea of Ryll-Nardzewski (see [13]).

LEMMA 3.5. *If B is a sigma-algebra and G is a sigma-positive subset of B, then there is a measure m on B such that $m(E) \geqq I(G)$ for each E in G.*

Proof. By 3.1, there is a finitely additive measure m' on B such that $m'(E) \geqq I(G)$ for each E in G. For each E in B, let $m(E) = \inf\{\sum_{n=1}^{\infty} m'(A_n) \colon A_1, A_2, \cdots$ is a pairwise disjoint sequence such that $\bigvee_{n=1}^{\infty} A_n = E\}$. Clearly, $0 \leqq m(E) \leqq m'(E)$ for each E in B. Let $A_1, A_2, \cdots$ be a pairwise disjoint sequence in B. If ε is a positive number, then for each n, there is a pairwise disjoint sequence $C_1^n, C_2^n, \cdots$ such that $A_n = \bigvee_{k=1}^{\infty} C_k^n$ and

$$m(A_n) \leqq \sum_{k=1}^{\infty} m'(C_k^n) < m(A_n) + (\varepsilon/2^n) .$$

Hence,

$$\sum_{n=1}^{\infty} m(A_n) \leqq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(C_k^n) < \sum_{n=1}^{\infty} m(A_n) + \varepsilon .$$

Since $\bigvee_{n=1}^{\infty} A_n = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} C_k^n$, by the definition of $m, m(\bigvee_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} m(A_n)$. If $m(\bigvee_{n=1}^{\infty} A_n) < \sum_{n=1}^{\infty} m(A_n)$, then there is a pairwise disjoint sequence $E_1, E_2, \cdots$ of elements of B such that $\bigvee_{n=1}^{\infty} A_n = \bigvee_{k=1}^{\infty} E_k$ and $m(\bigvee_{n=1}^{\infty} A_n) \leqq \sum_{k=1}^{\infty} m'(E_k) < \sum_{n=1}^{\infty} m(A_n)$. Since for each

$$n, \bigvee_{k=1}^{\infty} E_k \wedge A_n = A_n \wedge \bigvee_{k=1}^{\infty} E_k = A_n ,$$

(see [10], p. 28), $m(A_n) \leqq \sum_{k=1}^{\infty} m'(E_k \wedge A_n)$. Hence,

$$\sum_{n=1}^{\infty} m(A_n) \leqq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(E_k \wedge A_n) .$$

However, since for each i and k, $\bigvee_{n=1}^{i} E_k \wedge A_n \leqq E_k$, it follows that

$$\sum_{n=1}^{i} m'(E_k \wedge A_n) = m'\Big(\sum_{n=1}^{i} [E_k \wedge A_n]\Big) \leqq m'(E_k) .$$

Thus,

$$\sum_{n=1}^{\infty} m'(E_k \wedge A_n) \leqq m'(E_k) ,$$

and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(E_k \wedge A_n) \leqq \sum_{k=1}^{\infty} m'(E_k) < \sum_{n-1}^{\infty} m(A_n) \leqq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(E_k \wedge A_n)$$

which is a contradiction. It has been shown then that m is countably additive. Suppose now that $m(E) < I(G)$ for some E in G. There is a monotone sequence $E_1 \leqq E_2 \leqq \cdots$ such that $\bigvee_{n=1}^{\infty} E_n = E$, and $\{\sup\{m'(E_n): n \geqq 0\} < I(G)$, (this is a trivial alteration of the definition of m). Thus, for each n, $m'(E_n) < I(G)$. This implies that for each n, E_n is not in the set $\hat{G} = \{E \text{ in } B: D \leqq E \text{ for some } D \text{ in } G\}$. Since $B \backslash \hat{G}$ is a monotone class, E is $B \backslash \hat{G}$, which is a contradiction.

DEFINITION 3.6. Let B be a Boolean sigma-algebra, J be a sigma-ideal in B, and G be a subset of B. For each E in B, let $\bar{E}$ denote the equivalence class modulo J to which E belongs. The collection G is said to be *J-sigma-positive* whenever $\{\bar{E}: E \text{ is in } G\}$ is a sigma-positive subset of B/J.

PROPOSITION 3.7. Let B be a sigma-algebra and let J be a sigma-ideal in B. Let $\bar{E}$ denote the equivalence class modulo J of E for each E in B. If m is a measure on B such that $m(E) = 0$ for each E in J, and $\bar{m}(\bar{E}) = m(E)$ for each E in B, then $\bar{m}$ is a measure on B/J. Conversely, if $\bar{m}$ is a measure on B/J, and $m(E) = \bar{m}(\bar{E})$ for each E in B, then m is a measure on B such that $m(E) = 0$ for each E in J.

The above proposition is well known. The proof is a trivial alteration of a proof in [10], p. 65.

DEFINITION 3.8. If B is a Boolean sigma-algebra, J is a sigma-ideal in B and m is a measure on B, then m is called *J-null* whenever $m(E) = 0$ for each E in J.

COROLLARY 3-9. *If B is a sigma-algebra, J is a sigma-ideal in B and G is a J-sigma-positive subset of B, then there is a J-null measure on B and a positive real number a such that $m(A) \geqq a$ for each A in G.*

Proof. Let $\bar{G} = \{\bar{E}: E \text{ is in } G\}$. Then $\bar{G}$ is a sigma-positive subset of B/J. By 3.5, there is a measure $\bar{m}$ on B/J such that $\bar{m}(\bar{E}) \geqq I(\bar{G})$ for each E in G. Let $A = I(\bar{G})$ and let $m(E) = \bar{m}(\bar{E})$ for each E in B. By 3.6, $\bar{m}$ is a measure on B which vanishes on J. If E is in G, then $\bar{E}$ is in $\bar{G}$ and $m(E) = \bar{m}(\bar{E}) \geqq a$.

THEOREM 3.10. *Let X be a topological space, B be a sigma-field of subsets of X which contains a basis G of the topology of X, and J be a sigma-ideal in B which contains no open sets. Then there is a J-null measure m on B such that $m(U) > 0$ for each nonempty* *open subsets U of X if and only if $G\backslash\{\phi\}$ is contained in the union of a countable family of J-sigma-positive sub-collections of B.*

Proof. Suppose that m is a J-null measure on B which is positive on $G\backslash\{\phi\}$. For each n, let $B_n = \{E \text{ in } B\colon m(E) > (1/n)\}$. For each E in B, let $\bar{E}$ denote the equivalence class modulo J of E. Let $\bar{B}_n = \{\bar{E}\colon E \text{ is in } B_n\}$. By 3.7, if $\bar{m}(\bar{E}) = m(E)$ for each E in B, then $\bar{m}$ is a measure on B/I. By 3.2, $(1/n) \leqq \inf \bar{m}(\bar{E})\colon \bar{E}$ is in $\bar{B}_n \leqq I(\bar{B}_n)$. Thus, $\bar{B}_n$ is positive. Let $\bar{E}_1 \leqq \bar{E}_2 \leqq \cdots$ be a monotone sequence of elements of B/I such that for each k, and for each $\bar{E}$ in $\bar{B}_n$, it is not true that $\bar{E} \leqq \bar{E}_k$. Then for each k, $\bar{m}(\bar{E}_k) \leqq (1/n)$, which implies

$$\bar{m}\left(\sum_{k=1}^{\infty} \bar{E}_k\right) \leqq (1/n)\ ,$$

and hence it is not true that $\bar{E} \leqq \bigcup_{k=1}^{\infty} \bar{E}_k$ for some $\bar{E}$ in $\bar{B}_n$. Therefore, $\bar{B}_n$ is sigma-positive, and B_n is J-sigma-positive.

Conversely, if $\bigcup_{n=1}^{\infty} B_n$ contains $G\backslash\{\phi\}$ where for each n, B_n is J-sigma-positive, then for each n, there is a measure $\bar{m}_n$ on B/J such that $\bar{m}_n(\bar{E}) \geqq I(\bar{B}_n) > 0$ for each $\bar{E}$ in $\bar{B}_n$. Let $\bar{m} = \sum_{n=1}^{\infty} (1/2^n)\bar{m}_n$. Then for each $\bar{U}$ in $\bar{G}\backslash\{0\}$, $\bar{m}(\bar{U}) > 0$. If $m(E) = \bar{m}(\bar{E})$ for each E in B, then m vanishes on J and m is a positive on $G\backslash\{\phi\}$.

COROLLARY 3.11. *Let X be a compact Hausdorff space, B be the sigma-field of Baire subsets of X, and J be a sigma-ideal in B which contains no open sets. There is a regular Borel measure with support X such that $m(E) = 0$ for each E in J if and only if the collection G of nonempty open Baire subsets of X is contained in the union of a countable family of J-sigma-positive collections of Baire sets.*

Proof. According to 3.10, G is contained in the union of a countable family of J-sigma-positive subcollection if and only if there is a Baire measure m on X such that $m(U) > 0$ for each open Baire set U and $m(E) = 0$ for each E in J. The regular Borel extension of m is the regular Borel measure required.

The final theorem of this section combines the preceding results with those of § 2.

THEOREM 3.13. *Let X be a perfect compact Hausdorff space with topology G and with the property that each point contained in an open separable subset of X is a G_δ-point. Let J be the sigma-ideal of Borel sets which contains no nonempty perfect subsets. There is a nonatomic regular Borel measure whose support is X if and only if G is contained in a countable union of J-sigma-positive collections* *of Borel sets.*

Proof. Suppose that G is contained in a countable union of J-sigma-positive collections. By 3.10, there is a J-null Borel measure m' on X such that $m'(U) > 0$ for each nonempty open subset U of X. Let m'' be the restriction of m' to the Baire sets, and let m be the regular Borel extension of m''. Then m is a regular Borel measure which vanishes on Baire sets which contain no nonempty perfect subsets. Let X_1, X_2, m_1, m_2 be as in 2.3. Suppose that X_2 is nonempty. Then X_2 is separable and since $X_2 = \mathrm{cl}\,\mathrm{int}\,\mathrm{Supp}\,(m_2)$, $\mathrm{int}\,X_2$ is nonempty. It is clear then that $\mathrm{int}\,X_2$ is separable and locally compact in the relative topology. Thus, $\{x\}$ is a G_δ for each x in $\mathrm{int}\,X_2$, and hence $\{x\}$ is a Baire set for each x in $\mathrm{int}\,X_2$. Therefore, $m(\{x\}) = 0$ for each x in $\mathrm{int}\,X_2$. If U is an open subset of $X\backslash X_1$, then since $\mathrm{int}\,X_2$ contains $X\backslash X_1$, $\mathrm{int}\,X_2$ contains U. Since $m(\{x\}) = 0$ for each x in U, $m_2(U) = 0$. Since $U \cap \mathrm{Supp}\,m_2$ is empty, $m_1(U) = 0$. Thus, $m(U) = 0$, which contradicts the fact that m is positive on nonempty open Baire sets. Therefore, X_2 is empty, and $X_1 = \mathrm{Supp}\,(m_1) = X$.

Conversely, if m is a nonatomic regular Borel measure whose support is X, then m is a J-null Borel measure which is positive on nonempty open Borel sets. By 3.10, $G\backslash\{\phi\}$ is contained in the union of a countable family of J-sigma-positive subcollections.

4. **Normal measures and the projective resolution.** This section develops relationships between the regular Borel measures on a compact Hausdorff space X and the regular Borel measures on the projective resolution of X.

In [8], Gleason introduces the concept of the projective resolution of a compact Hausdorff space. If P is a compact Hausdorff space, then P is *extremally disconnected* if and only if the closure of each open subset of P is open. If X and Y are compact Hausdorff spaces, then a *continuous irreducible map* of X onto Y is a continuous function f from X onto Y such that Y is not the image under f of a proper closed subset of X. Gleason's result states: If X is a compact Hausdorff space, then there is an extremally disconnected compact Hausdorff space P (called the *projective resolution* of X) and a continuous irreducible map g (called the *Gleason map*) of P onto X, and if P' is an extremally disconnected compact Hausdorff space such that X is the image of P' under a continuous irreducible map g', then there is a homeomorphism h of P onto P' such that $g'h = g$.

If X is a topological space and U is an open subset of X, then U is called a regular open subset of X if and only if $U = \mathrm{int}\,\mathrm{cl}\,U$.

NOTATION. In this section, if Y is a compact Hausdorff space, then $R(Y)$ denotes the collection of regular open subsets of $Y, B_0(Y)$ denotes the sigma-algebra of Borel subsets of Y, and $I(Y)$ denote the sigma-ideal of meager (i.e., contained in a countable union of nowhere dense sets) Borel subsets of Y.

If Y is a compact Hausdorff space, then the collection $R(Y)$ is a Boolean algebra with the following operations: If U and V are in $R(X)$, then $U \vee V = \operatorname{int}\operatorname{cl}(U \cup V)$, $U \wedge V = U \cap V$, and the complement of U is $Y\backslash \operatorname{cl} U$; $0 = \phi$ and $1 = X$ (for a proof, see [10]). It is useful to note that each Borel subset of Y is equivalent modulo $I(Y)$ to a unique regular open set, and indeed $R(Y)$ is isomorphic to $B_0(Y)/I(Y)$, (see [10], p. 58).

In Gleason's construction, the projective resolution P of a compact Hausdorff space X turns out to be the Stone space of $R(X)$. The following sequence of lemmas throws some light upon the relationships involved.

LEMMA 4.1. *If P and X are compact Hausdorff spaces and g is a continuous irreducible map of P onto X, then for each nonempty open subset U of P, $g(U)$ has nonempty interior.*

Proof. Since U is nonempty, $P\backslash U$ is a proper closed subset of P and by the irreducibility of g, $g(P\backslash U)$ is a proper closed subset of X. If x is in $X\backslash g(P\backslash U)$, p is in P, and $g(p) = x$, then p is not in $P\backslash U$; hence p is in U and x is in $g(U)$. Therefore, $g(U)$ contains the nonempty open set $X\backslash g(P\backslash U)$.

In the remainder of this section, X is a compact Hausdorff space, P is the projective resolution of X, and g is the Gleason map of P onto X.

LEMMA 4.2. *If C is a nonempty clopen subset of P, then*

$$\operatorname{int} g^{-1}[g(C)] = C \quad \textit{and} \quad \operatorname{cl} g^{-1}[\operatorname{int} g(C)] = C\ .$$

Proof. Since P is extremally disconnected and $g(C)$ is closed, $\operatorname{int} g^{-1}[g(C)]$ is clopen. If $D = \operatorname{int} g^{-1}[(C)]\backslash C$ is nonempty, then $P\backslash D$ is a proper clopen subset of P containing C, and $g(P\backslash D) = X$, contradicting the irreducibility of g. Further, $g^{-1}[g(C)]$ contains the open set $g^{-1}[\operatorname{int} g(C)]$, and hence, $C = \operatorname{int} g^{-1}[(C)]$ contains $g^{-1}[\operatorname{int} g(C)]$. Since C is closed, C contains the clopen set $\operatorname{cl} g^{-1}[\operatorname{int} g(C)]$. If $D = C\backslash \operatorname{cl} g^{-1}[\operatorname{int} g(C)]$ is nonempty, then it is clopen, and by (9.1), $g(D)$ nonempty interior. But since D is disjoint from $g^{-1}[\operatorname{int} g(C)]$, $g(D)$ is contained in $g(C)\backslash \operatorname{int} g(C)$.

LEMMA 4.3. *If U is a regular open subset of X, then* $\mathrm{cl}\, g^{-1}(U) = \mathrm{int}\, g^{-1}(\mathrm{cl}\, U)$ *and* $\mathrm{cl}\, g^{-1}(U) \cap \mathrm{cl}\, g^{-1}(X \backslash \mathrm{cl}\, U)$ *is empty.*

Proof. Since $g^{-1}(U)$ is a subset of $g^{-1}(\mathrm{cl}\, U)$, the clopen set $\mathrm{cl}\, g^{-1}(U)$ is a subset of the closed set $g^{-1}(\mathrm{cl}\, U)$, hence, $\mathrm{cl}\, g^{-1}(U)$ is a subset of $\mathrm{int}\, g^{-1}(\mathrm{cl}\, U)$. Let

$$D = \mathrm{int}\, g^{-1}(\mathrm{cl}\, U) \backslash \mathrm{cl}\, g^{-1}(U) .$$

If D is nonempty, then $g(D)$ is contained in $\mathrm{cl}\,(U) \backslash U$ which is a nowhere dense set, that $g(D)$ has nonempty interior by 4.1. Suppose

$$\mathrm{cl}\, g^{-1}(U) \cap \mathrm{cl}\, g^{-1}(X \backslash \mathrm{cl}\, U) = \mathrm{int}\, g^{-1}[\mathrm{cl}\,(U)] \cap \mathrm{int}\, g^{-1}[\mathrm{cl}\,(X \backslash \mathrm{cl}\, U)] = A$$

is nonempty. Since A is clopen, $g(A)$ has nonempty interior, and since $g[\mathrm{int}\, g^{-1}(\mathrm{cl}\, U)]$ is a subset of $g[g^{-1}(\mathrm{cl}\, U)] = \mathrm{cl}\, U$ and

$$g[\mathrm{int}\, g^{-1}(\mathrm{cl}\,(X \backslash \mathrm{cl}\, U))]$$

is a subset of $\mathrm{cl}\,(X \backslash \mathrm{cl}\, U)$, $g(A)$ is a subset of $\mathrm{cl}\,(U) \cap \mathrm{cl}\,(X \backslash \mathrm{cl}\, U)$, a nowhere dense set.

LEMMA 4.4. *If A is a subset of X, then A is nowhere dense if and only if $g^{-1}(A)$ is nowhere dense.*

Proof. If A is nowhere dense, then so is $\mathrm{cl}\, A$. If $g^{-1}(\mathrm{cl}\, A)$ has nonempty interior, then by (4.1), $g[g^{-1}(\mathrm{cl}\, A)] = \mathrm{cl}\, A$ has nonempty interior. Hence, $g^{-1}(\mathrm{cl}\, A)$ is nowhere dense and contains $g^{-1}(A)$. Conversely, if D is a nowhere dense subset of P, then so is $\mathrm{cl}\, D$. Suppose that $\mathrm{cl}\, g(D)$ contains a nonempty open set V. Then $g^{-1}(V)$ is a nonempty open subset of $g^{-1}[\mathrm{cl}\, g(D)]$ which is contained in $g^{-1}[g(\mathrm{cl}\, D)]$. Since $\mathrm{cl}\, D$ contains no nonempty open set, $g^{-1}(V) \backslash \mathrm{cl}\, D$ contains a nonempty clopen set C. Since $g(\mathrm{cl}\, D)$ contains V which contains $g(C)$, $P \backslash g(C)$ maps onto X, which is a contradiction. Hence, $\mathrm{cl}\, g(D)$ is nowhere dense, and $g(D)$ is nowhere dense.

LEMMA 4.5. *If for each regular open subset U of X, $h(U) = \mathrm{cl}\, g^{-1}(U)$, then h is an isomorphism of the Boolean algebra $R(X)$ onto the Boolean algebra of clopen subsets of P.*

Proof. Since P is extremally disconnected and $g^{-1}(U)$ is open, $\mathrm{cl}\, g^{-1}(U)$ is clopen. If C is a nonempty clopen subset of P, then $g(C)$ is closed, and by 2.2, $\mathrm{int}\, g(C)$ is a regular open set. By 4.2,

$$h[\mathrm{int}\, g(C)] = \mathrm{cl}\, g^{-1}[\mathrm{int}\, g(C)] = C .$$

Thus, h is an onto map. If U and V are nonempty regular open subsets of X such that $h(U) = h(V)$, then suppose that $U \neq V$. Then either $U \cap (X \backslash \mathrm{cl}\, V)$ or $V \cap (X \backslash \mathrm{cl}\, U)$ is nonempty (otherwise, $\mathrm{cl}\, U$ contains V, $\mathrm{cl}\, U$ contains $\mathrm{cl}\, V$, $U = \mathrm{int}\,\mathrm{cl}\, U$ contains $\mathrm{int}\,\mathrm{cl}\, V = V$, and likewise V contains U). Suppose $W = U \cap (X \backslash \mathrm{cl}\, V)$ is nonempty. Then W is a regular open set, and $h(W)$ is a nonempty subset of both $h(U)$ and $h(X \backslash \mathrm{cl}\, V)$. But $h(U) = h(V)$, and by 4.3,

$$h(V) \cap h(X \backslash \mathrm{cl}\, V)$$

is empty, which is contradictory. Therefore, $h(U) = h(V)$ and h is one-to-one. To show that h preserves complementation, it is necessary only to show that for each regular open set V, $h(V) \cup h(X \backslash \mathrm{cl}\, V) = P$. If V is a nonempty regular open proper subset of X, then

$$\begin{aligned} \mathrm{cl}\, g^{-1}(V) \cup \mathrm{cl}\, g^{-1}(X \backslash \mathrm{cl}\, V) &= \mathrm{cl}[g^{-1}(V) \cup g^{-1}(X \backslash \mathrm{cl}\, V)] \\ &= \mathrm{cl}[g^{-1}(V \cup X \backslash \mathrm{cl}\, V)] . \end{aligned}$$

Since $X \backslash (V \cup X \backslash \mathrm{cl}\, V)$ is nowhere dense,

$$g^{-1}(X \backslash [V \cup (X \backslash \mathrm{cl}\, V)]) = P \backslash g^{-1}[V \cup (X \backslash \mathrm{cl}\, V)]$$

is nowhere dense, and the desired result follows. Suppose that U and V are nonempty regular open subsets of X. If

$$A = h(U \cap V) = \mathrm{cl}\, g^{-1}(U \cap V) = \mathrm{cl}[g^{-1}(U) \cap g^{-1}(V)] ,$$

then clearly both $h(U)$ and $h(V)$ contain A. Let D be a clopen subset of both $h(U)$ and $h(V)$. Then $h^{-1}(D)$ is contained in both $h^{-1}[h(U)]$ and $h^{-1}[h(V)]$. Thus $U \cap V$ contains $h^{-1}(D)$ and $A = h(U \cap V)$ contains D. Therefore, $A = h(U) \cap h(V)$. Therefore, h is one-to-one, onto, and h perserves complementation and intersection; thus, h is an isomorphism.

THEOREM 4.6. *There is a regular Borel measure whose support is X if and only if there is a regular Borel measure whose support is P.*

Proof. If μ is a regular Borel measure whose support is P, then $\nu(A) = \mu(g^{-1}(A))$ defines a regular Borel measure whose support is X, where g is the continuous irreducible map of P onto X.

Suppose ν is a regular Borel measure whose support is X. Let ϕ be the positive linear functional on $\{f \circ g : f \in C(X)\}$ defined by $\phi(f \circ g) = \nu(f)$. Then by [14], p. 20, ϕ has a positive extension ψ to all of $C(P)$. Let μ denote the regular Borel measure that corresponds to ψ. Then for each Borel set A in X, $\nu(A) = \mu(g^{-1}(A))$. Let U be a nonempty open set on P. The proof of Lemma 4.1 shows that $g^{-1}(X \backslash g(P \backslash U)) \subseteq U$ and if $U \neq P$, then $V = X \backslash g(P \backslash U)$ is nonempty.

Hence, $\mu(g^{-1}(V)) = \nu(V) > 0$. It follows that the support of μ is all of P.

REMARK. Suppose μ and ν are related by $\nu(A) = \mu(g^{-1}(A))$ for all Borel sets A in X. If ν is nonatomic and p is in P, then $\nu(g(p))=0$, and, thus, $0 \leqq \mu(p) \leqq \mu(g^{-1}(g(p))) = \nu(g(p)) = 0$ and μ is nonatomic.

DEFINITION 4.7. If Y is a compact Hausdorff space and m is a regular Borel measure on Y, then m is called a *normal* measure on Y if and only if for each meager Borel subset A of Y, $m(A) = 0$.

This concept applied to extremally disconnected spaces is equivalent to Dixmier's notation of normal measure (see [3]).

LEMMA 4.8. *If m is a normal measure on X, then m restricted to $R(X)$ is a measure on $R(X)$.*

Proof. It suffices to show that m is countably additive on $R(X)$. If $U_1, U_2, \cdots$ is a pairwise disjoint sequence of regular open sets, the least upper bound of this sequence in $R(X)$ is $\operatorname{int}\operatorname{cl}(\bigcup_{n=1}^{\infty} U_n)$. Since the boundary of an open set is nowhere dense,

$$m\Big(\bigcup_{n=1}^{\infty} U_n\Big) = m\Big(\operatorname{cl}\bigcup_{n=1}^{\infty} U_n\Big),$$

and it follows from 2.2 that $m(\operatorname{int}\operatorname{cl}\bigcup_{n=1}^{\infty} U_n) = m(\operatorname{cl}\bigcup_{n=1}^{\infty} U_n)$. Thus, the countable additivity on $R(X)$ follows from the countable additivity on the sigma-algebra of Borel sets.

THEOREM 4.9. *If m is a normal measure on X, then there is a unique normal measure m' on P such that for each Borel subset E of X, $m(E) = m'(g^{-1}[E])$.*

Proof. By 4.8, m defines a measure on $R(X)$. Let h be the isomorphism in 4.5 of $R(X)$ onto $R(P)$, let i be the isomorphism of $R(P)$ onto $B_0(P)/I(P)$, and let j be the natural homomorphism of $B_0(P)$ onto $B_0(P)/I(P)$. For each Borel subset F or P, let $m'(F) = m(h^{-1}i^{-1}j)(F)$. Clearly, $m'(F) = 0$ whenever F is a meager Borel subset of P. For each Borel subset E of X, let $f(E) = g^{-1}(E)$. By elementary properties of functions (see [12], p. 11), f is an isomorphism of $B_0(X)$ onto $B_0(P)$. If E is a Borel subset of X, then E is equivalent modulo $I(X)$ to a regular open subset U of X. Since the symmetric difference $U + E$ is in $I(X)$, by (4.4), $f(E + U) = f(E) + f(U)$ is in $I(P)$, and since $h(U) = \operatorname{cl} f(U)$, $h(U)$ is equivalent to $f(E)$ modulo $I(P)$. Hence, $j[f(E)]$ is the equivalence class of $h(U)$ and

$$m'[f(E)] = m(h^{-1}i^{-1}jf)(E) = m(U) = m(E) .$$

It remains to show that m' is a regular Borel measure. Suppose e is a positive real number and F is a Borel subset of P. There is a clopen set C such that the symmetric difference $C + F$ is meager. Hence, $F C$ is meager, and there are nowhere dense Borel sets $D_1, D_2, \cdots$ such that $F \backslash C = \bigcup_{n=1}^{\infty} D_n$. Now clearly,

$$g\Big(C \cup \bigcup_{n=1}^{\infty} \operatorname{cl} D_n\Big) = g(C) \cup \bigcup_{n=1}^{\infty} g(\operatorname{cl} D_n) = E$$

is a Borel subset of X, and $m(E) = m[\operatorname{int} g(C)] = m'(C)$. Since m is a regular Borel measure, there is an open set U containing E such that $m(U) < m(E) + e$. Thus,

$$\begin{aligned} m'(F) &= m'(C \cup F\backslash C) = m'\Big(C \cup \bigcup_{n=1}^{\infty} \operatorname{cl} D\Big) \leqq m'[g^{-1}(E)] = m(E) \\ &= m'(C) \leqq m(U) = m'[g^{-1}(U)] < m(E) + e = m'(E) + e . \end{aligned}$$

Since U contains

$$g(C) \cup g(F\backslash C) = g(C \cup F\backslash C) = g(C \cup F) ,$$

$g^{-1}(U)$ contain F. The uniqueness follows from the fact that the sigma-algebra of Baire subsets of P is generated by clopen subsets (see [12], p. 999). If $m''[g^{-1}(E)] = m(E)$ for each Borel set E, then m'' agrees with m' on the clopen subsets of P, and hence on the Baire subsets of P. By [11], p. 239, the regular Borel extension of m' restricted to the Baire subsets is unique.

COROLLARY 4.10. *There is a one-to-one, order and norm preserving correspondence between the collection $N(X)$ of normal measures on X and the collection $N(P)$ of normal measures on P.*

Proof. The mapping g of P onto X defines a continuous linear norm decreasing mapping $\hat{g}$ of $M(P)$ onto $M(X)$ such that for each Borel subset E of X, $\hat{g}(m)(E) = m[g^{-1}(E)]$ (see [15], p. 180). If $\hat{g}$ is restricted to $N(P)$, then the correspondence of the preceding theorem shows that $\hat{g}$ maps $N(P)$ one-to-one, onto $N(X)$. The preservation of order and norm is immediate.

COROLLARY 4.11. *There is a one-to-one, order and norm preserving correspondence between the collection of normal measures whose support is X and the collection of normal measures whose support is P.*

Proof. Let $\hat{g}$ be the mapping defined in the previous corollary.

If m is a normal measure whose support is P, then $\hat{g}(m)$ is a normal measure. If U is a nonempty open subset of X, then $g^{-1}(U)$ is a nonempty open subset of P and $\hat{g}(m)(U) = m[g^{-1}(U)] > 0$. If m is a normal measure on X and U is a nonempty open subset of P, then U contains a nonempty clopen set C and by (4.2), $\operatorname{cl} g^{-1}[\operatorname{int} g(C)] = C$. Thus, $g^{-1}(m)(U) \geqq g^{-1}(m)(C) \geqq g^{-1}(m)[g^{-1} \operatorname{int}\{g(C)\}] = m[\operatorname{int} g(C)] > 0$.

Recall that each signed regular Borel measure m on a compact Hausdorff space Y may be written as $m = m^+ - m^-$. If Y is a compact Hausdorff space and m_1 and m_2 are normal measures on Y, then $m = m_1 - m_2$ is a signed regular Borel measure on Y, and since $m_1 \geqq m^+ \geqq 0$ and $m_2 \geqq m^- \geqq 0$, m^+ and m^- are also normal measures on Y. Thus, $\{m: m$ is a signed regular Borel measure on Y such that m^+ and m^- are normal$\} = \{m: m = m_1 - m_2$ where m_1 and m_2 are normal measures on Y. This collection is called the collection of signed normal measures on Y and is denoted by $M'(Y)$. It is easy to show that $M'(Y)$ is a closed linear subspace $M(Y)$ for each compact Hausdorff space Y. It also follows easily from 4.10 that $M'(X)$ and $M'(P)$ are linearly isometric. For a different proof of this, see [2].

The following theorem summarizes some of the results in this section in combination with some results from the theory of measure algebras.

THEOREM 4.12. *The following statements are equivalent.*

(i) *There is a normal measure whose support is X.*

(ii) *There is a normal measure whose support is P.*

(iii) *There is a strictly positive (countably additive) measure of* $R(X)$ (resp. $R(P)$).

(iv) *There is a regular Borel measure on P whose support is P and every meager set in P is nowhere dense.*

(v) *There is a regular Borel measure on X whose support is X and every meager set in X is nowhere dense.*

Proof. Theorem 4.6 and 4.9 yield that (i) is equivalent to (ii). Since $R(P)$, $R(X)$, $B_0(X)/I(X)$, and $B_0(P)/I(P)$ are all isomorphic under an isomorphism that preserves countable operations, (i) equivalent to (iii) and (ii) is equivalent to (iii). Suppose there is a normal measure μ on P with support P. Let $B \subset P$ be a meager set. Then $\mu(B) = \mu(\operatorname{int} \operatorname{cl} B)$ and hence B is nowhere dense (see [3], p. 158). Thus (ii) implies (iv).

Suppose (iv) is true. Then by Theorem 4.6 there is a regular Borel measure whose support is X. If $B \subset X$ is meager, then by Lemma 4.4 $g^{-1}(B)$ is meager and thus $g^{-1}(B)$ is nowhere dense. Hence, by Lemma 4.4 again, B is nowhere dense and (iv) implies (v). A similar argument shows that (v) implies (iv). For, by Theorem 4.6 again, if there is a regular Borel measure whose support is X, then there is one whose support is P. Suppose $B \subset P$ is closed and nowhere dense. Then $g(B)$ is closed and nowhere dense. Thus, if $B \subset P$ is meager, then $g(B)$ is meager. It follows that if each meager set in X is nowhere dense, then each meager set in P is nowhere dense. Finally, (iv) implies (iii). For, in [13] Kelley shows that if a complete Boolean algebra A has a strictly positive finitely additive measure and if each meager set in the Stone space of A in nowhere dense, then A has a strictly positive countably additive measure. Thus, if μ is a regular Borel measure whose support is P and each meager set in P is nowhere dense, then μ induces a finitely additive strictly positive measure on $R(P)$, $R(P)$ is a complete Boolean algebra, the Stone space of $R(P)$ is P, and by the above cited result, $R(P)$ has a countably additive strictly positive measure.

BIBLIOGRAPHY

1. G. Birkhoff, *Lattice theory*, Am. Math. Soc. Colloquium Publications, Vol. 25, Providence, Rhode Island, 1961.

2. H. B. Cohen and H. E. Lacey, *On injective envelopes of Banach spaces*, Proceedings of the Conference on Projections and Related Topics, Clemson University, 1968, pp. 27-40. To appear in J. Final. Anal.

3. J. Dixmier, *Sur certains espaces consideres par M. H. Stone*, Sum. Bras. Math. **2** (1951), 151-181.

4. N. Dunford and J. T. Schwartz, *Linear operator*, Part I, Interscience Publishers, Inc., New York, 1964.

5. D. A. Edwards, *A class of topological Boolean algebras*, Proc. Lon. Math. Soc. (3) **13** (1963), 413-429.

6. R. E. Edwards, *Functional analysis*, Holt, Rinehart and Winston, New York, 1965.

7. H. Gaifman, *Concerning measures on Boolean algebras*, Pacific J. Math. **14** (1964), 61-73.

8. A. M. Gleason, *Projective topological spaces*, Illinois J. Math. **2** (1958), 482-489.

9. A. Grothendieck, *Une characterization vectoriellemetrique des Espaces* L^1, Canad. J. Math. **7** (1955), 552-561.

10. P. R. Halmos, *Lectures on Boolean algebras*, D. Van Nostrand Co., Inc., Princeton, N. J., 1963.

11. ———, *Measure theory*, D. Van Nostrand Co., Inc., Princeton, N. J., 1950.

12. J. L. Kelley, *General topology*, D. Van Nostrand Co., Inc., Princeton, N. J., 1955.

13———, *Measures on Boolean algebras*, Pacific J. Math. **9** (1959), 1165-1176.

14. J. L. Kelley, I. Namioka, and co-authors, *Linear topological spaces*, D. Van Nostrand Co., Inc., Princeton, N. J., 1963.

15. V. L. Klee, Jr., *Convex sets in linear spaces*, Duke Math. J. **18** (1951), 443-466.

16. D. Maharam, *An algebraic characterization of measure algebras*, Ann. of Math. **48** (1947), 154-167.

17. A. Pelczynski and Z. Semadeni, *Spaces of continuous functions III*, Studia Math. **18** (1959), 211-222.

18. R. Sikorski, *Boolean algebras*, second edition, Academic Press. Inc., New York, 1964.

Received July 19, 1967. The second author held a Resident Post-Doctoral Research Associateship at NASA-MSC Houston, Texas, during the preparation of part of this paper.

UNIVERSITY OF TEXAS AT AUSTIN