ON THE MULTIPLICATIVE PROPERTIES OF ARITHMETIC FUNCTIONS

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In this paper we define a generalization of the set $\mathscr{M}$ of all pairs of relatively prime natural numbers and then define a limit process to measure the multiplicativity of an arithmetic function with respect to this generalized set of pairs. In so doing we gain useful information about that most important special case, namely, functions which are multiplicative in the usual sense.

2. **Preliminary definitions and results.** By an *arithmetic function* we shall mean a real-valued function f whose domain is the set of natural numbers. We will deal only with arithmetic functions and, furthermore, we will assume throughout this paper that no function is eventually zero; that is, given any arithmetic function f and any number N, there is a natural number $k \geq N$ such that $f(k) \neq 0$.

Closely connected with the multiplicative properties of an arithmetic function is the concept of a basic sequence, which is defined as follows: A *basic sequence* is a set $\mathscr{B}$ of pairs of natural numbers (a, b) with the following three properties:

(i) If $(a, b) \in \mathscr{B}$, then $(b, a) \in \mathscr{B}$;

(ii) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$;

(iii) $(1, k) \in \mathscr{B}, k = 1, 2, 3, \cdots$.

We denote by $B_k (k = 1, 2, \cdots)$ the set of pairs $(a, b) \in \mathscr{B}$ such that $ab = k$.

Let an arithmetic function f and a basic sequence $\mathscr{B}$ be given. In order to measure the multiplicativity of f with respect to $\mathscr{B}$ we first define

$$\alpha_f(m, n) = \begin{cases} \dfrac{f(m)\,f(n) - f(mn)}{|f(m)\,f(n)| + |f(mn)|} \\ \qquad\qquad \text{if} \quad |f(m)\,f(n)| + |f(mn)| > 0 , \\ \qquad 0 \\ \qquad\qquad \text{if} \quad f(m)\,f(n) = f(mn) = 0 . \end{cases}$$

Next we set

$$\bar{\alpha}(k; f, \mathscr{B}) = \max \{\alpha_f(m, n) \mid (m, n) \in B_k\} ,$$

$$\underline{\alpha}(k; f, \mathscr{B}) = \min \{\alpha_f(m, n) \mid (m, n) \in B_k\} ,$$

(2.1)
$$\bar{\alpha}(f, \mathscr{B}) = \limsup_{k \to \infty} \bar{\alpha}(k; f, \mathscr{B}) ,$$

$$\underline{\alpha}(f, \mathscr{B}) = \liminf_{k \to \infty} \underline{\alpha}(k; f, \mathscr{B}) .$$

Finally, we define the *index of multiplicativity* of f *with respect*

to the basic sequence $\mathscr{B}$ (or, simply, the *index* of f) to be the common value of $\underline{\alpha}(f, \mathscr{B})$ and $\bar{\alpha}(f, \mathscr{B})$, if such a common value exists. In this case we denote the index of f by $I(f, \mathscr{B})$ or, simply, by $I(f)$.

Many of the multiplicative properties of f with respect to $\mathscr{B}$ will be independent of the sign of $I(f, \mathscr{B})$. Accordingly, we define

$$
\begin{aligned}
\beta_f(m, n) &= |\alpha_f(m, n)|, \\
\bar{\beta}(k; f, \mathscr{B}) &= \max\{\beta_f(m, n) \mid (m, n) \in B_k\}, \\
\underline{\beta}(k; f, \mathscr{B}) &= \min\{\beta_f(m, n) \mid (m, n) \in B_k\}, \\
\bar{\beta}(f, \mathscr{B}) &= \limsup_{k \to \infty} \bar{\beta}(k; f, \mathscr{B}), \\
\underline{\beta}(f, \mathscr{B}) &= \liminf_{k \to \infty} \underline{\beta}(k; f, \mathscr{B}).
\end{aligned}
$$

(2.2)

$J(f, \mathscr{B})$, the *absolute index* of f, is defined to be the common value of $\bar{\beta}(f, \mathscr{B})$ and $\underline{\beta}(f, \mathscr{B})$, if such a common value exists.

When no confusion is possible we will write $I(f)$, $J(f)$, for $I(f, \mathscr{B})$, $J(f, \mathscr{B})$, respectively. Occasionally we will talk about $I(f, \mathscr{B})$ or $J(f, \mathscr{B})$ without prefacing our discussion with a statement (which should then be supplied by the reader) such as "if $I(f, \mathscr{B})$ exists."

A necessary and sufficient condition that $I(f, \mathscr{B})$ exists and has the value φ is that $\lim_{k \to \infty} \alpha_f(m_k, n_k) = \varphi$ for every collection of pairs $\{(m_k, n_k)\}_{k=1}^{\infty} \subset \mathscr{B}$ such that $\lim_{k \to \infty} m_k n_k = \infty$. A similar statement holds for $J(f, \mathscr{B})$. If $I(f, \mathscr{B})$ exists, then $J(f, \mathscr{B})$ also exists and $J(f, \mathscr{B}) = |I(f, \mathscr{B})|$. Without additional assumptions, however, the existence of $J(f, \mathscr{B})$ does not imply the existence of $I(f, \mathscr{B})$, except in one case: $J(f, \mathscr{B}) = 0$ if and only if $I(f, \mathscr{B}) = 0$.

EXAMPLE 2.1. Let $\mathscr{S} = \bigcup_{k=1}^{\infty} S_k$ where $S_k = \{(1, k), (k, 1)\}$. If $f(1) = M > 1$ and $f(k) = (-1)^k$ for $k > 1$, then $J(f, \mathscr{S}) = (M-1)/(M+1)$ but $I(f, \mathscr{S})$ does not exist.

A number of consequences follow readily from the preceding definitions.

LEMMA 2.2. (a) *If there exists an increasing sequence $\{k_n\}_{n=1}^{\infty}$ of natural numbers such that $f(k_n) = 0$, then $\underline{\beta}(f, \mathscr{B}) = 0$ (in particular, if $J(f, \mathscr{B})$ exists and is not zero, then $f(k) \neq 0$ for all sufficiently large k).*

(b) *If $\underline{\alpha}(f, \mathscr{B}) > 0$ $[\bar{\alpha}(f, \mathscr{B}) < 0]$, then*

$$f(m)f(n) > f(mn) \qquad [f(m)f(n) < f(mn)]$$

provided mn is sufficiently large and $(m, n) \in \mathscr{B}$.

LEMMA 2.3. *If* $\bar{\beta}(f, \mathscr{B}) < 1$, *then there exists an integer* N *such that, for each* $k \geqq N$, *exactly one of the following must hold:*

(1) $f(k) = 0$ *and* $f(m)f(n) = 0$ *for every* $(m, n) \in B_k$,

(2) $f(m)f(n) \neq 0$ *for any* $(m, n) \in B_k$.

Moreover, if (2) *holds, then* $f(m)f(n)$ *has the same sign as* $f(k)$ *for every* $(m, n) \in B_k$.

LEMMA 2.4. (a) $f(1) \leqq 0$ *implies* $\bar{\beta}(f, \mathscr{B}) = 1$. *Thus, if* $J(f)$ *exists,* $f(1) \leqq 0$ *if and only if* $J(f) = 1$.

(b) *Suppose* $I(f)$ *exists and* $f(1) > 1$; *then* $I(f) > 0[I(f) < 0]$ *if and only if* $f(k) > 0[f(k) < 0]$ *for all sufficiently large* k.

(c) *Suppose* $I(f)$ *exists and* $f(1) < 1$; *then* $I(f) > 0[I(f < 0]$ *if and only if* $f(k) < 0[f(k) > 0]$ *for all sufficiently large* k.

Combining some of the results of the previous lemmas, we have

THEOREM 2.5. (1) *If* $J(f)$ *exists then*

$$J(f) = \frac{|\,f(1) - 1\,|}{|\,f(1)\,| + 1} \; ;$$

in particular, $J(f) = 0$ *if and only if* $f(1) = 1$;

(2) *If* $I(f)$ *exists and is not zero, there is an integer* N *such that*

(a) $f(k) \neq 0$ *for all* $k \geqq N$, *and*

(b) $f(k)$ *does not change sign for* $k \geqq N$.

We see from Example 2.1 that part (2) above does not hold if we replace $I(f)$ by $J(f)$. Under a slightly stronger hypothesis we are able to determine the sign of $f(k)$ for large values of k.

LEMMA 2.6. *If* $f(k) < 0$ *for all* $k \geqq N$ *and* $\bar{\beta}(f, \mathscr{B}) < 1$, *then* $\mathscr{B} = \mathscr{S}$.

Proof. The lemma follows easily from Lemma 2.3.

As an immediate consequence of this lemma and Theorem 2.5(2) we have

COROLLARY 2.6.1. *Suppose that* $0 < |I(f, \mathscr{B})| < 1$. *If* $\mathscr{B} \neq \mathscr{S}$, *then* $f(k) > 0$ *for all sufficiently large* k.

EXAMPLE 2.7. Let $\mathscr{B}$ be the basic sequence consisting of $\mathscr{S}$ together with all pairs of the form $(2^a, 3^b)$, $a \geqq 1$, $b \geqq 1$. Let $f(k) = M(M > 0, M \neq 1)$ for $k = 2^a 3^b$, $a \geqq 0$, $b \geqq 0$, and let $f(k) = -1$ otherwise. Then $\mathscr{B} \neq \mathscr{S}$ and $0 < J(f, \mathscr{B}) < 1$, but $f(k)$ keeps changing sign. Hence Corollary 2.6.1 is not valid with $I(f, \mathscr{B})$

replaced by $J(f, \mathscr{B})$.

The next lemma and the theorem following, which will conclude this section, relate the existence of $J(f, \mathscr{B})$ to the behavior of $f(m)f(n)/f(mn)$. We will deal with sequences $\{\rho_\nu\}_{\nu=1}^{\infty}$ where

$$(2.3) \qquad \rho_\nu = \frac{f(m_\nu)f(n_\nu)}{f(m_\nu n_\nu)}, \qquad (m_\nu, n_\nu) \in \mathscr{B},$$

$$\lim_{\nu \to \infty} m_\nu n_\nu = \infty, \qquad f(m_\nu n_\nu) \neq 0.$$

LEMMA 2.8. *If* $\bar{\beta}(f, \mathscr{B}) < 1$ *there exist constants* N, M_1, M_2 *such that*

$$0 < M_1 < \frac{f(m)f(n)}{f(mn)} < M_2$$

whenever $(m, n) \in \mathscr{B}$, $mn \geq N$, $f(mn) \neq 0$.

THEOREM 2.9. *If* (1) $f(k) \neq 0$ *for* $k \geq N$ *and* (2) *if every sequence* $\{\rho_\nu\}$ *satisfying* (2.3) *may be decomposed into two subsequences* $\{\varphi_\nu\}$, $\{\psi_\nu\}$ *(one of which may be empty) such that*

$$(2.4) \qquad \lim_{\nu \to \infty} \varphi_\nu = f(1), \qquad \lim_{\nu \to \infty} \psi_\nu = \frac{1}{f(1)},$$

then $J(f, \mathscr{B})$ *exists. Conversely, if* $J(f, \mathscr{B})$ *exists and* $J(f, \mathscr{B}) < 1$, *then every sequence* $\{\rho_\nu\}$ *satisfying* (2.3) *allows a decomposition (where, again, one subsequence may be empty) such that* (2.4) *holds.*

Proof. The first part of the theorem follows directly from definitions (2.2), the second part from Lemma 2.4, Theorem 2.5, and Lemma 2.8.

3. **The basic sequence.** It is clear that the size of the basic sequence $\mathscr{B}$ will play a crucial role in the existence of $I(f, \mathscr{B})$. For example, $I(f, \mathscr{S})$ will exist (and have the value 0) whenever $f(1) = 1$, while at the other extreme, the existence of $I(f, \mathscr{S})$, where $\mathscr{S}$ is the basic sequence consisting of all pairs (m, n) of natural numbers, imposes severe demands upon f.

For any finite set S, let us denote by $\sharp S$ the number of distinct elements in S. In order to make more precise the idea of "size" of a basic sequence $\mathscr{B}$, we make the following definitions: For a basic sequence $\mathscr{B} \equiv \bigcup_{k=1}^{\infty} B_k$, set

$$\delta_n(\mathscr{B}) = \frac{1}{n} \sum_{k=1}^{n} \frac{\sharp B_k}{d(k)},$$

where $d(k)$ represents the number of divisors of k,

$$\bar{\delta}(\mathscr{B}) = \limsup_{n\to\infty} \delta_n(\mathscr{B}) \, ,$$

$$\underline{\delta}(\mathscr{B}) = \liminf_{n\to\infty} \delta_n(\mathscr{B}) \, .$$

$\bar{\delta}(\mathscr{B})$ and $\underline{\delta}(\mathscr{B})$ will be called the *upper density* and *lower density* of $\mathscr{B}$, respectively. If the upper and lower densities have a common value, we will define the *density* of $\mathscr{B}$ to be that common value, and denote it by $\delta(\mathscr{B})$.

EXAMPLE 3.1. Let $\mathscr{M} \equiv \bigcup_{k=1}^{\infty} M_k$ be the basic sequence consisting of all pairs of natural numbers (a, b) for which a and b are relatively prime. We assert that $\underline{\delta}(\mathscr{M}) \geqq 6/\pi^2$. It is clear that $^\#M_k = 2^{\omega(k)}$ where $\omega(k)$ is the number of distinct prime divisors of k, and so $^\#M_k = d(k)$ whenever k is square-free. By [1, Th. 333], the number of square-free integers not exceeding x is $6x/\pi^2 + O(\sqrt{x})$. Hence

$$\sum_{k=1}^{n} \frac{^\#M_k}{d(k)} \geqq \sum_{\substack{k \leqq n \\ k \text{ square-free}}} 1 = 6n/\pi^2 + O(\sqrt{n}) \, ,$$

from which the assertion follows.

The above estimate for $\delta(\mathscr{M})$ is considerably too low and will be improved shortly.

For any set S of real numbers, let us define $S(k) = \{x \mid x \in S, x \leqq k\}$.

LEMMA 3.2. *If $\{b_k\}$ is a bounded sequence of real numbers, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{b_k}{d(k)} = 0 \, .$$

In particular, if $^\#B_k < M$ $(k = 1, 2, 3, \cdots)$, then $\delta(\mathscr{B}) = 0$.

Proof. For a fixed ε, $0 < \varepsilon < 1$, let A be the set of positive integers k for which it is *not* true that $2^{(1-\varepsilon)\log\log k} < d(k) < 2^{(1+\varepsilon)\log\log k}$. By [1, p. 359, Th. 432], $^\#A(n) = o(n)$. Hence

$$\frac{1}{n} \sum_{k \leqq n} \frac{1}{d(k)} = \frac{1}{n} \left(\sum_{\substack{k \leqq n \\ k \notin A}} \frac{1}{d(k)} + \sum_{k \in A(n)} \frac{1}{d(k)} \right)$$

$$\leqq \frac{1}{n} \sum_{\substack{k \leqq n \\ k \notin A}} 2^{-(1-\varepsilon)\log\log k} + \frac{^\#A(n)}{2n}$$

$$= \frac{1}{n} \sum_{\substack{k \leqq n \\ k \notin A}} (\log k)^{-(1-\varepsilon)\log 2} + o(1) \, .$$

The lemma now follows from the fact that if $c > 0$,

$$\sum_{\substack{k \le n \\ k \notin A}} (\log k)^{-c} \le \int_3^n (\log x)^{-c} dx + O(1) = O(n(\log n)^{-c}) \ .$$

From Corollary 2.6.1. and the above lemma we have

COROLLARY 3.2.1. *If* $0 < |I(f, \mathscr{B})| < 1$ *and* $\bar{\delta}(\mathscr{B}) > 0$, *then*
$f(k) > 0$ *for all sufficiently large* k.

EXAMPLE 3.3. Let $\mathscr{T}_m \equiv \bigcup_{k=1}^\infty T_k$ consist of the basic sequence
$\mathscr{S}$ together with all pairs of the form (m^a, m^b), $a \ge 1$, $b \ge 1$. Then
${}^{\#}T_{m^a} = a + 1$ but $\delta(\mathscr{T}_m) = 0$. Hence the boundedness of $\{{}^{\#}B_k\}$ is
sufficient to make $\delta(\mathscr{B}) = 0$, but it is not necessary.

Let us turn next to a more detailed discussion of the relationship
between the structure of a basic sequence $\mathscr{B}$ and its density. Suppose
that Φ is a set of pairs of natural numbers. We define the *basic
sequence generated by* Φ to be $\Gamma(\Phi) = \bigcap \mathscr{C}$ where the intersection is
taken over all basic sequences $\mathscr{C}$ which contain Φ. We will call a
pair (m, n) a *primitive pair* if both m and n are primes. If $m \ne n$,
then the primitive pair (m, n) is said to be of *type* I, otherwise it is
a primitive pair of *type* II.

Suppose that we represent m and n canonically: $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$,
$n = q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s}$. Then it is clear that (m, n) is a member of $\mathscr{B}$ if
and only if the primitive pairs (p_i, q_j) are all in $\mathscr{B}(i = 1, \cdots, r; j = 1, \cdots, s)$. Therefore, a basic sequence $\mathscr{B}$ is completely determined
by its subset of primitive pairs: $\mathscr{B} = \Gamma(\mathscr{B} \cap \varDelta)$, where $\varDelta$ represents
the set of all primitive pairs. In view of the preceding remark it
seems reasonable to expect that we should be able to make a statement
about $\delta(\mathscr{B})$ if we have enough information about the primitive pairs
in $\mathscr{B}$. To pursue this further we need some preliminary results.

Let A be a set of distinct natural numbers. We define the
asymptotic density of A to be $\underline{D}(A) = \liminf_{n \to \infty} {}^{\#}A(n)/n$. If ${}^{\#}A(n)/n$
has a limit as $n \to \infty$, we call that limit the *natural density* of A and
denote it by $D(A)$. The following lemma is a standard result (see, for
example, [2] p. 228–230) which we state here for convenience.

LEMMA 3.4. *Let* N *be a fixed positive integer. If each member
of* A *has* N *or fewer distinct prime factors, then* $D(A) = 0$.

We will say that a prime p is *finitely distributed* with respect
to a basic sequence $\mathscr{B}$ if p appears as a co-member in only a finite
number of primitive pairs of $\mathscr{B}$. Also, p is said to have *property
F* with respect to $\mathscr{B}$ if

(1) p is finitely distributed with respect to $\mathscr{B}$, and

(2)   every prime q with which p appears as a co-member of a
primitive pair in $\mathscr{B}$ is also finitely distributed with respect to $\mathscr{B}$.

THEOREM 3.5. *If there is a finite set $\{p_1, p_2, \cdots, p_M\}$ of distinct primes each of which has property F with respect to the basic sequence $\mathscr{B}$, then*

$$(3.1) \qquad \bar{\delta}(\mathscr{B}) \leqq \prod_{i=1}^{M} \left(1 - \frac{1}{p_i}\right).$$

Proof. We will prove the theorem for the case when $M = 2$, since this contains all the essentials of the general case. First we write

$$\frac{1}{n} \sum_{k=1}^{n} \frac{{}^{\sharp}B_k}{d(k)} = \frac{1}{n} (\Sigma_1 + \Sigma_2 + \Sigma_3) \frac{{}^{\sharp}B_k}{d(k)} ,$$

where Σ_1 is taken over all $k \leqq n$ such that ${}^{\sharp}B_k = 2$ and either $p_1 \mid k$ or $p_2 \mid k$; Σ_2 is taken over those $k \leqq n$ for which ${}^{\sharp}B_k > 2$ and such that either $p_1 \mid k$ or $p_2 \mid k$; and Σ_3 is taken over the remaining $k \leqq n$.

By Lemma 3.2, $\Sigma_1 \, {}^{\sharp}B_k/d(k) = o(n)$. To estimate Σ_2 we let A represent the set of all positive integers t for which ${}^{\sharp}B_t > 2$ and either $p_1 \mid t$ or $p_2 \mid t$. For a fixed member k in A, there are integers m and n with $m > 1$, $n > 1$, $mn = k$ and $(m, n) \in \mathscr{B}$. Since either p_1 or p_2 divides mn, let us assume that $p_1 \mid m$. Suppose that $q_1, q_2, \cdots, q_r$ are the only primes that appear as a co-member with p_1 in a pair in $\mathscr{B}$. Then $n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$, $a_i \geqq 0 (i = 1, 2, \cdots, r)$, and not all the a_i are zero. Suppose also that there are N_i primes which appear as co-member with $q_i (i = 1, 2, \cdots, r)$. If a prime q divides m, then q is a co-member with q_i for at least one $1 \leqq i \leqq r$. Hence there are at most $N_1 + N_2 + \cdots + N_r$ possible choices for q and therefore not more than $r + N_1 + N_2 + \cdots + N_r$ distinct prime factors in k. Similarly, if $p_2 \mid k$, we again get an upper bound for the number of distinct prime factors of k. It follows then from Lemma 3.4 that $D(A) = 0$. Hence

$$\lim_{n \to \infty} \frac{1}{n} \Sigma_2 \frac{{}^{\sharp}B_k}{d(k)} = \lim_{n \to \infty} \frac{1}{n} \sum_{k \in A(n)} \frac{{}^{\sharp}B_k}{d(k)} \leqq \lim_{n \to \infty} \frac{{}^{\sharp}A(n)}{n} = 0 .$$

Finally, since ${}^{\sharp}B_k \leqq d(k)$,

$$(3.2) \qquad \Sigma_3 \frac{{}^{\sharp}B_k}{d(k)} \leqq \sum_{\substack{k \leqq n \\ p_1 \nmid k, p_2 \nmid k}} 1 = n - \left(\left[\frac{n}{p_1}\right] + \left[\frac{n}{p_2}\right] - \left[\frac{n}{p_1 p_2}\right]\right).$$

Therefore

$$\bar{\delta}(\mathscr{B}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k \leqq n} \frac{{}^{\sharp}B_k}{d(k)}$$

$$(3.3) \qquad \leqq \limsup_{n \to \infty} \left\{1 - \frac{1}{n}\left(\left[\frac{n}{p_1}\right] + \left[\frac{n}{p_2}\right] - \left[\frac{n}{p_1 p_2}\right]\right)\right\}$$

$$= \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right),$$

which proves (3.1).

COROLLARY 3.5.1. *If $\{p_i\}$ is a sequence of distinct prime numbers such that*

(i) *Σp_i^{-1} diverges,*

(ii) *p_i has property F with respect to a basic sequence $\mathscr{B}$, then* $\delta(\mathscr{B}) = 0$.

COROLLARY 3.5.2. *If $\mathscr{B}$ is a basic sequence such that every prime p is finitely distributed with respect to $\mathscr{B}$, then $\delta(\mathscr{B}) = 0$.*

COROLLARY 3.5.3. *If Φ consists of*

(a) *a finite number of primitive pairs of type I, and*

(b) *any collection of primitive pairs of type II, then $\delta(\Gamma(\Phi)) = 0$.*

If we impose a restriction somewhat more severe than property F on the primes in Theorem 3.5, then the possibility of inequality in (3.1) can be eliminated. In particular, we have

THEOREM 3.6. *If $\{p_1, \cdots, p_M\}$ is a finite collection of distinct primes, let $\mathscr{B}[p_1, p_2, \cdots, p_M]$ denote the basic sequence $\mathscr{B} \equiv \bigcup_{k=1}^{\infty} B_k$ such that*

(i) *$^{\#}B_k = 2$ if $p_i \mid k$ for at least one $1 \leq i \leq M$, and*

(ii) *$^{\#}B_k = d(k)$ if $p_i \nmid k$, $i = 1, 2, \cdots, M$.*

Then

$$\delta(\mathscr{B}[p_1, \cdots, p_M]) = \prod_{i=1}^{M} \left(1 - \frac{1}{p_i}\right).$$

Proof. We remark first that condition (i) is equivalent to the fact that $p_i(i = 1, \cdots, M)$ does not appear in any primitive pair in $\mathscr{B}$, certainly a severe form of property F.

To prove the theorem we merely note that the inequality in (3.2) becomes an equality.

COROLLARY 3.6.1. *For any $0 \leq \gamma \leq 1$, there is a sequence $\{\mathscr{B}_m\}_{m=1}^{\infty}$ of basic sequences such that $\lim_{m \to \infty} \delta(\mathscr{B}_m) = \gamma$.*

Proof. If $\gamma = 1$, let $\mathscr{B}_m = \mathscr{S}$ $(m = 1, 2, \cdots)$. If $0 \leq \gamma < 1$, choose a sequence of distinct primes $\{p_i\}$ such that

$$\lim_{m \to \infty} \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right) = \gamma, \quad \text{and let} \quad \mathscr{B}_m = \mathscr{B}[p_1, p_2, \cdots, p_m].$$

Theorem 3.6 treats the question: what happens if each member of a finite set of primes appears in no primitive pair in $\mathscr{B}$? We may relax this restriction somewhat and ask: what happens if each member of a finite set of primes appears in no type II primitive pair in $\mathscr{B}$?

The answer to this question is given by

THEOREM 3.7. *If* $p_1, p_2, \cdots, p_N$ *are distinct primes such that* $(p_i, p_i) \notin \mathscr{B}$, $i = 1, 2, \cdots, N$, *then*

$$(3.4) \qquad \bar{\delta}(\mathscr{B}) \leq \prod_{i=1}^{N} (1 - \lambda_{p_i}) \,,$$

where

$$\lambda_x = 2 - \frac{1}{x} + 2x\left(1 - \frac{1}{x}\right) \log\left(1 - \frac{1}{x}\right) \qquad \text{for } x > 1 \,.$$

Proof. We shall prove the theorem for the case $N = 2$, since once again this contains all the essentials of the general case.

Let $p_1 = p$ and $p_2 = q$. If (p, p) and (q, q) are not in $\mathscr{B}$, then certainly (ps, pt) and (qs, qt) are also not in $\mathscr{B}$ for any natural numbers s and t. Hence, if $p^2 \mid k$, there are $d(k/p^2)$ pairs (ps, pt) with $p^2 st = k$, and so

$$(3.5) \qquad {}^{\#}B_k \leq d(k) - d(k/p^2) \,.$$

It follows then that

$$\bar{\delta}(\mathscr{B}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{{}^{\#}B_k}{d(k)}$$

$$(3.6) \qquad \leq \limsup_{n \to \infty} \left(1 - \frac{1}{n} \sum_{\nu \leq n/p^2} \frac{d(\nu)}{d(p^2\nu)} - \frac{1}{n} \sum_{\nu \leq n/q^2} \frac{d(\nu)}{d(q^2\nu)} \right.$$

$$\left. + \frac{1}{n} \sum_{\nu \leq n/p^2 q^2} \frac{d(\nu)}{d(p^2 q^2 \nu)}\right) \,,$$

where we have used the fact that any pair counted in both the first and second sums must be of the form (pqs, pqt).

Now if we mean by $a^t \| b$ that $a^t \mid b$ but $a^{t+1} \nmid b$, then

$$\sum_{\nu \leq n/p^2} \frac{d(\nu)}{d(p^2\nu)} = \sum_{i=0}^{K} \sum_{\substack{\nu \leq n/p^2 \\ p^i \| \nu}} \frac{d(\nu)}{d(p^2\nu)}$$

where $K = [\log n/\log p] - 2$. If $\nu = p^i w$ where $p \nmid w$, we have

$$\frac{d(\nu)}{d(p^2\nu)} = \frac{d(p^i)}{d(p^{i+2})} = \frac{i+1}{i+3} \,.$$

Therefore

$$\sum_{\nu \leq n/p^2} \frac{d(\nu)}{d(p^2\nu)} = \sum_{i=0}^{K} \frac{i+1}{i+3} \sum_{\substack{\nu \leq n/p^2 \\ p^i \| \nu}} 1$$

$$= \sum_{i=0}^{K} \frac{i+1}{i+3} \left(\frac{n}{p^{i+2}} - \frac{n}{p^{i+3}} + O(1)\right)$$

$$= \frac{n}{p^2}\left(1 - \frac{1}{p}\right) \sum_{i=0}^{K} \left(\frac{i+1}{i+3}\right) \frac{1}{p^i} + O(\log n) \,.$$

Thus, since

$$\sum_{i=0}^{\infty} \left(\frac{i+1}{i+3} \right) x^i = \frac{2-x}{x^2(1-x)} + \frac{2}{x^3} \log(1-x)$$

if $0 < |x| < 1$, and since

$$\sum_{i=K+1}^{\infty} \left(\frac{i+1}{i+3} \right) \frac{1}{p^i} = O\left(\frac{1}{n} \right),$$

we have

(3.7)
$$\sum_{\nu \leq n/p^2} \frac{d(\nu)}{d(p^2\nu)} = \frac{n}{p^2}\left(1 - \frac{1}{p} \right)\left\{ \left(2 - \frac{1}{p} \right) \Big/ \frac{1}{p^2}\left(1 - \frac{1}{p} \right) \right.$$
$$\left. + 2p^3 \log\left(1 - \frac{1}{p} \right) + O\left(\frac{1}{n} \right) \right\} + O(\log n)$$
$$= n\lambda_p + O(\log n) .$$

Relation (3.7) still remains valid, of course, if we replace p by q.

Next, by reasoning similar to that used above, we may show that

(3.8)
$$\sum_{\nu \leq n/p^2q^2} \frac{d(\nu)}{d(p^2q^2\nu)} = n\lambda_p\lambda_q + O(\log^2 n) .$$

Relation (3.4) now follows from relations (3.6)–(3.8), and so the proof is complete.

REMARK 3.8. With methods like those used in the above theorem, one may gain similar results for type I primitive pairs. One may show, for example, that if $(p, q) \notin \mathscr{B}$ for distinct primes p and q, then

$$\bar{\delta}(\mathscr{B}) \leq 1 - \frac{1}{2}\left(\lambda_p + \frac{1}{p} \right)\left(\lambda_q + \frac{1}{q} \right) + \lambda_p\lambda_q .$$

Thus, since $\bar{\delta}(\mathscr{B}) < 1$ whenever any primitive pair is missing from $\mathscr{B}$, we have

THEOREM 3.9. $\bar{\delta}(\mathscr{B}) = 1$ *if and only if* $\mathscr{B} = \mathscr{L}$.

Just as we were able to sharpen Theorem 3.5 when we knew exactly which primitive pairs were in $\mathscr{B}$, the same knowledge allows us to improve Theorem 3.7. To be precise, we have

THEOREM 3.10.   (a)   *If* $(p_1, p_1), \cdots, (p_N, p_N)$ *are the only primitive pairs not in* $\mathscr{B}$, *then*

$$\delta(\mathscr{B}) = \prod_{i=1}^{N} (1 - \lambda_p) .$$

(b)   *If* $\{(p_i, p_i)\}_{i=1}^{\infty}$ *are the only primitive pairs not in* $\mathscr{B}$, *then*

$$\delta(\mathscr{B}) = \prod_{i=1}^{\infty} (1 - \lambda_{p_i}) \ .$$

In particular, $\delta(\mathscr{A}) = \prod_p (1 - \lambda_p) \approx 0.8191 +$.

Proof. Part (a) follows immediately from the fact that relation (3.6) becomes an equality if (p, p) and (q, q) are the only primitive pairs not in $\mathscr{B}$.

To prove (b), let us denote by $\mathscr{B}^{(x)} \equiv \bigcup_{k=1}^{\infty} B_k^{(x)}$ the basic sequence generated by the set of all primitive pairs except those (p_i, p_i) for which $p_i < x$. Clearly

$$(3.9) \qquad \bar{\delta}(\mathscr{B}) \leqq \bar{\delta}(\mathscr{B}^{(x)}) = \prod_{p_i < x} (1 - \lambda_{p_i})$$

for any x. If we let $x \to \infty$ in (3.9), then we have

$$(3.10) \qquad \bar{\delta}(\mathscr{B}) \leqq \prod_{i=1}^{\infty} (1 - \lambda_{p_i}) \ ,$$

On the other hand, suppose that N is any fixed positive integer and that $n > N$. We may write

$$\underline{\delta}(\mathscr{B}) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k \leqq n} \frac{{}^{\#}B_k}{d(k)} = \liminf_{n \to \infty} \frac{1}{n} (\Sigma_1 + \Sigma_2) \frac{{}^{\#}B_k}{d(k)} \ ,$$

where Σ_1 is taken over those $k \leqq n$ such that $p_i^2 \nmid k$ for any $p_i \geqq N$; Σ_2 is taken over those $k \leqq n$ such that $p_i^2 \mid k$ for at least one $p_i \geqq N$.

Now if k is counted by Σ_1, then ${}^{\#}B_k = {}^{\#}B_k^{(N)}$. Moreover, since each integer counted by Σ_2 is divisible by the square of a prime $p \geqq N$, the number of integers counted by Σ_2 is certainly not more than

$$\frac{n}{N^2} + \frac{n}{(N + 1)^2} + \frac{n}{(N + 2)^2} + \cdots = O\left(\frac{n}{N}\right) .$$

Hence

$$\underline{\delta}(\mathscr{B}) = \liminf_{n \to \infty} \frac{1}{n} \left(\Sigma_1 \frac{{}^{\#}B_k^{(N)}}{d(k)} + \Sigma_2 \frac{{}^{\#}B_k}{d(k)} \right)$$

$$\geqq \liminf_{n \to \infty} \frac{1}{n} \sum_{k \leqq n} \frac{{}^{\#}B_k^{(N)}}{d(k)} + O\left(\frac{1}{N}\right)$$

$$= \prod_{p_i < N} (1 - \lambda_{p_i}) + O\left(\frac{1}{N}\right) .$$

If we let $N \to \infty$, we have

$$(3.11) \qquad \underline{\delta}(\mathscr{B}) \geqq \prod_{i=1}^{\infty} (1 - \lambda_{p_i}) \ .$$

The theorem now follows from (3.10) and (3.11).

4. **Nonsingular functions.** An arithmetic function f will be called *nonsingular* with respect to a basic sequence $\mathscr{B}$ if $I(f, \mathscr{B})$ exists and is zero; the set of all such functions will be denoted by $N(\mathscr{B})$. All other functions will be called *singular* with respect to $\mathscr{B}$. We shall fix our attention on certain subsets of $N(\mathscr{B})$:

(1) An arithmetic function f is said to be *eventually multiplicative* with respect to $\mathscr{B}$, $f \in EM(\mathscr{B})$, if there is an integer N such that $f(m)f(n) = (fmn)$ whenever $(m, n) \in \mathscr{B}$ and $mn \geq N$. When we wish to emphasize the role of the integer N, we will write $f \in EM(\mathscr{B}, N)$.

(2) An arithmetic function f is said to be *multiplicative* with respect to $\mathscr{B}$, $f \in M(\mathscr{B})$, if $f(m)f(n) = f(mn)$ for every $(m, n) \in \mathscr{B}$.

We note that f is multiplicative in the usual sense if $f \in M(\mathscr{M})$ and that f is a character if $f \in M(\mathscr{L})$. Since $f(1) = 1$ if $f \in N(\mathscr{B})$, it is clear that the sets $EM(\mathscr{B}, N)$ and $M(\mathscr{B})$ are identical if $N = 1, 2, 3$, or 4.

Since $M(\mathscr{B}) \subset EM(\mathscr{B}) \subset N(\mathscr{B})$, we will concern ourselves in this section with the conditions that ensure inclusions in the other direction. First, however, we present some examples to show that $M(\mathscr{B})$ is indeed a proper subset of $EM(\mathscr{B})$ and that $EM(\mathscr{B})$ is a proper subset of the set of nonsingular functions, $N(\mathscr{B})$.

The motivation for our first example is as follows: It is clear that if there is to be a function f such that $I(f, \mathscr{B}) = 0$ but $f \notin EM(\mathscr{B})$, then there must be a sequence $\{(m_k, n_k)\}_{k=1}^{\infty}$ of pairs in $\mathscr{B}$ such that $\alpha(m_k, n_k) \neq 0$ but $\alpha(m_k, n_k) \to 0$ as $m_k n_k \to \infty$. Obviously, this imposes severe restrictions on $f(m_k)$, $f(n_k)$ and $f(m_k n_k)$. If, moreover, $(t, m_k) \in \mathscr{B}$, then $\alpha(t, m_k)$ must be small if tm_k is large, and this affects $f(t)$. It seems plausible, then, that a function with the desired properties would be most likely to exist with respect to a basic sequence $\mathscr{B}$ for which the number of integers t such that $(t, m_k) \in \mathscr{B}$ is severely limited. This does not necessitate making $\partial(\mathscr{B})$ small (as we shall see), but can be done by "separating" the pairs (m_k, n_k) from the rest of $\mathscr{B}$.

EXAMPLE 4.1. Let p and q be distinct primes and let $\mathscr{B}$ be the basic sequence generated by the primitive pair (p, q) and all other primitive pairs in which neither p nor q is a member. In other words, $\mathscr{B} = \Gamma(\mathscr{B}[p, q] \cup (p, p))$, where $\mathscr{B}[p, q]$ was defined in Theorem 3.6. Define f by: $f(p^a) = 1 + (1/a)(a \geq 1)$, $f(q^b) = 1 + (1/b)(b \geq 1)$, $f(p^a q^b) = 1 + (1/a) + (1/b)(a \geq 1, b \geq 1)$, and $f(n) = 1$ otherwise. It is clear that $\alpha(m, n) = 0$ if neither p nor q divides mn. If p or q does divide k, then either B_k contains only the trivial pairs $(1, k)$ and $(k, 1)$ or $k = p^a q^b (a \geq 1, b \geq 1)$, and in the latter case B_k contains,

in addition to the two trivial pairs, only the pairs (p^a, q^b) and (q^b, p^a). Clearly $\alpha(p^a, q^b) = (1/ab)/2\{1 + (1/a) + (1/b) + (1/2ab)\}$ which is not zero, but which approaches zero as $p^a q^b \to \infty$. Hence $f \in N(\mathscr{B})$ but $f \notin EM(\mathscr{B})$. Finally, we note that

$$\delta(\mathscr{B}) \geqq \delta(\mathscr{B}[p, q]) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)$$

which can be made arbitrarily close to 1.

EXAMPLE 4.2. (a) Let $\mathscr{B} = \mathscr{T}_2$ (see Example 3.3). Define f by $f(n) = 1$ if $n \neq 2^a$ $(a > 1)$, $f(2^a) = 0$ for $a > 1$. Then $\alpha(1, k) = 0$ for all k and $\alpha(2^a, 2^b) = 0$ if $a + b \geqq 3$, so $f \in EM(\mathscr{B})$. But $f(2)f(2) \neq f(4)$, hence $f \notin M(\mathscr{B})$.

(b) Let $\mathscr{B}$ be generated by the pair $(2, 3)$ and define f by: $f(n) = n$ if $n \neq 6$; $f(6) = w$ where w is arbitrary except that $w \neq 6$. Then $f \in EM(\mathscr{B}, 7)$ but $f \notin M(\mathscr{B})$.

The failure of each of the functions in Example 4.2 to be multiplicative was due both to the small size of $\mathscr{B}$ and to the values of k for which $f(k) = 0$. When $\mathscr{B}$ contains a type II primitive pair, the function in part (a) is in many ways characteristic of functions which are in $EM(\mathscr{B})$ but not in $M(\mathscr{B})$. In fact, when $\mathscr{B}$ is generated by a single type II primitive pair (as is the case here), we shall see from the proof of Theorem 4.5 that we really have very little choice in the construction of f. For the time being, however, we note that the zero set $Z_f = \{k \mid f(k) = 0\}$ in part (a) is small (in fact, $D(Z_f) = 0$), but that Z_f contains all the numbers 2^t, $t > 1$.

The following theorem is an immediate consequence of the definition of nonsingularity.

THEOREM 4.3. If f is nonsingular with respect to $\mathscr{B}$ and if there exists $K > 0$ such that either $\alpha(m, n) = 0$ or $\beta(m, n) > K$ provided $(m, n) \in \mathscr{B}$ and mn is sufficiently large, then $f \in EM(\mathscr{B})$. In particular, a nonsingular function which assumes only finitely many values is eventually multiplicative with respect to $\mathscr{B}$.

Theorem 4.3 imposes severe demands on f but says little about the basic sequence $\mathscr{B}$. The next lemma, which will be most useful in what follows, depends on the properties of both f and $\mathscr{B}$.

LEMMA 4.4. Let f be a nonsingular function with respect to $\mathscr{B}$ and suppose that $(m, n) \in \mathscr{B}$. If there exists a strictly increasing sequence $\{a_i\}_{i=1}^{\infty}$ of natural numbers such that
(1) $(a_i, mn) \in \mathscr{B}$,
(2) $f(a_i) \neq 0$,
then $f(mn) = f(m)f(n)$.

Proof. We will consider two cases.

Case 1. $f(a_i mn) = 0$ for $a_i \geq N_1$. Let N_2 be the integer given in Lemma 2.3 and let $N = \max\{N_1, N_2\}$. If $a_i \geq N$ then

$$f(a_i)f(mn) = 0 ;$$

hence $f(mn) = 0$. Moreover, $f(a_i n)f(m) = 0$. If $f(m) \neq 0$, then $f(a_i n) = f(a_i)f(n) = 0$; hence $f(n) = 0$. Thus $f(mn) = 0 = f(m)f(n)$.

Case 2. Without loss of generality we may assume $f(a_i mn) \neq 0$ for all i. Since f is nonsingular, $J(f) = 0$ and $f(1) = 1$. It follows from Theorem 2.9 that

$$f(a_i)f(mn)/f(a_i mn) , \qquad f(a_i m)f(n)/f(a_i mn) ,$$

and $f(a_i)f(m)/f(a_i m)$ all approach 1 as $i \to \infty$. Hence $f(mn)/f(m)f(n) \to 1$ and so $f(m)f(n) = f(mn)$.

We note from Example 4.2(a) that Lemma 4.4 is not valid if we omit assumption (2) from the hypothesis.

As an application of the preceding lemma we now prove

THEOREM 4.5. *If f is nonsingular with respect to $\mathscr{B}$ and if $(p, p) \in \mathscr{B}$ for some prime p, then there is an integer N_p such that*

$$f(p^m)f(p^n) = f(p^{m+n})$$

provided $m + n \geq N_p$. If also there is a strictly increasing sequence $\{r_j\}_{j=1}^{\infty}$ of natural numbers such that $f(p^{r_j}) \neq 0$, then $f(p) \neq 0$ and

$$f(p^n) = f^n(p)$$

for all $n \geq 0$.

Proof. Let N be the integer given in Lemma 2.3 and suppose $p^t \geq N$. If $f(p^t) \neq 0$, then $f(p^m)f(p^n) \neq 0$ for any non-negative integers m and n such that $m + n = t$. Hence $f(p^m) \neq 0$ for any $0 \leq m \leq t$. We conclude that either
  (i)   there exists an integer N_p such that $f(p^t) = 0$ for $t \geq N_p/2$, or
  (ii) $f(p^t) \neq 0$ for all $t \geq 0$.
If case (i) holds, then $f(p^{m+n}) = 0 = f(p^m)f(p^n)$ provided $m + n \geq N_p$. On the other hand, if case (ii) holds, we may apply Lemma 4.4 with $a_i = p^i$, from which it follows that $f(p^{m+n}) = f(p^m)f(p^n)$ for all $m \geq 0$, $n \geq 0$.
Finally, case (i) cannot hold if $f(p^{r_j}) \neq 0$ for arbitrarily large values of r_j. That completes the proof.

COROLLARY 4.5.1. (a) *If $\mathscr{B}$ is generated by a finite set of type II primitive pairs, then $f \in N(\mathscr{B})$ implies $f \in EM(\mathscr{B})$.*

(b) *Suppose $\mathscr{B}$ is generated by an arbitrary set of type II primitive pairs and $f \in N(\mathscr{B})$. If, for each prime p for which $(p, p) \in \mathscr{B}$, there exists a strictly increasing sequence $\{r_{p,j}\}_{j=1}^{\infty}$ of positive integers such that $f(p^{r_{p,j}}) \neq 0$, then $f \in M(\mathscr{B})$.*

As noted earlier, the zero-set Z_f will play an important part in determining whether a function in $EM(\mathscr{B})$ is also in $M(\mathscr{B})$. Roughly speaking, if $\delta(\mathscr{B})$ is large enough, then for any $(m, n) \in \mathscr{B}$ there will be (many) large values of t for which $(mn, t) \in \mathscr{B}$. Moreover, if $f \in EM(\mathscr{B})$, then $f(m)f(t) = f(mt)$, $f(n)f(t) = f(nt)$, and $f(mn)f(t) = f(mnt)$. Now if Z_f is small enough, $f(t)$, $f(mt)$, $f(nt)$, and $f(mnt)$ will be nonzero. Thus $f(m)$, $f(n)$, and $f(mn)$ will be determined by $f(k)$ for large values of k, and the multiplicative properties of f for these large values will carry over to $f(mn)$. (The trouble spots in Example 4.2 should now be clear. Even though $D(Z_f) = 0$ in part (a), $\mathscr{B}$ was small enough for $f(k)$ to be zero for all k for which $(4, k) \in \mathscr{B}$. And although $Z_f = \varnothing$ in part (b), $\mathscr{B}$ was so small then that there simply were no t such that $(6, t) \in \mathscr{B}$, $t \neq 1$).

In making the above remarks more precise, the cornerstone of our arguments will be the easily proved

THEOREM 4.6. *Suppose that $f \in EM(\mathscr{B}, N)$ and let $C_N = \prod_{p < N/2} p$. If there exists an integer $a \geq N/2$ such that*
  (i) *$(a, C_N) \in \mathscr{B}$,*
  (ii) *$f(a) \neq 0$,*
then f is multiplicative with respect to $\mathscr{B}$.

Proof. Choose m and n such that $mn < N$, $(m, n) \in \mathscr{B}$. Since $f(1) = 1$, we may assume $m > 1$ and $n > 1$; therefore $m < N/2$ and $n < N/2$ and we may decompose mn canonically: $mn = q_1^{\gamma_1}q_2^{\gamma_2} \cdots q_r^{\gamma_r}$, where each q_i is a prime less than $N/2$. Thus each $q_i \mid C_N$.

Since $(a, C_N) \in \mathscr{B}$, we have $(a, q_i) \in \mathscr{B}$, $1 \leq i \leq r$; hence (a, mn), (a, m), (a, n), and (am, n) are all in $\mathscr{B}$. Finally, since $f \in EM(\mathscr{B}, N)$ and $amn > am \geq N$, we have

$$f(a)f(mn) = f(amn) = f(am)f(n) = f(a)f(m)f(n) .$$

Thus $f(mn) = f(m)f(n)$ and the proof is complete.

The following corollary and the next three theorems all depend on Theorem 4.6. They illustrate the remarks preceding that theorem and emphasize the importance of the relationship between Z_f and $\mathscr{B}$.

COROLLARY 4.6.1. *Suppose $f \in EM(\mathscr{B}, N)$ and, as before, let $C_N = \prod_{p < N/2} p$. Also let $A = \{a \mid (a, C_N) \in \mathscr{B}\}$. If*

(1) $\underline{D}(A) = \theta > 0$,

(2) $\bar{D}(Z_f) < \theta$,

then f is multiplicative with respect to $\mathscr{B}$.

Proof. Let $\eta = (\theta - \bar{D}(Z_f))/3 > 0$ and choose K such that $x > K$ implies

$$\frac{{}^{\#}Z_f(x)}{x} < \bar{D}(Z_f) + \eta < \theta - \eta < \frac{{}^{\#}A(x)}{x} .$$

Then ${}^{\#}A(x) - {}^{\#}Z_f(x) > (\theta - \bar{D}(Z_f) - 2\eta)x = \eta x$ provided $x > K$. Thus ${}^{\#}A(x) - {}^{\#}Z_f(x) \to \infty$ as $x \to \infty$, and the corollary follows.

THEOREM 4.7. *Suppose $f \in EM(\mathscr{B}, N)$ and that*

$$\bar{\delta}(\mathscr{B}) \geqq 1 - \lambda_M ,$$

where $M = [N/2] + 1$ and λ_M is as defined in Theorem 3.7. If $p_1, \cdots, p_K$ are the primes less than $N/2$, then $f \in M(\mathscr{B})$ provided

$$\liminf_{n \to \infty} \frac{{}^{\#}Z_f(n)}{\log^K(n/M)} < \frac{1}{K!(\log p_1 \cdots \log p_K)} .$$

Proof. Assume $N \geqq 5$, since otherwise $EM(\mathscr{B}, N)$ and $M(\mathscr{B})$ are identical. If $(M, p) \notin \mathscr{B}$ for a prime $p < N/2$, then $(q, p) \notin \mathscr{B}$ for some prime $q \leqq M$. It is a simple exercise to show that, for fixed $y \geqq 2$, the function

$$F(x) = 1 - \frac{1}{2}\Big(\lambda_y + \frac{1}{y}\Big)\Big(\lambda_x + \frac{1}{x}\Big) + \lambda_y \lambda_x$$

increases for $x \geqq 2$, and it is also easy to show that

$$1 - \frac{1}{2}\Big(\lambda_x + \frac{1}{x}\Big)^2 + \lambda_x^2 < 1 - \lambda_x$$

for $x \geqq 2$. Therefore, if $p \neq q$, the above and Remark 3.8 yield

$$\bar{\delta}(\mathscr{B}) \leqq 1 - \frac{1}{2}\Big(\lambda_p + \frac{1}{p}\Big)\Big(\lambda_q + \frac{1}{q}\Big) + \lambda_p \lambda_q$$

$$< 1 - \frac{1}{2}\Big(\lambda_M + \frac{1}{M}\Big)^2 + \lambda_M^2 < 1 - \lambda_M .$$

If, on the other hand, $p = q$, Theorem 3.7 gives

$$\bar{\delta}(\mathscr{B}) \leqq 1 - \lambda_p < 1 - \lambda_M .$$

In either case, $\bar{\delta}(\mathscr{B}) < 1 - \lambda_M$, which contradicts the hypothesis. Hence $(M, p) \in \mathscr{B}$ for every prime $p < N/2$ and so $(M, C_N) \in \mathscr{B}$.

Suppose now that $f \notin M(\mathscr{B})$. Then, by Theorem 4.6, $f(M) = 0$. If $t \geq 2$ is divisible only by primes $p < N/2$, then, by the above, $(t, M) \in \mathscr{B}$, and so $f(tM) = f(t)f(M) = 0$. Since there are

$$\frac{\log^K (n/M)}{K!(\log p_1 \cdots \log p_K)} + O(\log^{K-1} (n/M))$$

integers $t \leq n/M$ divisible only by $p_1, \cdots, p_K$, there are at least that many integers $k \leq n$ such that $f(k) = 0$. Therefore,

$$\liminf_{n \to \infty} \frac{{}^\sharp Z_f(n)}{\log^K (n/M)} \geq \frac{1}{K!(\log p_1 \cdots \log p_K)} ,$$

and the proof is complete.

If we know something about the structure of the basic sequence $\mathscr{B}$, we may be able to weaken the conditions of the preceding theorem. For instance, recalling from Example 3.1 that $\mathscr{M}$ is the basic sequence consisting of all pairs (a, b) such that a and b are relatively prime, we have

THEOREM 4.8. *If* $D(Z_f) = 0$, *then* $f \in EM(\mathscr{M})$ *implies* $f \in M(\mathscr{M})$.

Proof. Suppose that $f \in EM(\mathscr{M}, N)$ and $A = \{a \mid (a, C_N) \in \mathscr{M}\}$, with C_N again as given in Theorem 4.6. Then $a \in A$ if and only if a is not divisible by any prime $p < N/2$. Now it is easy to show that $D(A) = \prod_{p<N/2}\{1 - (1/p)\}$ and, since $\bar{D}(Z_f) = 0 < D(A)$, the conclusion follows from Corollary 4.6.1.

THEOREM 4.9. *If* $f \in EM(\mathscr{B}, N)$ *and* $\bar{\delta}(\mathscr{B}) = 1$, *then* f *is a character.*

Proof. Choose $a \geq N/2$ such that $f(a) \neq 0$. By Theorem 3.9, $\mathscr{B} = \mathscr{L}$; hence $(a, p) \in \mathscr{B}$ for every $p < N/2$. It follows now from Theorem 4.6 that $f \in M(\mathscr{L})$.

In anticipation of its importance in connection with singular functions, we close this section with a short discussion about a particular type of nonsingular function: An arithmetic function f will be called a *unit* with respect to $\mathscr{B}$ if (i) $f \in N(\mathscr{B})$ and (ii) there exists an integer N such that $f(k)$ is not zero and does not change sign for $k \geq N$. A unit f will be called *positive* or *negative* depend-

ing on the sign of $f(k)$ for large values of k.

From Lemma 2.3 we may easily prove

LEMMA 4.10. *If there is an integer N such that*
    (1) $f(k) < 0$ *for $k \geqq N$,*
    (2) $\bar{\beta}(f, \mathscr{B}) < 1$,
then $\mathscr{B} = \mathscr{S}$. In particular, negative units can exist only with respect to $\mathscr{S}$.

A straightforward application of Lemma 4.4 yields the following useful theorem on units.

THEOREM 4.11. (a) *Suppose f is a unit with respect to $\mathscr{B}$ and $(m, n) \in \mathscr{B}$. If there is an integer $a > 1$ such that $(a, mn) \in \mathscr{B}$, then*
    (1) $f(m) > 0,\ f(n) > 0$,
    (2) $f(mn) = f(m)f(n)$.
  (b) *If f is a unit with respect to $\mathscr{B}$ and $\bar{\delta}(\mathscr{B}) = 1$, then*
    (1) $f(k) > 0$ *for all k,*
    (2) *f is a character.*
  (c) *If f is a unit with respect to $\mathscr{M}$, then*
    (1) $f(k) > 0$ *for all k,*
    (2) *f is multiplicative with respect to $\mathscr{M}$.*

Finally, from Corollary 4.5.1 we have

THEOREM 4.12. *If $\mathscr{B}$ is generated by an arbitrary collection of type* II *primitive pairs and if f is a unit with respect to $\mathscr{B}$, then f is multiplicative with respect to $\mathscr{B}$.*

5. Singular functions. We conclude this paper with a discussion of those singular functions for which $I(f, \mathscr{B})$ exists. As the following theorem shows, these functions are closely connected with the set of units on $\mathscr{B}$.

THEOREM 5.1. *Let an arithmetic function f and a basic sequence $\mathscr{B}$ be given. Then $I(f, \mathscr{B})$ exists and $0 < |I(f, \mathscr{B})| < 1$ if and only if*

$$f = Mf^* ,$$

where (i) *f^* is a unit with respect to $\mathscr{B}$,*
    (ii) $M > 0,\ M \neq 1$.
Moreover, the decomposition $f = Mf^*$ is unique.

Proof. The uniqueness of the decomposition described above, if such a decomposition exists, is immediate. For suppose

$$0 < |I(f, \mathscr{B})| < 1 \quad \text{and} \quad f = Mf^* ,$$

where f^* is a unit with respect to $\mathscr{B}$. Then

$$f(1) = Mf^*(1) = M \quad \text{and} \quad f^*(k) = f(k)/f(1) .$$

Assume now that $0 < |I(f, \mathscr{B})| < 1$. We know from Lemma 2.4 that $f(1) > 0$ and from Theorem 2.5 that $f(1) \neq 1$. Let $M = f(1)$ and let $f^*(k) = f(k)/f(1)$. Again by Theorem 2.5, $f(k)$ does not change sign for sufficiently large k and so $f^*(k)$ has the same property.

If $\mathscr{B} = \mathscr{S}$ it follows from the above that $I(f^*, \mathscr{S})$ exists and, since $f^*(1) = 1$, f^* is a unit with respect to $\mathscr{S}$.

If $\mathscr{B} \neq \mathscr{S}$, Corollary 2.6.1 implies $f(k) > 0$ for all sufficiently large k, hence $I(f, \mathscr{B}) = (f(1) - 1)/(f(1) + 1)$. Therefore, from Lemma 2.3, we have

$$(5.1) \qquad \frac{f(1) - 1}{f(1) + 1} - \varepsilon < \frac{\dfrac{f(m)f(n)}{f(mn)} - 1}{\dfrac{f(m)f(n)}{f(mn)} + 1} < \frac{f(1) - 1}{f(1) + 1} + \varepsilon$$

whenever $(m, n) \in \mathscr{B}$, $mn \geq N(\varepsilon)$. Consequently, if ρ_ν is defined by (2.3), then $\rho_\nu \to f(1)$. But then if $\rho_\nu^* = f^*(m_\nu)f^*(n_\nu)/f^*(m_\nu n_\nu)$, $\rho_\nu^* \to 1$. It follows from Theorem 2.9 that $J(f^*, \mathscr{B})$ exists. Finally, since $f^*(1) = 1$, $J(f^*, \mathscr{B}) = I(f^*, \mathscr{B}) = 0$.

Conversely, suppose $f = Mf^*$, where $M > 0$, $M \neq 1$, f^* a unit with respect to $\mathscr{B}$. Since $f(k)$ does not change sign for all sufficiently large k, $I(f, \mathscr{S})$ exists. By Theorem 2.5, $0 < |I(f, \mathscr{S})| < 1$. If $\mathscr{B} \neq \mathscr{S}$, it follows from $J(f^*, \mathscr{B}) = 0$ (and so $f^*(1) = 1$) and Theorem 2.9 that $\rho_\nu^* \equiv f^*(m_\nu)f^*(n_\nu)/f^*(m_\nu n_\nu) \to 1$ when

$$(m_\nu, n_\nu) \in \mathscr{B} , \qquad m_\nu n_\nu \to \infty , \qquad f^*(m_\nu n_\nu) \neq 0 .$$

But then if ρ_ν satisfies (2.3), $\rho_\nu = M\rho_\nu^* \to M = f(1)$. Therefore, by Theorem 2.9, $J(f, \mathscr{B})$ exists. Since $0 < J(f, \mathscr{B}) < 1$ (Theorem 2.5) and since $f(k) > 0$ for all sufficiently large k (f^* must be a positive unit), we have with the aid of Lemma 2.3

$$(5.2) \qquad \alpha_f(m_\nu, n_\nu) = \frac{\rho_\nu - 1}{\rho_\nu + 1} \to \frac{M - 1}{M + 1} .$$

Thus $I(f, \mathscr{B})$ exists and $0 < |I(f, \mathscr{B})| < 1$.

A partial analogue to Theorem 5.1 will hold when $I(f, \mathscr{B}) = \pm 1$. The proof is straightforward and is omitted.

THEOREM 5.2. (a) *If $\mathscr{B} = \mathscr{S}$ and $f(1) \neq 0$, then $I(f, \mathscr{B})$ exists and has the value ± 1 if and only if $f = Mf^*$, where f^* is a unit with respect to $\mathscr{S}$ and $M < 0$. Moreover, the decomposition is unique.*

(b) *If $\mathscr{B} \neq \mathscr{S}$ and $f = Mf^*$, where f^* is a unit with respect to $\mathscr{B}$ and $M < 0$, then $I(f, \mathscr{B})$ exists and has the value $+1$.*

EXAMPLE 5.3. (a) If $I(f) = +1$ then $f^* \equiv f/f(1)$ may or may not be a unit. For example, let $\mathscr{B} = \Gamma((2, 2))$, $f(1) = -1$, $f(2^a) = -k$ where $k > 0$, $k \neq 1$, $a > 0$, and let $f(t)$ be any negative number otherwise. Then $I(f^*, \mathscr{B})$ does not exist. On the other hand, let $\mathscr{B}$ be arbitrary and let $f(k) = -1$ for all k. Then f^* is a unit with respect to $\mathscr{B}$.

(b) If $I(f, \mathscr{B}) = -1$ and $\mathscr{B} \neq \mathscr{S}$, then $f(k) > 0$ for sufficiently large k and $f(1) \leqq 0$. Hence (assuming $f(1) < 0$) $f^*(k) \equiv f(k)/f(1) < 0$ for large k and so f^* can never be a unit with respect to $\mathscr{B}$.

Although, as the preceding example shows, it is not usually possible to express a function f for which $I(f) = -1$ as a multiple of a unit, we can characterize such functions in terms of the behavior of $f(m)f(n)/f(mn)$.

THEOREM 5.4. *$I(f, \mathscr{B})$ exists and has the value -1 if and only if*

(1) $f(1) \leqq 0$,

(2) *there is an integer N such that*

(a) $f(k) > 0$ *for* $k \geqq N$,

(b) $f(m)f(n)/f(mn) \to 0$ *as* $mn \to \infty$, $(m, n) \in \mathscr{B}$, $m \geqq N$, $n \geqq N$,

(c) *if* $1 < m_0 < N$ *and* $f(m_0) > 0$, *then* $f(n)/f(m_0 n) \to 0$ *as* $n \to \infty$, $(m_0, n) \in \mathscr{B}$.

Proof. If $I(f, \mathscr{B}) = -1$, parts (1) and (2a) follow from Lemma 2.4. Parts (2b) and (2c) are easy consequences of part (2a) and definition (2.1). The converse also follows readily from definition (2.1).

The following example shows that (2c) may not be omitted from the above.

EXAMPLE 5.5. Let $\mathscr{B} = \Gamma((2, 3))$. Define $f(k) = 1$ if $k = 2$, 3^n, or $2 \cdot 3^n$ for $n \geqq 1$; $f(k) = -1$ if $k = 1$ or 4; $f(k) = k^k$ otherwise. Conditions (1), (2a), and (2b) are satisfied; (2c) is not. $I(f, \mathscr{B})$ does not exist.

Theorem 5.1 has an interesting application with regard to the index

of a product of two arithmetic functions. First we need a lemma.

LEMMA 5.6. *If f and g are units with respect to $\mathscr{B}$, then fg is also a unit with respect to $\mathscr{B}$.*

Proof. The lemma follows readily from Theorem 2.9.

Suppose now that either (i) f is a unit with respect to $\mathscr{B}$ or (ii) $0 < |I(f, \mathscr{B})| < 1$. Then $f(k)$ does not change sign for sufficiently large k, and it is useful to define a function ω by: $\omega(f) = +1$ if $f(k) > 0$ for all large k and $\omega(f) = -1$ if $f(k) < 0$ for large k.

THEOREM 5.7. *Suppose that f and g are arithmetic functions such that either f is a unit with respect to $\mathscr{B}$ or $0 < |I(f, \mathscr{B})| < 1$, and similarly for g. Then $I(fg, \mathscr{B})$ exists and*

$$I(fg) = \omega(f)\omega(g) \frac{\omega(f)I(f) + \omega(g)I(g)}{1 + \omega(f)\omega(g)I(f)I(g)} \; .$$

In particular, if $\mathscr{B} \neq \mathscr{S}$,

$$I(fg) = \frac{I(f) + I(g)}{1 + I(f)I(g)} \; .$$

Proof. The theorem is easily proved by considering the two cases $\mathscr{B} = \mathscr{S}$ and $\mathscr{B} \neq \mathscr{S}$ and, within each case, by considering the various combinations of f and g. We illustrate one of the possibilities:

Suppose $\mathscr{B} \neq \mathscr{S}$, $0 < |I(f, \mathscr{B})| < 1$, and $0 < |I(g, \mathscr{B})| < 1$. From Theorem 5.1 we know that $f = Mf^*$ and $g = Kg^*$, where f^* and g^* are units, $M > 0$, $M \neq 1$, $K > 0$, $K \neq 1$. Therefore $fg = MK(f^*g^*)$ and, by Lemma 5.6, f^*g^* is also a unit with respect to $\mathscr{B}$. Since

$$I(f) = (M - 1)/(M + 1) \, , \qquad I(g) = (K - 1)/(K + 1) \, ,$$

and $I(fg) = (MK - 1)/(MK + 1)$, we have

$$M = (1 + I(f))/(1 - I(f))$$

and $K = (1 + I(g))/(1 - I(g))$, and the desired relation follows if we substitute these values in the expression for $I(fg)$.

COROLLARY 5.7.1. *Suppose $\mathscr{B} \neq \mathscr{S}$. If $0 < |I(f, \mathscr{B})| < 1$ and $I(g, \mathscr{B}) = -I(f, \mathscr{B})$, then fg is a unit with respect to $\mathscr{B}$.*

The extension from the product of two functions to the product

of any finite number of arithmetic functions is easily accomplished by induction. For let $\sigma_m[x_1, \cdots, x_n]$ be the m-th elementary symmetric function of the variables $x_1, \cdots, x_n$. That is, $\sigma_0[x_1, \cdots, x_n] = 1$ and $\sigma_m[x_1, \cdots, x_n] = \sum_{1 \leq i_1 < \cdots < i_m \leq n} x_{i_1} \cdots x_{i_m}$ for $1 \leq m \leq n$. Then we have

THEOREM 5.8. *Let* $f_1, f_2, \cdots, f_n$ *be arithmetic functions such that, for each* $1 \leq i \leq n$, *either* f_i *is a unit with respect to* $\mathscr{B}$ *or* $0 < |I(f_i, \mathscr{B})| < 1$. *Also, for* $0 \leq m \leq n$, *let*

$$\sigma_m^n = \sigma_m[\omega(f_1)I(f_1), \cdots, \omega(f_n)I(f_n)] \ .$$

Then $I(f_1 \cdots f_n)$ *exists and*

$$I(f_1 \cdots f_n) = \begin{cases} \omega(f_1) \cdots \omega(f_n) \dfrac{\sigma_1^n + \sigma_3^n + \cdots + \sigma_{n-1}^n}{\sigma_0^n + \sigma_2^n + \cdots + \sigma_n^n} \\ \qquad\qquad\qquad\qquad\qquad \textit{if } n \textit{ is even} \ , \\[2ex] \omega(f_1) \cdots \omega(f_n) \dfrac{\sigma_1^n + \sigma_3^n + \cdots + \sigma_n^n}{\sigma_0^n + \sigma_2^n + \cdots + \sigma_{n-1}^n} \\ \qquad\qquad\qquad\qquad\qquad \textit{if } n \textit{ is odd} \ . \end{cases}$$

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