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REPRESENTABLE DISTRIBUTIVE NOETHER LATTICES

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Recently, Bogart showed that a certain class of distributive Noether lattices, namely regular local ones, are embeddable in the lattice of ideals of an appropriate Noetherian ring. In this paper a characterization of the distributive Noether lattices which are representable as the complete lattice of ideals of a Noetherian ring is obtained.

We observe that if $L(R)$ is the lattice of ideals of a ring R (commutative with 1) and if A, B and C are elements of $L(R)$ with $A \not\leqq B$ and $A \not\leqq C$, then there exists a principal element $E \in L(R)$ with $E \leqq A$, $E \not\leqq B$ and $E \not\leqq C$. If a Noether lattice L has this property, then we will say that L satisfies the *weak union condition*. (The term union condition has been used elsewhere for a stronger property.) With this definition, then, the main result of this paper is that a distributive Noether lattice L is representable as the lattice of ideals of a Noetherian ring if, and only if, L satisfies the weak union condition.

We adopt the terminology of [2] and we assume throughout that L is a Noether lattice.

LEMMA 0. *If L is local, and if the maximal element $P \in L$ is principal, then every element $A \neq 0$ of L is a power $P^n (0 \leqq n)$ of P.*

Proof. If $A \neq 0$, then by the Intersection Theorem [2] there exists a largest integer n such that $A \leqq P^n$. Then

$$A = A \wedge P^n = (A: P^n)P^n ,$$

so since $A \not\leqq P^{n+1}$, it follows that $A: P^n = I$, and therefore that $A = P^n$.

LEMMA 1. *Assume L is distributive and satisfies the weak union condition. If L is local and if the maximal element of L is principal, or if 0 is prime and every element $A \neq 0$ has a primary decomposition involving only powers of maximal primes, then L is representable as the lattice of ideals of a Noetherian ring.*

Proof. Assume L is local with maximal element P, and that P is principal. Let (R, M) be a regular local ring of altitude one. If 0 is prime in L, then the powers of P are distinct, and L is isomorphic to the lattice of ideals of R. If 0 is not prime in L, and if k is the least positive integer such that $P^k = P^{k+1}$, then L is isomorphic to the lattice of ideals of $R \mid M^k$.

Now, assume that 0 is prime and that every element $A \neq 0$ has a primary decomposition $P_1^{e_1} \cap \cdots \cap P_k^{e_k}$, where each P_i is maximal. Then every prime $P \neq 0$ is maximal, so the P_i in any decomposition $A = P_1^{e_1} \cap \cdots \cap P_k^{e_k}$ are just the minimal primes over A. Since 0 is prime in L, it follows that distinct powers of maximal primes are distinct. Then by the comaximality of distinct primes, it follows that every element $A \neq 0$ has a factorization as a product of primes [2], and since the primes involved are maximal, the factorizations are unique.

Now, let α be the cardinality of the collection $\mathscr{P}$ of maximal primes in L, and let K be a field of cardinality $\beta \geqq \alpha$. Let A be a subset of K of cardinality α, and let S be the complement in $K[x]$ of the union of the prime ideals $(a + x)$, $a \in A$. Then S is a multiplicatively closed subset of $K[x]$ which doesn't meet any of the prime ideals $(a + x)$, and which meets every other prime ideal. Hence $K[x]_S$ is a Dedekind Domain with α maximal primes [3].

We let φ be a one-one correspondence between the maximal primes of L and the maximal primes of $K[x]_S$, and extend φ to a map of L onto the lattice of ideals of $K[x]_S$ by taking 0 to 0 and products to products. Then since L is distributive and distinct nonzero primes are comaximal, we have

$$\text{(i)} \quad \left(\prod_1^n P_i^{e_i}\right) \cdot \left(\prod_1^n P_i^{f_i}\right) = \prod_1^n P_i^{e_i + f_i}$$

$$\text{(ii)} \quad \left(\prod_1^n P_i^{e_i}\right) \wedge \left(\prod_1^n P_i^{f_i}\right) = \left(\bigwedge_1^n P_1^{e_i}\right) \wedge \left(\bigwedge_1^n P_i^{f_i}\right)$$

$$= \bigwedge_1^n P_i^{\max(e_i, f_i)} = \prod_1^n P_i^{\max(e_i, f_i)}, \quad \text{and}$$

$$\text{(iii)} \quad \left(\prod_1^n P_i^{e_i}\right) \vee \left(\prod_1^n P_i^{f_i}\right) = \left(\bigwedge_1^n P_i^{e_i}\right) \vee \left(\bigwedge_1^n P_i^{f_i}\right)$$

$$= \bigwedge_1^n P_i^{\min(e_i, f_i)} = \prod_1^n P_i^{\min(e_i, f_i)},$$

for distinct primes P_i and for $e_i, f_i \geqq 0$.

Since the lattice of ideals of a Dedekind domain also has these properties [3], it follows that φ is an isomorphism of L onto the lattice of ideals of $K[x]_S$.

To reduce the general case to the cases covered by Lemma 1, we require the following lemmas.

LEMMA 2. *If L is distributive and satisfies the weak union condition, and if $D \in L$, then $L \mid D$ and L_D are distributive and satisfy the weak union condition.*

Proof. The proof is immediate for $L \mid D$, as is the distributivity of L_D. If $\{A\}, \{B\}$ and $\{C\}$ are elements of L_D with $\{A\} \not\leqq \{B\}$ and

$\{A\} \not\leqq \{C\}$, then $A_D \not\leqq B_D$ and $A_D \not\leqq C_D$. So there exists a principal element $E \in L$ with $E \leqq A_D$, $E \not\leqq B_D$ and $E \not\leqq C_D$. Then $\{E\}$ is principal with $\{E\} \leqq \{A\}$, $\{E\} \not\leqq \{B\}$ and $\{E\} \not\leqq \{C\}$.

LEMMA 3. *If L is a distributive local Noether lattice which satisfies the weak union condition, then the maximal element P of L is principal.*

Proof. Let $A_1, \cdots, A_k$ be a minimal collection of principal elements with join P. If $k > 1$, then $P \not\leqq A_1 \vee \cdots \vee A_{k-1}$ and $P \not\leqq A_k$, so there exists a principal element $A \leqq P$ with $A \not\leqq A_1 \vee \cdots A_{k-1}$ and $A \not\leqq A_k$. Then

$$\begin{aligned} A = A \wedge P &= A \wedge [(A_1 \vee \cdots \vee A_{k-1}) \vee A_k] \\ &= ((A_1 \vee \cdots \vee A_{k-1}) \wedge A) \vee (A_k \wedge A) \\ &= ((A_1 \vee \cdots \vee A_{k-1})\colon A \vee (A_k\colon A))A \,. \end{aligned}$$

Since $A \neq 0$, it follows from the Intersection Theorem [2] that

$$(A_1 \vee \cdots \vee A_{k-1})\colon A \vee A_k\colon A = I \,,$$

which is a contradiction since L is local. Hence $k = 1$.

We are now ready to prove the following

THEOREM 4. *If L is a distributive Noether lattice, then L is representable as the lattice of ideals of a Noetherian ring if and only if, L satisfies the weak union condition.*

Proof. Since the lattice of ideals of any ring satisfies the weak union condition, the "only if" is clear. Hence, assume L is a distributive Noether lattice which satisfies the weak union condition. Let

$$0 = Q_1 \cap \cdots \cap Q_s \cap \cdots \cap Q_k$$

be a normal decomposition of 0 in which Q_i is P_i-primary. We assume that $P_1, \cdots, P_s$ are nonmaximal elements of L and that $P_{s+1}, \cdots, P_k$ are maximal.

By Lemmas 2 and 3 and the Principal Ideal Theorem [2], if P is any prime in L, then P has height no greater than one, so every prime is either maximal or minimal. Further, if $P' < P$ are primes, then by Lemma 0, 0 is prime in L_P, so $O_P = P' = \bigwedge_1^\infty P^n$. It follows from this that 0 has no embedded primes, that the primaries Q_i, $1 \leqq i \leqq s$, are the P_i, and that no prime P contains two distinct minimal primes. Further, since every element, except possibly 0, of L_P is a power of the maximal element, we have that the P-primary elements of the maximal primes P are precisely the powers P^n of P.

Then for each $i, s+1 \leqq i \leqq k$, there exists a positive integer e_i with $Q_i = P_i^{e_i}$. Hence $0 = P_1 \cap \cdots \cap P_s \cap P_{s+1}^{e_{s+1}} \cap \cdots \cap P_k^{e_k}$. Then since the P_i are pairwise comaximal we have

$$L \cong L \mid P_1 \oplus \cdots \oplus L \mid P_s \oplus L \mid P_{s+1}^{e_{s+1}} \oplus \cdots \oplus L \mid P_k^{e_k} ,$$

where each summand is of the type considered in Lemma 1.

Since the lattice of ideals of a direct sum $R_1 \oplus \cdots \oplus R_n$ of rings is isomorphic to the direct sum of the lattices of ideals of the rings, the result now follows.

It is easily seen from the decomposition

$$L \cong L \mid P_1 \oplus \cdots \oplus L \mid P_s \oplus L \mid P_{s+1}^{e_{s+1}} \oplus \cdots \oplus L \mid P_k^{e_k} ,$$

in the proof of Theorem 4 that every element of L is a product of primes and that the maximal elements of L are meet principal (in fact that every element is principal). Also, it is seen that the decomposition above characterizes the distributive Noether lattices which are representable as the lattice of ideals of a Noetherian ring. These observations lead to the following theorem which is stated without proof since the proof is similar to that of Theorem 4.

THEOREM 5. *The following are equivalent for a Noether lattice L:*

(i) *L is distributive and representable as the lattice of ideals of a Noetherian ring*

(ii) *L is distributive and satisfies the weak union condition*

(iii) *For every maximal element P, L_P is linear*

(iv) *Every element A of L different from I is a product of primes*

(v) *Every maximal element P of L satisfies the condition $A \wedge P = (A: P)P$, for all A in L*

(vi) *L is the direct sum $L = L_1 \oplus \cdots \oplus L_n$ of Noether lattices L_i, where for each i, either L_i is local with a principal maximal element, or 0 is prime in L_i and every element $A \neq I$ is a (unique) product of primes.*

REFERENCES

1. K. P. Bogart, *Structure theorem for regular local Noether lattices*, Mich. J. Math. **15** (1968), 167-176.

2. R. P. Dilworth, *Abstract commutative ideal theory*, Pacific J. Math. **12** (1962), 481-498.

3. O. Zariski, and P., Samuel, *Commutative algebra*, I, D. Van Nostrand Co., Inc., Princeton.

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