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SUMS OF AUTOMORPHISMS OF A PRIMARY ABELIAN GROUP

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This paper is concerned with the problem: For which abelian groups G does the group of automorphisms of G generate the ring of endomorphisms of G? R. S. Pierce has shown that if G is a 2-primary group, all of whose finite Ulm invariants are equal to one, then the subring of $E(G)$ generated by the group of automorphisms of G is properly contained in $E(G)$. The groups considered in this paper are p-primary abelian groups where p is a fixed prime number greater than two. The paper is divided into four parts. In §1 the following theorem is proved:

THEOREM. If G is a countable reduced p-primary, $(p > 2)$, abelian group, then every endomorphism of G is a sum of two automorphisms.

The second part gives an extension of this theorem to the case where G is the direct sum of such groups. In §3 the result of this theorem is established for torsion complete abelian groups, using some known results about their endomorphism rings. Finally an example is given (for an arbitrary prime p) of a reduced p-primary abelian group G for which there are endomorphisms in $E(G)$ that are not sums of automorphisms.

NOTATION. Throughout this paper p will represent a fixed prime number greater than two. Mainly the notation will follow that of [3] and [5]. All groups considered are assumed to be p-primary abelian groups. If G is such a group, we define for each ordinal β the subgroup G_β as follows: $G_1 = pG = \{x \in G \mid x = pg, g \in G\}$, $G_\beta = pG_{\beta-1}$ if $\beta - 1$ exists and $G_\beta = \bigcap_{\alpha<\beta} G_\alpha$ if β is a limit ordinal. A primary group G is said to be divisible if $pG = G$ and is said to be reduced if 0 is its only divisible subgroup. The height function h_G for a reduced p-group G is defined by the conditions

$$h_G(x) = \alpha \text{ if } x \in G_\alpha \text{ and } x \notin G_{\alpha+1}$$
$$h_G(0) = \infty \text{ where } \infty > \alpha \text{ for each ordinal } \alpha .$$

If there is no possibility of confusion, we will write $h(x)$ for $h_G(x)$.

A subgroup H of G is pure in G if $p^nG \cap H = p^nH$ for each $n \geqq 0$.

For every integer $e \geqq 0$ we define $G[p^e]$ as

$$G[p^e] = \{x \in G \mid p^e x = 0\} .$$

If $x \in G$, $o(x)$ is defined to be the order of x.

Finally for any subset X of G, let $\{X\}$ denote the subgroup of G which is generated by X.

1. Countable groups. One of the fundamental theorems in abelian groups is Ulm's theorem, [7], which states that any two countable reduced p-primary abelian groups having the same Ulm invariants are isomorphic. (For a definition of the Ulm invariants see [5], p. 27.) A proof of this theorem is given in [5]. Roughly speaking the concept of the proof is to build an isomorphism between the two groups G and H. It is clear that one could by similar methods build an automorphism of the group G, taking G and H to be the same group. Given an endomorphism of G we might then ask, "Could one not build two automorphisms of G in such a way that their sum is always the given endomorphism?" This is indeed possible and we get the following theorem.

THEOREM 1.1. *If G is a countable, reduced p-primary abelian group, then every endomorphism of G is a sum of two automorphisms.*

The proof of 1.1 is based on several lemmas. We begin with the following definition.

DEFINITION. If S is a subgroup of G, an element $x \in G - S$ is said to be proper with respect to S if $h(x + s) \leqq h(x)$ for all $s \in S$.

Let S be a subgroup of G, let $x, y \in G$ with $h(x) < h(y)$, and let n be an integer prime to p. It is an immediate consequence of this definition that if any one of $x, x + y$, or nx is proper with respect to S, then so are the other two.

Now suppose that S and T are subgroups of G and that θ is a height preserving isomorphism between them. Assume that there exist x and $z \in G$, having height α, and satisfying

(i) $px \in S$ and $\theta(px) = pz$,

(ii) x and z are proper with respect to S and T respectively. Then θ can be extended to a height preserving isomorphism θ' between $\{S, x\}$ and $\{T, z\}$ by defining

$$\theta'(s + nx) = \theta(s) + nz .$$

Some of the lemmas that follow can be found in [5], § 11, and so, mainly, the proofs will be omitted.

LEMMA 1.2. *Let S and T be subgroups of G and let θ be a height preserving isomorphism between them. Assume that $x \in G$ satisfies*

(i) *x is proper with respect to S,*
(ii) *if $w \in x + S$ is proper with respect to S then $h(pw) \leqq h(px)$,*
(iii) *$px \in S$ and $h(px) = h(x) + 1$.*

Then there is a $z \in G$ with $pz = \theta(px)$ and $h(z) = h(x)$. Furthermore any z with these two properties will be proper with respect to T.

If S is any subgroup of G, define S_α by $S_\alpha = S \cap G_\alpha$. Define the subgroup S_α^* of S_α to be

$$S_\alpha^* = \{x \in S_\alpha \mid px \in G_{\alpha+2}\} .$$

The quotient $S_\alpha^*/S_{\alpha+1}$ can be considered as a vector space over Z_p.

LEMMA 1.3. *If S is a Subgroup of G then there exists a monomorphism ζ^* of $S_\alpha^*/S_{\alpha+1}$ into $G_\alpha[p]/G_{\alpha+1}[p]$.*

Since the technique involved in the proof of 1.3 is vital to other parts of this paper the proof will be outlined.

Proof. If $x \in S_\alpha^*$ then $px \in G_{\alpha+2}$ and there is a $y \in G_{\alpha+1}$ such that $px = py$. The mapping

$$x \longrightarrow x - y + G_{\alpha+1}[p]$$

has $S_{\alpha+1}$ as its kernel.

LEMMA 1.4. *Let S and ζ^* be as in 1.3. If $x \in G_\alpha[p] - G_{\alpha+1}[p]$ then $x + G_{\alpha+1}[p] \notin \operatorname{Im} \zeta^*$ if and only if x is proper with respect to S.*

LEMMA 1.5. *Let S and T be finite subgroups of G and let θ be a height preserving isomorphism between them. Suppose there is an $x \in G$ of height α, which is proper with respect to S, and such that $h(px) > \alpha + 1$. Then there exists $z \in G_\alpha[p]$ such that $h(z) = \alpha$ and z is proper with respect to T. Furthermore z can be chosen so that it is also proper with respect to S and has the following form*

$$z = x + s_0 + w$$

where $s_0 \in S_\alpha^*$ and $h(w) > \alpha$.

Proof. Choose $v \in G_{\alpha+1}$ such that $x - v \in G_\alpha[p]$. If x is proper with respect to T we let $s_0 = 0$ and $w = -v$. Otherwise let ζ^* and ξ^* be the monomorphisms of $S_\alpha^*/S_{\alpha+1}$ and $T_\alpha^*/T_{\alpha+1}$ into $G_\alpha[p]/G_{\alpha+1}[p]$ respectively. Both Im ξ^* and Im ζ^* have the same finite dimension and since, by 1.4,

$$x - v + G_{\alpha+1}[p] \in \operatorname{Im} \xi^* - \operatorname{Im} \zeta^*$$

it follows that there is a $y \in G_\alpha[p]$ such that

$$y + G_{\alpha+1}[p] \in \operatorname{Im} \zeta^* - \operatorname{Im} \xi^* .$$

From 1.3 we see that y has the form

$$y = s_0 + u,\ s_0 \in S_\alpha^* \quad \text{and} \quad h(u) > \alpha .$$

We let $z = x + s_0 - (u + v)$.

We can now proceed with a proof of 1.1. Let $\psi \in E(G)$ and enumerate G as $G = \{x_1 = 0, x_2, x_3, \cdots\}$. Suppose that we have finite subsets S, T, and T' and height preserving isomorphisms θ between S and T and θ' between S and T' such that $\theta(s) + \theta'(s) = \psi(s)$ for each $s \in S$. Assume that

$$x_m \notin S \cap T \cap T' .$$

We must extend θ and θ' to height preserving isomorphisms $\bar{\theta}$ and $\bar{\theta}'$ between finite subgroups $\bar{S}$ containing S, $\bar{T}$ containing T, and $\bar{T}'$ containing T' such that $x_m \in \bar{S} \cap \bar{T} \cap \bar{T}'$ and for all $s \in \bar{S}$, $\bar{\theta}(s) + \bar{\theta}'(s) = \psi(s)$. The proof is broken into two cases: $x_m \notin S$ and $x_m \notin T \cap T'$.

Case I. $x_m \notin S$. We may assume that we have an x which is proper with respect to S, $px \in S$ and such that if $y \in x + S$ and $h(y) = h(x)$ then $h(py) \leqq h(px)$. Let $h(x) = \alpha$, $\theta(px) = y$, and $\theta'(px) = y'$.

If $h(y) = \alpha + 1$, we can use 1.2 to get a z which is proper with respect to T and such that $h(z) = \alpha$ and $pz = \theta(px)$. Define $z' = \psi(x) - z$. Then $h(z') = \alpha$ and $pz' = \theta'(px)$. Again using 1.2 we see that z' is proper with respect to T'. Letting $\bar{S} = \{S, x\}$, $\bar{T} = \{T, z\}$, and $\bar{T}' = \{T', z'\}$, the extensions $\bar{\theta}$ and $\bar{\theta}'$ of θ and θ' follow. Since $z + z' = \psi(x)$ it is clear that $\bar{\theta}(s) + \bar{\theta}'(s) = \psi(s)$ for all $s \in \bar{S}$.

Suppose then that $h(y) > \alpha + 1$. By 1.5 there exists $z_0 \in G_\alpha[p]$, of height α, which is proper with respect to T. We have $h(pz_0) = h(0) > \alpha + 1$. Since $\theta'\theta^{-1}$ is a height preserving isomorphism between T and T' we can apply 1.5 again to obtain $z_1 \in G_\alpha[p]$, of height α, which is proper with respect to both T and T'. Choose u and $u' \in G_{\alpha+1}$ such that $pu = y$ and $pu' = y'$.

If $h(\psi(x)) > \alpha$ let

$$z = z_1 + u \quad \text{and} \quad z' = \psi(x) - z .$$

A few easy calculations show that with these definitions of z and z' the desired extensions follow.

Thus we may suppose that $h(\psi(x)) = \alpha$. Letting $w = u + u'$ we see that $h(\psi(x) - w) = \alpha$ and $\psi(x) - w \in G_\alpha[p]$. Now if $\psi(x) - w$ is proper with respect to both T and T' choose v such that $2v = \psi(x) - w$. Then $h(v) = \alpha$, $v \in G_\alpha[p]$, and v is proper with respect to both T and

T'. Define

$$z = v + u \quad \text{and} \quad z' = v + u'$$

to get the extensions of θ and θ'. Therefore we may assume that $\psi(x) - w$ is not proper with respect to T. With z_1 as defined above it follows from 1.4 that $\psi(x) - w - z_1$ is proper with respect to T. Then with the definitions

$$z = \psi(x) - w - z_1 + u \quad \text{and} \quad z' = z_1 + u'$$

we have completed the proof of Case I.

Case II. $x_m \notin T \cap T'$. To be explicit, let us say that $x_m \notin T$. Then as before, the problem reduces to that of having a z which is proper with respect to T and $pz \in T$. Let $h(z) = \alpha$ and $pz = y$. The case $h(y) = \alpha + 1$ is handled just as before. Use 1.2 to get x and let $z' = \psi(x) - z$.

Consider then the situation when $h(y) > \alpha + 1$. We use 1.5 to select $w_0 \in G_\alpha[p]$, of height α, such that w_0 is proper with respect to both T and T' and has the form

$$w_0 = z + t_0 + u_0 ,$$

where $t_0 \in T_\alpha^*$ and $h(u_0) > \alpha$.

Let ζ^* and γ^* be the monomorphisms of $S_\alpha^*/S_{\alpha+1}$ and $T_\alpha'^*/T'_{\alpha+1}$ into $G_\alpha[p]/G_{\alpha+1}[p]$ respectively as given by 1.3. Let $S^\#$ be a complimentary summand of Im ζ^* in $G_\alpha[p]/G_{\alpha+1}[p]$:

$$G_\alpha[p]/G_{\alpha+1}[p] = \operatorname{Im} \zeta^* \oplus S^\# .$$

Let ψ^* be the endomorphism of $G_\alpha[p]/G_{\alpha+1}[p]$ induced by ψ. Choose $x_1 + G_{\alpha+1}[p] \in S^\#$ such that $x_1 \notin G_{\alpha+1}[p]$. Since $p > 2$ there is a $k \not\equiv 0 \pmod p$ such that

$$k(w_0 + G_{\alpha+1}[p]) + \psi^*(x_1 + G_{\alpha+1}[p]) \notin \operatorname{Im} \gamma^* .$$

Let l satisfy

$$kl \equiv 1 (\operatorname{mod} p) .$$

Then

$$(w_0 + \psi(lx_1)) + G_{\alpha+1}[p] \notin \operatorname{Im} \gamma^* .$$

It follows that lx_1 is proper with respect to S and $w_0 + \psi(lx_1)$ is proper with respect to T'. Define

$$w = w_0 - u_0 = z + t_0 .$$

Then $pw \in T$ and $h(\theta^{-1}(pw)) > \alpha + 1$. Choose $v \in G_{\alpha+1}$ such that $pv = \theta^{-1}(pw)$. Define x and w' by

$$x = -lx_1 + v$$
$$w' = \psi(x) - w \,.$$

It is routine to check that the subgroups $\{S, x\}$, $\{T, w\}$, and $\{T', w'\}$ give rise to the desired extensions of θ and θ'. This completes the proof of 1.1.

2. **Direct sums of countable groups.** In this section we will be considering subgroups G_λ, H_β, etc., of a group G, where λ and β are members of an index set Λ. These are not the subgroups defined in the notation. To avoid confusion, we will denote the subgroups G_α, defined previously, by $p^\alpha G$, $p^\alpha H_\beta$, etc.

The purpose of this section is to extend Theorem 1.1 to reduced p-primary abelian groups G that are direct sums of countable groups. In the next lemma and theorem, however, we make no restriction on the summands except that they be countable.

LEMMA 2.1. *Let $G = \sum_{\lambda\in\Lambda} G_\lambda$ where $|G_\lambda| \leqq \aleph_0$ for all $\lambda \in \Lambda$ and let $\psi \in E(G)$. Then $\Lambda = \bigcup_{\gamma\in\Lambda} I_\gamma$ where $|I_\gamma| \leqq \aleph_0$ for each $\gamma \in \Lambda$ and*

$$\psi(\textstyle\sum_{\lambda\in I_\gamma} G_\lambda) \subseteqq \sum_{\lambda\in I_\gamma} G_\lambda \,.$$

Proof: Let γ be a fixed element of Λ and define $\Gamma_0 = \{\gamma\}$. Having defined the countable subset Γ_h of Λ we define Γ_{h+1} as follows. Since $\sum_{\lambda\in\Gamma_h} G_\lambda$ is a countable set there is a countable subset Γ_{h+1} of Λ which contains Γ_h and satisfies

$$\psi(\textstyle\sum_{\lambda\in\Gamma_h} G_\lambda) \subseteqq \sum_{\lambda\in\Gamma_{h+1}} G_\lambda \,.$$

Define

$$I_\gamma = \textstyle\bigcup_{h<\omega} \Gamma_h \,.$$

THEOREM 2.2. Let *$G = \bigoplus \sum_{\lambda\in\Lambda} G_\lambda$ where each G_λ is countable and let $\psi \in E(G)$. Then G can be written as $G = \bigcup_{\beta\in\Lambda} H_\beta$ where each H_β has the following properties.*

(i) *$H_\beta \subseteqq H_{\beta+1}$ for all $\beta \in \Lambda$,*

(ii) *$H_\beta = \bigcup_{\alpha<\beta} H_\alpha$ if β is a limit cardinal,*

(iii) *$H_\beta = H_{\beta-1} \oplus C_\beta$, where C_β is countable, if β is not a limit ordinal,*

(iv) *for each $\beta \in \Lambda$, $\psi(H_\beta) \subseteqq H_\beta$.*

Proof. Let $\Lambda = \bigcup_{\gamma\in\Lambda} I_\gamma$ as in 2.1. Define $\Gamma_0 = I_0$ and having defined Γ_α for all $\alpha < \beta$, define $\Gamma_\beta = \Gamma_{\beta-1} \cup I_\beta$ if $\beta - 1$ exists and $\Gamma_\beta = \bigcup_{\alpha<\beta} \Gamma_\alpha$ if β is a limit ordinal. Now define $H_\beta = \bigoplus \sum_{\lambda\in\Gamma_\beta} G_\lambda$. We note that if β is not a limit ordinal

$$H_\beta = H_{\beta-1} \oplus \textstyle\sum_{\lambda\in K_\beta} G_\lambda$$

where $K_\beta = I_\beta - \Gamma_{\beta-1}$. It is easily seen that the assertions (i)-(iv) are satisfied.

THEOREM 2.3. *If G is a reduced p-primary abelian group, $(p > 2)$, and if G is a direct sum of countable groups, then every endomorphism of G is a sum of two automorphisms.*

Proof. Let $G = \bigcup_{\gamma \in \Lambda} H_\gamma$ as is given in 2.2 and let $\psi \in E(G)$. We define inductively, for each $\gamma \in \Lambda$, automorphisms θ_γ and θ'_γ such that $\psi \mid H_\gamma = \theta_\gamma + \theta'_\gamma$ and if $\alpha < \gamma$, $\theta_\gamma \mid H_\alpha = \theta_\alpha$ and $\theta'_\gamma \mid H_\alpha = \theta'_\alpha$. For $\gamma = 0$, 1.1 gives θ_0 and θ'_0. Assume that θ_γ and θ'_γ have been defined for all $\gamma < \beta$. If β is a limit ordinal we define $\theta_\beta = \bigcup_{\gamma<\beta} \theta_\gamma$ and $\theta'_\beta = \bigcup_{\gamma<\beta} \theta'_\gamma$. Assume that $\beta - 1$ exists. Then $H_\beta = H_{\beta-1} \oplus C_\beta$. Let π_1 and π_2 be the projections of H_β onto $H_{\beta-1}$ and C_β respectively. Then $\pi_2(\psi \mid C_\beta) \in E(C_\beta)$ and by 1.1 there exist ϕ and ϕ', automorphisms of C_β, such that $\pi_2(\psi \mid C_\beta) = \phi + \phi'$. For each $c \in C$ choose $v_c \in H_{\beta-1}$ such that $2v_c = \pi_1\psi(c)$. Plainly v_c defines a homomorphism of C_β into $H_{\beta-1}$. Now define θ_β and θ'_β on H_β by

$$\theta_\beta(x + c) = \theta_{\beta-1}(x) + \phi(c) + v_c$$
$$\theta'_\beta(x + c) = \theta'_{\beta-1}(x) + \phi'(c) + v_c$$

where $x \in H_{\beta-1}$ and $c \in C_\beta$. It is routine to check that θ_β and θ'_β satisfy the required conditions.

3. **An application.** In this section we show that if G is a torsion complete group, (in the p-adic topology), then each endomorphism of G is a sum of two automorphisms. This result will follow as a corollary of a theorem in [6].

NOTATION. $B_n = \oplus \sum_{i \in I_n} \{b_{in}\}$, for some index set I_n and $o(b_{in}) = p^{n+1}$.

$$B = \oplus \textstyle\sum_{n<\omega} B_n$$
$$\bar{B} = \text{torsion subgroup of } \textstyle\prod_{n<\omega} B_n$$
$$C_n = \{p^n\bar{B}, B_n, B_{n+1}, \cdots\}\,.$$

It is not difficult to show that

$$(1) \qquad \bar{B} = \oplus \textstyle\sum_{k<n} B_k \oplus C_n\,.$$

Let π_n be the natural projection of $\bar{B}$ onto C_n as determined by (1). Define $\rho_n = \pi_n - \pi_{n+1}$. Then ρ_n is a projection of $\bar{B}$ onto B_n. Define λ_n: $E(\bar{B}) \to E(B_n[p])$ by

$$(\lambda_n\phi)(x) = \rho_n(\phi(x)) \quad \text{for all} \quad x \in B_n[p]\,.$$

LEMMA 3.1. *λ_n is a ring homomorphism of $E(\bar{B})$ into $E(B_n[p])$.*

Proof. If $\phi \in E(\bar{B})$ then plainly $\lambda_n\phi$ maps $B_n[p]$ to $B_n[p]$. It is also clear that λ_n is a group homomorphism of $E(\bar{B})$ to $E(B_n[p])$. Note now that $x \in C_n[p]$ implies $\phi(x) \in C_n[p]$. Suppose that $\phi, \psi \in E(\bar{B})$. For $i \in I_n$, we can write

$$\psi(p^n b_{in}) = \sum_{j \in I_n} \alpha_{ij} p^n b_{jn} + x$$

where $x \in C_{n+1}[p]$. Consequently

$$\lambda_n \psi(p^n b_{in}) = \sum_{j \in I_n} \alpha_{ij} p^n b_{jn}$$

so that

$$\psi(p^n b_{in}) = \lambda_n \psi(p^n b_{in}) + x \ .$$

Then

$$\lambda_n \phi\psi(p^n b_{in}) = \rho_n \phi(\lambda_n \psi(p^n b_{in}) + x) = (\lambda_n \phi)(\lambda_n \psi) p^n b_{in}) \ .$$

LEMMA 3.2. *Define*

$$\lambda : E(\bar{B}) \longrightarrow \prod_{n<\omega} E(B_n[p])$$

by

$$\lambda : \phi \longrightarrow (\lambda_0\phi, \lambda_1\phi, \cdots, \lambda_n\phi, \cdots) \ .$$

Then λ is a ring epimorphism.

Proof. Certainly λ is a ring homomorphism. Let

$$(\phi'_0, \phi'_1, \cdots, \phi'_n, \cdots) \in \pi_{n<w} E(B_n[p]) \ ,$$

where $\phi'_n \in (B_n[p])$. Extend ϕ'_n to $\phi^\#_n \in E(B_n)$. Now define $\phi \in E(\bar{B})$ as follows. If $x \in \bar{B}$, $x = \lim_m \sum_{k<m} \rho_k(x)$, (the limit being taken in the p-adic topology) and hence the elements

$$\sum_{k<m} \phi^\#_k \rho_k(x) \ , \qquad m = 1, 2, \cdots$$

form a Cauchy sequence in $\bar{B}$, with bounded order. Define $\phi(x)$ to be the limit of this sequence. If $x \in B_n[p]$ then $\phi(x) = \phi'_n(x) \in B_n[p]$ and hence

$$\lambda\phi = (\phi'_0, \phi'_1, \cdots, \phi'_n, \cdots) \ .$$

LEMMA 3.3. *Let R be a ring and $J(R)$ the Jacobson radical of R. If each element of $R/J(R)$ is a sum of n units then each element of R is a sum of n units.*

Proof. If $y + J(R)$ is a unit then it has an inverse $z + J(R)$ and $zy + J(R) = 1 + J(R)$. This implies that $zy = 1 - r$ for some $r \in J(R)$

and hence zy is a unit. Thus $((zy)^{-1}z)y = 1$ so that y has a left inverse. Similarly y has a right inverse and hence y is a unit. It now follows easily that if $x + J(R)$ is a sum of n units in $R/J(R)$ then x is a sum of n units in R.

THEOREM 3.4. $\prod_{n<\omega} E(B_n[p]) \cong E(\bar{B})/J(E(\bar{B}))$.

Proof. Let λ be as in 3.2. We prove Ker $\lambda = J(E(\bar{B}))$. Suppose $\lambda(\phi) = 0$. Then for each $n, x \in B_n[p]$ implies $\phi(x) \in C_{n+1}$. Hence, if $x \neq 0$, $h(\phi(x)) > h(x)$. That is,

$$\phi \in H(\bar{B}) = \{\phi \in E(\bar{B}) \mid x \in \bar{B}[p], x \neq 0 , \quad \text{implies } h(\phi(x)) > h(x)\} .$$

Conversely, if $\phi \in H(\bar{B})$, then clearly $\lambda\phi = 0$. The result now follows from [6], p. 287, which states that $J(E(\bar{B})) = H(\bar{B})$.

COROLLARY 3.5. *If $\phi \in E(\bar{B})$ then ϕ is the sum of two automorphisms.*

Proof. $B_n[p]$ is a direct sum of countable groups and therefore, by 2.3, each $\phi_n \in E(B_n[p])$ is a sum of two automorphisms from which it follows that each element of $E(\bar{B})/J(E(\bar{B}))$ is a sum of two units. consequently each element of $E(\bar{B})$ is a sum of two units.

4. An example. In this section we exhibit, for an arbitrary prime p, a reduced p-primary group G for which the group of automorphisms does not generate the ring of endomorphisms. The group G will in fact be without nonzero elements of infinite height and have a countable basic subgroup.

The notation will be as in § 3 except that we require each index set I_n to be countably infinite.

Define τ on $\bar{B}$ by the condition

$$\tau(b_{m\,n}) = b_{m+1\,n} .$$

Plainly this property uniquely determines an endomorphism of $\bar{B}$. Let R be the subring of $E(\bar{B})$ generated by the identity and τ.

LEMMA 4.1. *If $\phi \in R - pR$ then ϕ is one-to-one.*

Proof. Let $\phi = \sum_{i=0}^{r} k_i\tau^i$ and suppose $\phi(x) = 0$ for some $x \in \bar{B}[p] - \{0\}$. We show that $\phi \in pR$. Indeed suppose otherwise. Then we can assume that p does not divide k_r. Write $x = \sum_{n<\omega} (\sum_{m<\omega} a_{mn}p^n b_{mn})$ where $a_{mn} \in Z$ and, for each n, almost all a_{mn} are zero. We have

$$0 = \phi(x) = \sum_{n<\omega}\Big(\sum_{m<\omega} a_{mn}p^n\Big(\sum_{i=0}^{r} k_i b_{m+i\,n}\Big)\Big) .$$

Since $x \neq 0$, we can choose n so that $\sum_{m<\omega} a_{mn}p^n b_{mn} \neq 0$ and hence, for this n, there is a largest $m = M$ such that p does not divide a_{Mn}. It then follows that

$$\sum_{m<\omega} a_{mn}p^n\left(\sum_{i=0}^{r} k_i b_{m+i\ n}\right) = a_{Mn}k_r p^n b_{M+r\ n} + \sum_{j<M+r} c_j p^n b_{jn} ,$$

for some integers c_j, and from this expression it follows that $\phi(x) \neq 0$.

Let $\mathscr{C}$ be the collection of subrings S of $E(\bar{B})$ which satisfy, for all nonnegative integers e, the condition

C_e: $\phi \in S$ and $\phi((p^n\bar{B})[p^e]) = 0$ for some n implies $\phi \in P^eS$.

We note that if S is a subring of $E(\bar{B})$ which satisfies C_1 then $S \in \mathscr{C}$. Clearly our subring $R \in \mathscr{C}$.

Let $\bar{R}$ be the closure of R in the p-adic topology on $E(\bar{B})$.

LEMMA 4.2. *If $\phi \in \bar{R} - p\bar{R}$ then ϕ is one-to-one.*

Proof. Let $\{\phi_n\}_{n=1}^{\infty}$ be a Cauchy sequence in R which converges to ϕ. We can assume that, for all $n > 0$, $\phi_n - \phi_{n+1} \in p^nR$. Now note that if $\phi_r \in pR$ for some r then $\phi_n \in pR$ for all n. Hence we could write $\phi_n = p\psi_n$, $\psi_n \in R$. Then

$$\phi = \lim_n \phi_n = \lim_n p\psi_n = p \lim_n \psi_n = p\psi,\ \psi \in \bar{R} .$$

Thus we can assume no $\phi_n \in pR$.

Let $x \in \bar{B}[p]$, $x \neq 0$. We can write $x = b_m + y$ where $0 \neq b_m \in B_m$ and $h(y) > m$. Since the conditions $\theta \in R - pR$ and $0 \neq b_n \in B_n$ always imply $0 \neq \theta(b_n) \in B_n$ it follows that $h(\phi_n(x)) \leqq m$ for all n. Hence

$$\phi(x) = \lim \phi_n(x) \neq 0 .$$

It is clear that $\bar{R} \in \mathscr{C}$. Furthermore $\bar{R}$ is a closed separable subring of $E(\bar{B})$. Thus $\bar{R}$ satisfies the hypothesis of the following theorem by A. L. S. Corner [1].

THEOREM 4.3. *Let C be a torsion complete p-group with unbounded countable basic subgroup B, and let Φ be a separable closed subring of $E(C)$ such that $\Phi(B) \subseteqq B$ and, for all positive integers e, Φ satisfies C_e. Then there is a pure subgroup G of C containing B such that*

$$E(G) = \Phi \oplus E_s(G) .$$

Here $E_s(G)$ is the subring of small endomorphisms of G. Note, too, that if G is pure in C we can consider $E(G)$ as being embedded in $E(C)$

Now if G is a pure subgroup of $\bar{B}$ corresponding to $\bar{R}$ in this theorem we can write

$$E(G) = \bar{R} \oplus E_S(G) .$$

We then get the following epimorphism of $E(G)$

$$\gamma: E(G) \to \bar{R} \to \bar{R}/p\bar{R} \simeq R/pR \simeq Z_p[X]$$

where X is an indeterminate. The only units in $Z_p[X]$ are the nonzero constant polynomials. Thus, the nonconstant polynomials cannot be written as sums of units and therefore, their pre-images in $E(G)$ cannot be written as sums of automorphisms.

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