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COMPARISON OF HAAR SERIES WITH GAPS WITH TRIGONOMETRIC SERIES

J. R. McLAUGHLIN AND J. J. PRICE

We study Haar series with gaps and show striking differences between these series and lacunary trigonometric series. For example, we prove that under certain gap conditions Haar series are finite series almost everywhere.

Haar's orthonormal system $\{\chi_m(t)\}$ is defined as follows on $[0, 1]$: $\chi_0(t) \equiv 1$ and for $m = 2^n + k$ with $0 \leqq k < 2^n$, $n = 0, 1, \cdots$

$$\begin{aligned}\chi_m(t) &= 2^{n/2}, t \in (k/2^n, (k + 1/2)/2^n) ,\\ &= -2^{n/2}, t \in ((k + 1/2)/2^n, (k + 1)/2^n) ,\\ &= 0, t \notin [k/2^n, (k + 1)/2^n] ,\end{aligned}$$

and at the three remaining points we let $\chi_m(t)$ be equal to the average of the right and left hand limits. Thus, in contrast to the trigonometric system, if $2^n \leqq m < 2^{n+1}$, the Haar function $\chi_m(t)$ is supported on an interval of Length 2^{-n} and

$$\int_0^1 |\chi_m(t)|\, dt = 2^{-n/2} .$$

For $f \in L(0, 1)$ we call

$$a_m(f) = \int_0^1 f(t)\chi_m(t)dt, \; m = 0, 1, \cdots$$

the Haar-Fourier coefficients of f and $\sum_{m=0}^{\infty} a_m(f)\chi_m(t)$ the Haar-Fourier series of f.

P. L. Ul'janov has noted [8, p. 42] that if $\{m_k\}$ is an increasing sequence of positive integers for which $\sum (m_k)^{-1}$ converges, and if the gap series $\sum a_{m_k} x_{m_k}(t)$ is the Haar-Fourier series of a bounded function, then the series converges absolutely almost everywhere (cf. [9, p. 247]). The following theorem strengthens this result.

THEOREM 1. (i) *If $\{a_k\}$ is any sequence of real numbers and $\{m_k\}$ is an increasing sequence of positive integers such that $\sum (m_k)^{-1}$ converges, then $\sum_{k=1}^{\infty} a_k\chi_{m_k}(t)$ is a finite series for almost every $t \in [0, 1]$.*

(ii) *If $\sum (m_k)^{-1}$ diverges, then there exists a sequence of real numbers $\{a_k\}$ and an increasing sequence of positive integers $\{n_k\}$ satisfying*

(a) $\displaystyle\sum_{k=1}^{N} \frac{1}{n_k} \leqq \sum_{k=1}^{N} \frac{1}{m_k}$ *for* $N = 1, 2, \cdots$,

(b) $\sum a_k \chi_{n_k}(t)$ *is the Haar-Fourier series of* $f \in L^p$, *for all* $p \in [1, \infty)$,

(c) $\sum |a_k \chi_{n_k}(t)|$ *diverges for almost every* $t \in [0, 1]$.

Proof. Part (i). Let E_m denote the support of $\chi_m(t)$ on $[0, 1]$ for $m = 2^n + k$ with $0 \leqq k < 2^n$, $n = 0, 1, \cdots$. Then

$$\frac{1}{m} \leqq \frac{1}{2^n} = \mu(E_m) < \frac{2}{m}$$

where $\mu(E_m)$ denotes the measure of E_m. Thus, $\sum_{k=1}^{\infty} \mu(E_{m_k})$ converges and consequently $\mu(\limsup_k E_{m_k}) = 0$ [5, p. 40, Exercise 6].

Part (ii). Choose a sequence of real numbers $\{b_k\}$ satisfying

$$\sum b_k^2 < \infty \quad \text{and} \quad \sum |b_k| = \infty \,. \tag{1}$$

Set

$$f(t) = \sum_{k=1}^{\infty} b_k r_{p_k}(t) = \sum_{k=1}^{\infty} b_k (2^{p_k})^{-1/2} \sum_{m=2^{p_k}}^{2^{p_k+1}-1} \chi_m(t) \tag{2}$$

where $r_m(t)$ denotes the mth Rademacher function [1, p. 51] and $\{p_k\}$ is an increasing sequence of positive integers. Now let $\{a_k\}$ and $\{n_k\}$ be defined by the right side of (2). Then if $\{p_k\}$ increases fast enough (a) holds. Also, since $\sum a_k^2$ converges, the right hand side of (2) is the Haar-Fourier series of its sum $f(t)$ [1, p. 47]. The remaining properties follow from (1) by well-known properties of Rademacher series [9, p. 213].

REMARK 1. It would be interesting to know if in condition (b) in Theorem 1 one might replace $f \in L^p$, for all $p \in [1, \infty)$, by f continuous or even f bounded.

REMARK 2. A. M. Olevskiĭ has proved [6, p. 1382] that for every complete orthonormal system (and hence the Haar system) there exists a continuous functions whose Fourier series is absolutely divergent almost everywhere.

It is known [3, p. 243] that if a lacunary trigonometric series is the Fourier series of a function f, then $f \in L^q$ for every $q \in [1, \infty)$.

This result is not valid for Haar series as we now prove.

THEOREM 2. *For every* $p \geqq 1$, *there exists a function* $f \in L^p$ *with Haar-Fourier series* $\sum a_{m_k} \chi_{m_k}(t)$ *where* $m_{k+1}/m_k = 2$, $k = 1, 2, \cdots$, *and such that for every* $q > p$, $f \notin L^q$.

Proof. Define

$$f(t) = (2^n \cdot n^{-2})^{1/p} \quad \text{if} \quad t \in (2^{-n}, 2^{-n+1}),\ n = 1, 2, \cdots .$$

Then

$$\int_0^1 |f(t)|^p\, dt = \sum_{n=1}^{\infty} (2^n \cdot n^{-2}) \cdot 2^{-n} < \infty ,$$

but if $q > p$,

$$\int_0^1 |f(t)|^q\, dt = \sum_{n=1}^{\infty} (2^n \cdot n^{-2})^{q/p} \cdot 2^{-n} = \infty .$$

Also, the Haar-Fourier series of f is

$$a_0(f) + \sum_{k=0}^{\infty} a_{2^k}(f)\chi_{2^k}(t) .$$

If a lacunary trigonometric series is a Fourier series with Fourier coefficients $\{c_k\}$, then $\sum c_k^2$ converges [9, p. 203]. As Theorem 2 shows, for Haar-Fourier series, this need not be. We can even obtain a stronger result.

THEOREM 3. *Let $\{a_k\}$ be any sequence of real numbers. Then there is a function in $L(0, 1)$ with a gap Haar-Fourier series*

$$\sum_{k=1}^{\infty} a_k \chi_{m_k}(t) . \tag{3}$$

Proof. If $m = 2^n + k$ with $0 \leqq k < 2^n$, $n = 0, 1, \cdots$, then

$$\int_0^1 |\chi_m(t)|\, dt = 2^{-n/2} < 2m^{-1/2}$$

and so there is a sequence of positive integers $\{m_k\}$ increasing so fast that

$$\sum_{k=1}^{\infty} |a_k| \int_0^1 |\chi_{m_k}(t)|\, dt < \infty .$$

Hence, series (3) is the Haar-Fourier series of its sum by Lebesgue's dominated convergence theorem.

If a lacunary trigonometric series converges to zero in a set of positive measure, then all the coefficients of the series equal zero [3, p. 265]. For Haar series this result is not valid. In fact, we have the following.

THEOREM 4. *For every $t_0 \in [0, 1]$, there exists a gap Haar-Fourier series $\sum a_{m_k}\chi_{m_k}(t)$, where $m_{k+1}/m_k \geqq 2$, which converges to zero for*

$t \neq t_0$ and diverges for $t = t_0$.

Proof. If $t_0 = 1$, we set $a_0 = -1$, $a_{2^{n+1}-1} = 2^{n/2}$ for $n = 0, 1, \cdots$, and $a_m = 0$ otherwise.

If $t_0 \in [0, 1)$, then for the sequence of integers $\{k_n\}$ satisfying

$$(k_n)2^{-n} \leqq t_0 < (k_n + 1)2^{-n}\,, \qquad n = 0, 1, \cdots$$

we set

$$\begin{aligned} a_m &= 1\,, \quad m = 0 \\ &= (-1)^{k_n+1}2^{n/2}\,, \quad m = 2^n + k_n\,, \quad n = 0, 1, \cdots \\ &= 0 \quad \text{otherwise}\,. \end{aligned}$$

Then, using the fact (which is easily proved inductively) that

$$\begin{aligned} \sum_{n=0}^{2^n-1} a_m\chi_m(t) &= 2^n\,, \quad t \in ((k_n)2^{-n}, (k_n + 1)2^{-n}) \\ &= 0\,, \quad t \notin [(k_n)2^{-n}, (k_n + 1)2^{-n}] \end{aligned}$$

for $n = 0, 1, \cdots$, we obtain our desired result.

COROLLARY. *A nonempty set is a set of multiplicity for Haar series.*

REMARK 3. G. Faber had previously shown [4, p. 111] that the point $t_0 = 1/2$ was a set of multiplicity for Haar series. Also F. G. Arutjunjan and A. A. Talaljan noted Theorem 4 for $t_0 = 0$ [2, p. 1405]. On the other hand, M. B. Petrovskaja proved that the empty set is a set of uniqueness for Haar series [7, p. 797].

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PENNSYLVANIA STATE UNIVERSITY AND
PURDUE UNIVERSITY