PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 3, 1968

ON $(J, M, \mathfrak{m})$-EXTENSIONS OF ORDER SUMS OF DISTRIBUTIVE LATTICES

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In the first section of this paper a characterization of the order sum of a family $\{L_\alpha\}_{\alpha \in S}$ of distributive lattices is given which is analogous to the characterization of a free distributive lattice as one generated by an independent set. We then consider the collection Q of order sums obtained by taking different partial orderings on S. A natural partial ordering is defined on Q and its maximal and minimal elements are characterized.

Let J and M be collections of nonempty subsets of a distributive lattice L, and $\mathfrak{m}$ a cardinal. We define a $(J, M, \mathfrak{m})$-extension (ψ, E) of L, where E is a $\mathfrak{m}$-complete distributive lattice and $\psi: L \rightarrow E$ is a (J, M)-monomorphism. In the last section we define a $\mathfrak{m}$-order sum of a family of distributive lattices $\{L_\alpha\}_{\alpha \in S}$. The main result here is that the $\mathfrak{m}$-order sum exists if the order sum L of $\{L_\alpha\}_{\alpha \in S}$ has a $(J, M, \mathfrak{m})$-extension, where J and M are certain collections of subsets of L. These results are analogous to R. Sikorski's work in Boolean algebras (e.g., [6]).

1. Order sums. Let S be a fixed set and $\{L_\alpha\}_{\alpha \in S}$ a fixed collection of distributive lattices. From [2] it follows that for each poset $P = (S, \leqq)$, there exists a pair $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$, where $L(P)$ is a distributive lattice, and for each $\alpha \in S$, $\varphi_\alpha : L_\alpha \rightarrow L(P)$ is a monomorphism such that:

(1.1) L is generated by $\bigcup_{\alpha \in S} \varphi_\alpha(L_\alpha)$.

(1.2) If $\alpha < \beta$ then $\varphi_\alpha(x) < \varphi_\beta(y)$, for all $x \in L_\alpha$ and $y \in L_\beta$.

(1.3) If M is a distributive lattice and $\{f_\alpha : L_\alpha \rightarrow M\}_{\alpha \in S}$ is a family of homomorphisms such that $f_\alpha(x) \leqq f_\beta(y)$ whenever $\alpha < \beta$, $x \in L_\alpha$ and $y \in L_\beta$, then there exists a homomorphism $f: L(P) \rightarrow M$ such that $f\varphi_\alpha = f_\alpha$ for each $\alpha \in S$.

The pair $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ will be called an *order sum of* $\{L_\alpha\}_{\alpha \in S}$ *over* P.

Let $\boldsymbol{P}$ be the family of all posets of the form $(S, \leqq)$ and let $Q = \{(\{\varphi_\alpha\}_{\alpha \in S}, L(P)) \mid P \in \boldsymbol{P}\}$. For $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ and $(\{\theta_\alpha\}_{\alpha \in S}, L(P'))$ in Q we write

(1.4) $(\{\varphi_\alpha\}_{\alpha \in S}, L(P)) \leqq (\{\theta_\alpha\}_{\alpha \in S}, L(P'))$ provided:

(1.5) there is a homomorphism $f: L(P') \rightarrow L(P)$ such that $f\theta_\alpha = \varphi_\alpha$ for each $\alpha \in S$.

Note that (1.5) implies f is an epimorphism. If f is an isomorphism, we say that $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ is *isomorphic* with $(\{\varphi_\alpha\}_{\alpha \in S}, L(P'))$. Isomorphism in this sense is an equivalence relation $\simeq$, and [2, Th. 1.2] implies that any two order sums over P are isomorphic. By identifying isomorphs, (1.4) determines a partial ordering on the equivalence classes of $Q/\simeq$.

DEFINITION 1.1. Suppose $P \in \boldsymbol{P}$ and $\{N_\alpha\}_{\alpha \in S}$ is a family of sublattices of a distributive lattices N. The family $\{N_\alpha\}_{\alpha \in S}$ is called *P-independent* if whenever $\alpha_1, \cdots, \alpha_m$ are distinct elements of S, $a_{m+1}, \cdots, \alpha_n$ are distinct elements of S and $x_i \in N_{\alpha_i}$ for $i = 1, \cdots, n$ then

(1.6) $x_1 \cdot \cdots \cdot x_m \leqq x_{m+1} + \cdots + x_n$ if and only if

(1.7) for some i and j, either $\alpha_i < \alpha_j$ or $\alpha_i = \alpha_j$ and $x_i \leqq x_j$, where $1 \leqq i \leqq m$ and $m + 1 \leqq j \leqq n$.

LEMMA 1.2. *Suppose N and M are distributive lattices and $\{N_\alpha\}_{\alpha \in S}$ is a collection of sublattices of N such that $\cup_{\alpha \in S} N_\alpha$ generates N. A necessary and sufficient condition for a family $\{f_\alpha : N_\alpha \to M\}_{\alpha \in S}$ of homomorphisms to have a common extension on N is that if $\alpha_1, \cdots, \alpha_m$ are distinct members of S, $\alpha_{m+1}, \cdots, \alpha_n$ are distinct members of S, $x_i \in N_{\alpha_i}$ for $i=1, \cdots, n$ and*

(1.8) $x_1 \cdot \cdots \cdot x_m \leqq x_{m+1} + \cdots + x_n$ *then*

(1.9) $f_{\alpha_1}(x_1) \cdot \cdots \cdot f_{\alpha_m}(x_m) \leqq f_{\alpha_{m+1}}(x_{m+1}) + \cdots + f_{\alpha_n}(x_n)$.

Proof. The necessity is clear. Now if $x \in N_\alpha \cap N_\beta$ then by (1.9), $x \leqq x$ implies that $f_\alpha(x) = f_\beta(x)$. So the function f: $\cup_{\alpha \in S} N_\alpha \to M$ defined by $f(x) = f_\alpha(x)$ if $x \in L_\alpha$ makes sense and has the property that if A and B are finite nonempty subsets of $\cup_{\alpha \in \alpha S} N_\alpha$, then $\Pi_N(A) \leqq \Sigma_N(B)$ implies $\Pi_M f(A) \leqq \Sigma_M f(B)$. By [1, Lemma 1.7], f can be extended to a homomorphism $f' : N \to M$. This is the required extension.

THEOREM 1.3. *The pair $(\{\theta_\alpha\}_{\alpha \in S}, L)$ is the order sum of $\{L_\alpha\}_{\alpha \in S}$ over $P \in \boldsymbol{P}$ if and only if $\{\theta_\alpha : L_\alpha \to L\}_{\alpha \in S}$ is a family of monomorphisms such that:*

(1.10) $\cup_{\alpha \in S} \theta_\alpha(L_\alpha)$ *generates L, and*

(1.11) $\{\theta_\alpha(L_\alpha)\}_{\alpha \in S}$ *is P-independent.*

Proof. For the sufficiency suppose first that $\alpha < \beta$. By (1.11) $\theta_\alpha(x) \leqq \theta_\beta(y)$ for all $x \in L_\alpha$, $y \in L_\beta$. But if $\theta_\beta(y) \leqq \theta_\alpha(x)$ then $\beta \leqq \alpha$. Hence (1.2) is satisfied. Now assume the hypothesis of (1.3). It is sufficient to show that the family $\{f_\alpha \theta_\alpha^{-1} : \theta_\alpha(L_\alpha) \to M\}_{\alpha \in S}$ has a common extension on L. So if

$$\theta_{\alpha_1}(x_1) \cdot \cdots \cdot \theta_{\alpha_m}(x_m) \leqq \theta_{\alpha_{m+1}}(x_{m+1}) + \cdots + \theta_{\alpha_n}(x_n)$$

where $\alpha_1, \cdots, \alpha_m$ are distinct and $\alpha_{m+1}, \cdots, \alpha_n$ are distinct then by (1.11) there exists p, q such that $\alpha_p < \alpha_q$ or $\alpha_p = \alpha_q$ and $\theta_{\alpha_p}(x_p) \leqq \theta_{\alpha_q}(x_q)$, where $1 \leqq p \leqq m$ and $m+1 \leqq q \leqq n$. In any case $f_{\alpha_p}(x_p) \leqq f_{\alpha_q}(x_q)$ and so

$$\Pi_{i=1}^{m} f_{\alpha_i} \theta_{\alpha_i}^{-1} \theta_{\alpha_i}(x_i) \leqq \Sigma_{j=m+1}^{n} f_{\alpha_j} \theta_{\alpha_j}^{-1} \theta_{\alpha_j}(x_j) .$$

The result now follows from Lemma 1.2. The converse is essentially [2, Th. 1.9].

The set $\boldsymbol{P}$ can be partially ordered as follows. If $P, P' \in \boldsymbol{P}$ then $P \leqq P'$ provided $P' \subseteqq P$, as sets of ordered pairs. It is immediate that $\boldsymbol{P}$ has a greatest element—the trivial partial ordering on S. Also, it can be shown that P is minimal in $\boldsymbol{P}$ if and only if P is a chain.

THEOREM 1.4. $\boldsymbol{P} \cong Q/\simeq$.

Proof. It is sufficient to show that for $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$, $(\{\theta_\alpha\}_{\alpha \in S}, L(P')) \in Q$:

(1.12) $P \leqq P'$

if and only if

(1.13) $(\{\varphi_\alpha\}_{\alpha \in S}, L(P)) \leqq (\{\theta_\alpha\}_{\alpha \in S}, L(P'))$.

If $P \leqq P'$, then $\{\varphi_\alpha : L_\alpha \rightarrow L(P)\}_{\alpha \in S}$ is a family of homorphisms with the property that if $\alpha < \beta$ (in P') then $\varphi_\alpha(x) < \varphi_\beta(y)$ for all $x \in L_\alpha$, $y \in L_\beta$. So by (1.3), we have (1.13). Conversely, suppose (1.5) holds and $\alpha < \beta$ (in P'). Letting $x \in L_\alpha$ and $y \in L_\beta$, we have $\theta_\alpha(x) < \theta_\beta(y)$ so $\varphi_\alpha(x) = f\theta_\alpha(x) \leqq f\theta_\beta(y) = \varphi_\beta(y)$. Since $\{\varphi_\alpha(L_\alpha)\}_{\alpha \in S}$ is P-independent, $\alpha \leqq \beta$ (in P). It follows that $P' \subseteqq P$.

COROLLARY 1.5. *$(\{\varphi_\alpha\}_{\alpha \in S}, L(P))/\simeq$ is the greatest element in $Q/\simeq$ if and only if $L(P)$ is the free product of $\{L_\alpha\}_{\alpha \in S}$. Furthermore, $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))/\simeq$ is minimal in $Q/\simeq$ if and only if $L(P)$ is an ordinal sum of $\{L_\alpha\}_{\alpha \in S}$.*

Proof. The definitions of free product and ordinal sum can be found in [7, § 9] and [2, Definition 1.3]. The result then follows from Theorem 1.4 and the remark following Theorem 1.3.

For the remainder of this section, let $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ be a fixed member of Q.

A lattice L is said to be *conditionally implicative* if for each pair $x, y \in L$ such that $x \nleqq y$ there is an element $x \rightarrow y$ with the property that $x \cdot z \leqq y$ if and only if $z \leqq x \rightarrow y$. Note that conditionally implicative lattices are distributive. The following theorem, which we stated without proof in [2], is the converse of [2, Th. 2.5].

THEOREM 1.6. *If* $L(P)$ *is conditionally implicative then* L_α *is conditionally implicative for each* $\alpha \in S$.

Proof. Let $x, y \in L_\alpha$ and $x \not\leqq y$. Then $\varphi_\alpha(x) \rightarrow \varphi_\alpha(y)$ exists in $L(P)$ and equals a sum of m products, each of the form

$$\varphi_{\gamma_1}(x_1) \cdot \cdots \cdot \varphi_{\gamma_n}(x_n) .$$

We can assume $\gamma_i \not\leqq \gamma_j$ for $i \neq j$. Now

$$\varphi_\alpha(x)(\varphi_{\gamma_1}(x_1) \cdot \cdots \cdot \varphi_{\gamma_n}(x_n)) \leqq \varphi_\alpha(x)(\varphi_\alpha(x) \rightarrow \varphi_\alpha(y)) \leqq \varphi_\alpha(y) .$$

By (1.11) there exists p such that $\gamma_p < \alpha$ or $\gamma_p = \alpha$ and $xx_p \leqq y$. But in any case $\varphi_\alpha(x)\varphi_{\gamma_p}(x_p) \leqq \varphi_\alpha(y)$. Hence

(1.14) $\varphi_{\gamma_p}(x_p) \leqq \varphi_\alpha(x) \rightarrow \varphi_\alpha(y)$.

Choosing an element $\varphi_{\beta_j}(y_j)$, that satisfies (1.14), from each of the m summands of $\varphi_\alpha(x) \rightarrow \varphi_\alpha(y)$, we have:

$$\sum_{j=1}^{m} \varphi_{\beta_j}(y_j) \leqq \varphi_\alpha(x) \rightarrow \varphi_\alpha(y) \leqq \sum_{j=1}^{m} \varphi_{\beta_j}(y_j) ,$$

and so $\varphi_\alpha(x) \rightarrow \varphi_\alpha(y) = \sum_{j=1}^{p}\varphi_{\beta_j}(y_j)$, where $\beta_i \not\leqq \beta_j$ for $i \neq j$. For each j, $\varphi_\alpha(x)\varphi_{\beta_j}(y_j) \leqq \varphi_\alpha(x)(\varphi_\alpha(x) \rightarrow \varphi_\alpha(y)) \leqq \varphi_\alpha(y)$, and since $x \not\leqq y$, we have: $\beta_j \leqq \alpha$ for $j = 1, \cdots, p$. But $\varphi_\alpha(y) \leqq \varphi_\alpha(x) \rightarrow \varphi_\alpha(y) = \varphi_{\beta_1}(y_1) + \cdots + \varphi_{\beta_p}(y_p)$. Hence there exists j_0 such that $\alpha \leqq \beta_{j_0}$. Since $\alpha = \beta_{j_0}$ and $\alpha > \beta_j$ for $j \neq j_0$, we have $\varphi_\alpha(x) \rightarrow \varphi_\alpha(y) = \varphi_\alpha(x_{j_0})$. From the fact that φ_α is a monomorphism, it is now easy to show that $x \rightarrow y = x_{j_0}$.

The following property of φ_α will be needed in § 3. Note that the power of a set H is denoted by $|H|$.

DEFINITION 1.7. Let L and M be distributive lattices and $\mathfrak{m}$ a cardinal. A homomorphism h: $L \rightarrow M$ is called a $\mathfrak{m}$ *– homomorphism* provided:

If $H \subseteqq L$, $0 < |H| \leqq \mathfrak{m}$, and $\Sigma_L(H)$ exists then $\Sigma_M h(H)$ exists and equals $h(\Sigma_L(H))$; and similarly for products. The homomorphism is *complete* if it is a $\mathfrak{m}$-homomorphism for each cardinal $\mathfrak{m}$.

LEMMA 1.8. *Each monomorphism* $\varphi_\alpha : L_\alpha \rightarrow L(P)$ *of* $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ *is complete.*

Proof. Let $H \subseteqq L_\alpha$ and suppose $x = \Sigma_{L_\alpha}(H)$ exists. Clearly $\varphi_\alpha(y) \leqq \varphi_\alpha(x)$ for all $y \in H$. Now suppose that $\Sigma_{L(P)}(H_1) \cdot \cdots \cdot \Sigma_{L(P)}(H_n)$ is an upper bound for $\varphi_\alpha(H)$, where $H_i \subseteqq \cup_{\alpha \in S}\varphi_\alpha(L_\alpha)$ for $i = 1, \cdots, n$.

We can assume $H_1 = \{\varphi_{\alpha_1}(x_1 \cdot \cdots \cdot \varphi_{\alpha_m}(x_m)\}$ where $x_i \in L_{\alpha_i}$ and $\alpha_k \neq \alpha_j$ for $k \neq j$. Suppose:

(1.15) there exists $j \in \{1, \cdots, m\}$ such that $\alpha < \alpha_j$. Then $\varphi_\alpha(x) < \varphi_{\alpha_j}(x_j)$ so

(1.16) $\varphi_\alpha(x) \leqq \Sigma_{L(P)}(H_1)$.

Now suppose that (1.15) does not hold. Since $\varphi_\alpha(y) \leqq \varphi_{\alpha_1}(x_1) + \cdots + \varphi_{\alpha_m}(x_m)$ for each $y \in H$, and $\alpha_j \neq \alpha_k$ for $j \neq k$, there exists α_j such that $\alpha = \alpha_j$ and $\varphi_\alpha(y) \leqq \varphi_{\alpha_j}(x_j)$ for all $y \in H$. Hence $x_j \in L_\alpha$ and $y \leqq x_j$ for all $y \in S$. So $x \leqq x_j$ and therefore (1.16) is valid regardless of the validity of (1.15). Applying this argument to each H_i, we have $\varphi_\alpha(x) \leqq \Sigma_{L(P)}(H_1) \cdot \cdots \cdot \Sigma_{L(P)}(H_n)$, and so $\varphi_\alpha(\Sigma_{L_\alpha}(H)) = \Sigma_{L(P)}\varphi_\alpha(H)$. Similarly for products.

2. **$(J, M, \mathfrak{m})$-extensions.** Throughout this section, let L be a distributive lattice, and $\mathfrak{m}$ a fixed infinite cardinal. Also let J and M be collections of nonempty subsets of L such that

(2.1) $|H| \leqq \mathfrak{m}$ for each $H \in J$ and each $H \in M$.

(2.2) $\Sigma_L(H)$ exists for each $H \in J$ and $\Pi_L(H)$ exists for each $H \in M$.

DEFINITION 2.1. If L' is a distributive lattice then a homomorphism $f\colon L \to L'$ is a *(J, M)-homomorphism* provided:

(2.3) If $H \in J$ then $\Sigma_{L'}f(H)$ exists and equals $f(\Sigma_L(H))$.

(2.4) If $H \in M$ then $\Pi_{L'}f(H)$ exists and equals $f(\Pi_L(H))$.

DEFINITION 2.2. The pair (ψ, E) is called a *$(J, M, \mathfrak{m})$-extension* of L provided:

(2.5) E is a $\mathfrak{m}$-complete distributive lattice.

(2.6) $\psi\colon L \to E$ is a (J, M)-monomorphism.

(2.7) $\psi(L)$ $\mathfrak{m}$-generates E (i.e., E is the smallest $\mathfrak{m}$-complete sublattice of E that contains $\psi(L)$).

Every distributive lattice has a $(\phi, \phi, \mathfrak{m})$-extension: the smallest $\mathfrak{m}$-ring of subsets of the Stone space X of L that contains all of the compact-open sets of X, together with the correspondence that associates elements of L with compact-open sets of X. If $J(M)$ is the collection of all subsets of L of power $\leqq \mathfrak{m}$ which have a sum (product) in L then a $(J, M, \mathfrak{m})$-extension of L is called a *$\mathfrak{m}$-regular extension.* Note that in this case, ψ is a $\mathfrak{m}$-homomorphism. In [5], Crawley has constructed an example of a distributive lattice which can not be regularly imbedded in any complete distributive lattice. In this example if we take I to be countable then L will have no $\aleph_0$-regular extension.

A sufficient condition for L to have a $(J, M, \mathfrak{m})$-extension is that L be conditionally implicative. Indeed, it is easily verified that the MacNeille completion [3, p. 58] of such a lattice is also conditionally implicative and hence distributive. Note that the category of conditionally implicative lattices includes the categories of Boolean algebras, chains, free and finite distributive lattices, and pseudo Boolean algebras. Another sufficient condition for L to have a $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extension is that

$$(2.8) \qquad y \sum_{i \in I} x_i = \sum_{i \in I} y x_i \text{ and } y + \prod_{i \in I} x_i = \prod_{i \in I} (y + x_i)$$

whenever the left sides exist and $|I| \leqq \mathfrak{m}$. This follows from [4, Lemma 2].

If (ψ, E) and (ψ', E') are $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extensions of L, then we write

(2.9) $(\psi, E) \leqq (\psi', E')$

provided there is a $\mathfrak{m}$-homomorphism $h: E' \to E$ such that $h\psi' = \psi$. Clearly h is onto. If h is an isomorphism we say (ψ, E) is *isomorphic* with (ψ', E'). Isomorphism in this sense is an equivalence relation $\simeq$, and by identifying isomorphs, (2.9) determines a partial ordering on the equivalence classes of $K/\simeq$ where K is the set of $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extensions of L.

By generalizing the method in [6, p. 166], we now investigate the class K.

DEFINITION 2.3. A congruence relation R on a $\mathfrak{m}$-complete lattice M is called a $\mathfrak{m}$-*congruence relation* on M if whenever I is an index set of power $\leqq \mathfrak{m}$ and $(x_i, y_i) \in R$ for each $i \in I$ then

$$(\Sigma\{x_i \mid i \in I\}, \Sigma\{y_i \mid i \in I\}) \in R \text{ and } (\Pi\{x_i \mid i \in I\}, \Pi\{y_i \mid i \in I\}) \in R \,.$$

For a $\mathfrak{m}$-congruence relation R on a $\mathfrak{m}$-complete lattice M, let $[x]_R$ be the equivalence class containing $x \in M$, and let

$$M/R = \{[x] \mid x \in M\} \,.$$

The following theorem is easily verified.

THEOREM 2.4. *If R is a $\mathfrak{m}$-congruence relation on a $\mathfrak{m}$-complete lattice M then M/R is partially ordered as follows*: $[x]_R \leqq [y]_R$ *provided there exists* $x', y' \in M$ *such that* $(x, x') \in R$, $x' \leqq y'$ *and* $(y', y) \in R$. *Furthermore, M/R is a $\mathfrak{m}$-complete lattice such that if* $H \subseteqq M$ *and* $0 < |H| \leqq \mathfrak{m}$ *then* $\Sigma_{M/R}\{[x]_R \mid x \in H\} = [\Sigma_M(H)]_R$ *and* $\Pi_{M/R}\{[x]_R \mid x \in H\} = [\Pi_M(H)]_R$. *If M is distributive so is M/R.*

Let $\mathfrak{n}$ be the power of the distributive lattice L and let F be the free $\mathfrak{m}$-complete distributive lattice with $\mathfrak{n}$ generators. That is, F satisfies:

(2.10) F is a $\mathfrak{m}$-complete distributive lattice and is $\mathfrak{m}$-generated by a subset G of power $\mathfrak{n}$.

(2.11) If $h: G \to M$ is a function, where M is a $\mathfrak{m}$-complete distributive lattice, then h can be extended to a $\mathfrak{m}$-homomorphism on F.

By (2.11), G has the property that if G_1, G_2 are finite nonempty subsets of G and $\Pi_F(G_1) \leqq \Sigma_F(G_2)$, then $G_1 \cap G_2 \neq \phi$. So the sublattice F' generated by G is freely generated by G, and there is an epimorphism g: $F' \to L$. Let $\boldsymbol{R}$ be the set of $\mathfrak{m}$-congruence relations R on F such that:

(2.12) If $x, y \in F'$ then $(x, y) \in R \Leftrightarrow g(x) = g(y)$.

(2.13) If $H \subseteqq F'$, $|H| \leqq \mathfrak{m}$, $g(H) \in \boldsymbol{J}$, $x \in F'$, and $g(x) = \Sigma_L g(H)$ then $(x, \Sigma_F(H)) \in R$.

(2.14) If $H \subseteqq F'$, $|H| \leqq \mathfrak{m}$, $g(H) \in \boldsymbol{M}$, $x \in F'$ and $g(x) = \Pi_L g(H)$ then $(x, \Pi_F(H)) \in R$.

For each $R \in \boldsymbol{R}$, let F'_R be the sublattice $\{[x]_R \mid x \in F'\}$ of F/R. By (2.12), the mapping $g_R : F'_R \to L$ defined by:

(2.15) $g_R([x]_R) = g(x)$ for each $x \in F'$ is an isomorphism. Define $\psi_R : L \to F/R$ by $\psi_R = i_R g_R^{-1}$ where $i_R : F'_R \to F/R$ is the inclusion map. We have

(2.16) $\psi_R g(x) = [x]_R$ for each $x \in F'$.

THEOREM 2.5. *For each $R \in \boldsymbol{R}$, the pair $(\psi_R, F/R)$ is a $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extension of L.*

Proof. First F/R is $\mathfrak{m}$-complete by Theorem 2.4. Let $G \in J$, then $|G| \leqq \mathfrak{m}$ and $\Sigma_L(G)$ exists. Since g is onto L there exists $\{x\} \cup H \subseteqq F'$ such that $|H| \leqq \mathfrak{m}$, $g(H) = G$ and $g(x) = \Sigma_L g(H)$. By (2.13), $(x, \Sigma_F(H)) \in \boldsymbol{R}$ so

$$\begin{aligned}\psi_R(\Sigma_L(G)) &= [x]_R = [\Sigma_F(H)]_R = \Sigma_{F/R}\{[y]_R \mid y \in H\}\\ &= \Sigma_{F/R}\psi_R g(H) = \Sigma_{F/R}\psi_R(G)\,.\end{aligned}$$

A similar argument for $G \in \boldsymbol{M}$ implies that ψ_R is a $(\boldsymbol{J}, \boldsymbol{M})$-monomorphism. Finally since

$$\psi_R(L) = \psi_R g(F') = F'_R$$

and F' $\mathfrak{m}$-generates F, we have $\psi_R(L)$ $\mathfrak{m}$-generates F/R.

THEOREM 2.6. *For each $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extension (ψ, E) of L, there exists $R \in \boldsymbol{R}$ such that $(\psi, E) \simeq (\psi_R, F/R)$.*

Proof. By (2.11), the mapping $\psi g : F' \to E$ can be extended to a $\mathfrak{m}$-homomorphism k of F onto E. Define a relation R on F by $(x, y) \in R$ if $k(x) = k(y)$. It is easily verified that $R \in \boldsymbol{R}$ so that by Theorem 2.5, $(\psi_R, F/R)$ is a $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extension of L. Next, define $h : F/R \to E$ by $h([x]_R) = k(x)$ for each $x \in F$. Then h is an isomorphism. Let $y \in L$, then there is an $x \in F'$ such that $g(x) = y$, so

$$h\psi_R(y) = h\psi_R g(x) = h([x]_R) = k(x) = \psi g(x) = \psi(y) .$$

It follows that $(\psi, E) \simeq (\psi_R, F/R)$.

THEOREM 2.7. *If* $(\psi_R, F/R)$ *and* $(\psi_{R'}, F/R')$ *are* $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$*-extensions of L then*

$$(\psi_R, F/R) \leqq (\psi_{R'}, F/R')$$

if and only if

$$R' \subseteqq R .$$

Consequently, $K/\simeq$ *is isomorphic with* $\boldsymbol{R}$ *(partially ordered by the converse of inclusion).*

Proof. Suppose there is a $\mathfrak{m}$-epimorphism $h: F/R' \to F/R$ such that $h\psi_{R'} = \psi_R$. For each $x \in F'$, $h([x)]_{R'}) = h\psi_{R'}g(x) = \psi_R g(x) = [x]_R$. But, in fact, $\{x \in F \mid h([x]_{R'}) = [x]_R\}$ is a $\mathfrak{m}$-sublattice of F containing F'. So $h([x]_{R'}) = [x]_R$ for each $x \in F$. Thus if $(x, y) \in R'$ then $[x]_R = h([x]_{R'}) = h([y]_{R'} = [y]_R$, i.e., $R' \subseteqq R$. For the converse, define $h: F/R' \to F/R$ by $h([x]_{R'}) = [x]_R$ for each $x \in F$. The hypothesis implies h is a $\mathfrak{m}$-homomorphism. Since $h\psi_{R'} = \psi_R$, the result follows.

COROLLARY 2.8. *The intersection* $\rho = \bigcap_{R \in \boldsymbol{R}} R$ *is an element of* $\boldsymbol{R}$ *and hence the equivalence class containing* $(\psi_\rho, F/\rho)$ *is the greatest element in* $K/\simeq$. *Here it is assumed* $\boldsymbol{R} \neq \varnothing$.

Proof. Conditions (2.12), (2.13), and (2.14) are satisfied by ρ.

DEFINITION 2.9. A $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extension (ψ, E) of L is said to be *free* provided that for each $\mathfrak{m}$-complete distributive lattice L' and each $(\boldsymbol{J}, \boldsymbol{M})$-homomorphism $f: L \to L'$, there exists a $\mathfrak{m}$-homomorphism $h: E \to L'$ such that $f = h\psi$.

The main result of this section is then:

THEOREM 2.10. *If L has a* $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$*-extension then L has a free* $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$*-extension:* $(\psi_\rho, F/\rho)$.

Proof. As in the proof of Theorem 2.6, the mapping $fg: F' \to L'$ can be extended to a $\mathfrak{m}$-homomorphism $h': F \to L'$. Define a relation R' on F by $(x, y) \in R'$ if $h'(x) = h'(y)$. We first show that $R' \cap \rho \in \boldsymbol{R}$. Clearly $R' \cap \rho$ is a $\mathfrak{m}$-congruence relation. For (2.12), (2.13), and (2.14), first let $x, y \in F'$. Since $\rho \in \boldsymbol{R}$, $(x, y) \in R' \cap \rho$ implies $g(x) = g(y)$. Conversely if $g(x) = g(y)$ then $fg(x) = fg(y)$ so $(x, y) \in \rho \cap R'$. If

$H \subseteqq F'$, $|H| \leqq \mathfrak{m}$, $g(H) \in \boldsymbol{J}$, $x \in F'$ and $g(x) = \Sigma_L g(H)$ then since $\rho \in R$, $(x, \Sigma_F(H)) \in \rho$. But f is a $(\boldsymbol{J}, \boldsymbol{M})$-homomorphism so $fg(x) = f(\Sigma_L g(H)) = \Sigma_{L'} fg(H)$. Hence $h'(x) = \Sigma_{L'} h'(H) = h'(\Sigma_F(H))$, i.e., $(x, \Sigma_F(H)) \in \rho \cap R'$. Similarly for (2.14). Now $\rho \cap R' \in \boldsymbol{R}$ so $\rho \subseteqq R'$. Hence we can define $h : F/\rho \to L'$ by $h([x]_\rho) = h'(x)$ for each $x \in F$. It follows that h is a $\mathfrak{m}$-homomorphism and $f = h\psi_\rho$.

3. $\mathfrak{m}$**-order sums.** In this section $\{L_\alpha\}_{\alpha \in S}$ is a fixed set of distributive lattices, $\mathfrak{m}$ is a fixed infinite cardinal and P is a partial ordering on S.

DEFINITION 3.1. The pair $(\{\psi_\alpha\}_{\alpha \in S}, E)$ is said to be a $\mathfrak{m}$-*order sum of* $\{L_\alpha\}_{\alpha \in S}$ *over* P provided E is a $\mathfrak{m}$-complete distributive lattice, and for each $\alpha \in S$, $\psi_\alpha : L_\alpha \to E$ is a $\mathfrak{m}$-monomorphism such that:

(3.1) E is $\mathfrak{m}$-generated by $\bigcup_{\alpha \in S} \psi(L_\alpha)$.

(3.2) If $\alpha < \beta$ then $\psi_\alpha(x) < \psi_\beta(y)$ for each $x \in L_\alpha$ and $y \in L_\beta$.

(3.3) If L' is a $\mathfrak{m}$-complete distributive lattice and $\{f_\alpha : L_\alpha \to L'\}_{\alpha \in S}$ is a collection of $\mathfrak{m}$-homomorphisms such that whenever $\alpha < \beta$ then $f_\alpha(x) \leqq f_\beta(y)$ for all $x \in L_\alpha$, $y \in L_\beta$, then there exists a $\mathfrak{m}$-homomorphism $f : E \to L'$ such that $f\psi_\alpha = f_\alpha$ for each $\alpha \in S$.

It follows that the $\mathfrak{m}$-order sum is essentially unique—if it exists. Note also that if P is the trivial ordering on S and $|L_\alpha| = 1$ for each $\alpha \in S$ then E is the free $\mathfrak{m}$-complete distributive lattice with $|S|$ generators. We now investigate the existence question.

Let $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ be the order sum of $\{L_\alpha\}_{\alpha \in S}$ over P. Let $\boldsymbol{J}$ be the class of all sets of the form $\varphi_\alpha(H)$ where

(3.4) $\alpha \in S$, $H \subseteqq L_\alpha$, $|H| \leqq \mathfrak{m}$, $H \neq \phi$

and such that $\Sigma_{L_\alpha}(H)$ exists. Let $\boldsymbol{M}$ be the class of all sets of the form $\varphi_\alpha(H)$ satisfying (3.4) and such that $\Pi_{L_\alpha}(H)$ exists. Note that since φ_α is a complete monomorphism (Lemma 1.8), conditions (2.1) and (2.2) of § 2 are satisfied.

THEOREM 3.2. *If* $L(P)$ *has a* $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-*extension then* $\{L_\alpha\}_{\alpha \in S}$ *has a* $\mathfrak{m}$-*order sum over* P.

Proof. By Theorem 2.10, $L(P)$ has a free $(\boldsymbol{J}, \boldsymbol{M}, \mathfrak{m})$-extension (ψ, E). We. will show that $(\{\psi\varphi_\alpha\}_{\alpha \in S}, E)$ is the required $\mathfrak{m}$-order sum. Let $H \subseteqq L_\alpha$, $0 < |H| \leqq \mathfrak{m}$ and suppose that $\Sigma_{L_\alpha}(H)$ exists. Then $\varphi_\alpha(H) \in \boldsymbol{J}$. Since ψ is a $(\boldsymbol{J}, \boldsymbol{M})$-monomorphism and φ_α is complete,

$$\psi\varphi_\alpha(\Sigma_{L_\alpha}(H)) = \Sigma_E \psi\varphi_\alpha(H) .$$

Similary for products. So $\psi\varphi_\alpha$ is a $\mathfrak{m}$-monomorphism. Since $\bigcup_{\alpha \in S} \varphi_\alpha(L_\alpha)$ generates $L(P)$ and $\psi(L(P))$ $\mathfrak{m}$-generates E, it follows that

$\bigcup_{\alpha \in S} \psi\varphi_\alpha(L_\alpha)$ $\mathfrak{m}$-generates E. Finally, let L' be a $\mathfrak{m}$-complete distributive lattice and $\{f_\alpha : L_\alpha \to L'\}_{\alpha \in S}$ a family of $\mathfrak{m}$-homomorphisms with the property that $\alpha < \beta$ implies $f_\alpha(x) \leqq f_\beta(y)$ for all $x \in L_\alpha$, $y \in L_\beta$. By (1.3) three exists a homomorphism $f' : L(P) \to L'$ such that $f'\varphi_\alpha = f_\alpha$ for each $\alpha \in S$. Since φ_α is complete, f' is a $(\boldsymbol{J}, \boldsymbol{M})$-homomorphism. But (ψ, E) is a free-$(\boldsymbol{J}, \boldsymbol{M})$-extension, so there exists a $\mathfrak{m}$-homomorphism $f : E \to L'$ such that $f' = f\psi$. Thus $f\psi\varphi_\alpha = f_\alpha$ for each $\alpha \in S$.

COROLLARY 3.3. *If $\{L_\alpha\}_{\alpha \in S}$ is a collection of conditionally implicative lattices (or lattices satisfying (2.8)), then $\{L_\alpha\}_{\alpha \in S}$ has a $\mathfrak{m}$-order sum over P for each partial ordering P on S.*

Proof. This is immediate from Theorem 3.2 and the remarks following Definition 2.2.

A necessary condition for the $\mathfrak{m}$-order sum $(\{\psi_\alpha\}_{\alpha \in S}, E)$ over P of $\{L_\alpha\}_{\alpha \in S}$ to exist is that each L_α have a free $\mathfrak{m}$-regular extension (consider the smallest m-complete sublattice of E that contains $\psi_\alpha(L_\alpha)$). A case in which an $\mathfrak{m}$-order sum has a rather simple structure is obtained in the next theorem. For the definition of ordinal sum, see [2, Definition 1.3].

THEOREM 3.4. *Suppose S is finite and P is a chain in $\boldsymbol{P}$. If (ψ_α, E_α) is a free $\mathfrak{m}$-regular extension of L_α for each $\alpha \in S$, then $(\{i_\alpha\psi_\alpha\}_{\alpha \in S}, E)$ is the $\mathfrak{m}$-order sum of $\{L_\alpha\}_{\alpha \in S}$ over P, where E is the ordinal sum of $\{E_\alpha\}_{\alpha \in S}$ and $i_\alpha : E_\alpha \to E$ is the inclusion map for each $\alpha \in S$.*

Proof. We can assume that $S = \{1, 2, \cdots, n\}$ with the usual ordering and $\{E_\alpha\}_{\alpha \in S}$ is a pair-wise disjoint family. Clearly, for $H \subseteqq E$, $0 < |H| < \mathfrak{m}$, we have $\Sigma_E(H) = \Sigma_{E_\beta}(H \cap E_\beta)$ where $\beta = \max\{\alpha \in S \mid H \cap E_\alpha \neq \phi\}$. It is evident that E is a $\mathfrak{m}$-complete distributive lattice, $\mathfrak{m}$-generated by $\bigcup_{\alpha \in S} i_\alpha\psi_\alpha(L_\alpha)$. Now assume the hypothesis of (3.3). Since (ψ_α, E_α) is a $\mathfrak{m}$-regular extension of L_α, there exists a $\mathfrak{m}$-homomorphism $g_\alpha : E_\alpha \to L'$ such that $g_\alpha\psi_\alpha = f_\alpha$ for each $\alpha \in S$. The function $g : E \to L'$ defined by $g(x) = g_\alpha(x)$ for $x \in E_\alpha$ has the property $g\psi_\alpha = f_\alpha$ for each $\alpha \in S$. To show g preserves order, suppose $\alpha < \beta$, x is a fixed element in L_α and let $F = \{y \in E_\beta \mid g_\alpha\psi_\alpha(x) \leqq g_\beta(y)\}$. Then

(i) $\psi_\beta(L_\beta) \subseteqq F$ and

(ii) F is a $\mathfrak{m}$-complete sublattice of E_β.

It follows that $F = E_\beta$ and

$$g_\alpha\psi_\alpha(x) \leqq g_\beta(y) \quad \text{for} \quad x \in L_\alpha,\ y \in E_\beta\,.$$

Now let y be a fixed element of E_β and let $G = \{z \in E_\alpha \mid g_\alpha(z) \leqq g_\beta(y)\}$. Then

(iii) $\psi_\alpha(L_\alpha) \subseteqq G$ and

(iv) G is a $\mathfrak{m}$-complete sublattice of E_α.

It follows that $G = E_\alpha$ and that for $x \in L_\alpha$, $y \in L_\beta$, $g(x) \leqq g(y)$. Finally, to show g is a $\mathfrak{m}$-homomorphism, let $H \subseteqq E$, $0 < |H| < \mathfrak{m}$, and set $\beta = \max\{\alpha \in S \mid H \cap E_\alpha \neq \phi\}$. Then

$$\begin{aligned}\Sigma_{L'} g(H) \leqq g(\Sigma_E(H)) &= g(\Sigma_{E_\beta}(H \cap E_\beta)) = g_\beta(\Sigma_{E_\beta}(H \cap E_\beta)) \\ &= \Sigma_{L'} g_\beta(H \cap E_\beta) \leqq \Sigma_{L'} g(H) .\end{aligned}$$

So

$$\Sigma_{L'} g(H) = g(\Sigma_E(H)) .$$

I wish to thank Dr. A. Horn and the referee for their valuable comments concerning this paper.

References

1. R. Balbes, *Projective and injective distributive lattices*, Pacific J. Math. **21** (1967), 405-420.

2. R. Balbes and A. Horn, *Order sums of distributive lattices*, Pacific J. Math. **21** (1967), 421-435.

3. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. **25**, (1964).

4. C. C. Chang and A. Horn, *On the representation of α-complete lattices*, Fund. Math. **51** (1962), 253-258.

5. P. Crawley, *Regular embeddings which preserve lattice structure*, Proc. Amer. Math. Soc. **13** (1962), 748-752.

6. R. Sikorski, *Boolean Algebras*, Academic Press, 1964.

7. ———, *Products of abstract algebras*, Fund. Math. **39** (1952).

Received July 10, 1967.

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