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SOME INEQUALITIES FOR STARSHAPED AND CONVEX FUNCTIONS

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Necessary and sufficient conditions are obtained on a function G of bounded variation such that $\phi\left(\int x(t)dG(t)\right) \leqq \int \phi(x(t))dG(t)$ for all increasing x for which $x(t_0) = 0$ for some specified t_0, and all convex ϕ for which $\phi(0) = 0$; the conditions are otherwise independent of ϕ and x. Similar results are obtained when the inequality is reversed. Necessary and sufficient conditions for both directions of inequality are also obtained when ϕ is starshaped and $\phi(0) = 0$.

The relationship to previous results is sketched. Applications to statistical tolerance limits are indicated.

Several inequalities are known that give necessary and sufficient conditions for a signed measure μ to satisfy

$$\int \phi(x)d\mu(x) \geqq 0 \tag{1.1}$$

for all functions ϕ in a given convex cone. For example, such results were obtained by Hardy, Littlewood and Pólya [7] for the cone of convex functions, and by Karlin and Novikoff [9], Ziegler [17] and Karlin and Studden [10] for cones of generalized convex functions.

By changing variables in such a result, it is easy to obtain conditions on μ in order that

$$\int \phi(x(t))d\mu(t) \geqq 0 \tag{1.2}$$

for all ϕ in the given convex cone, where x is an increasing function. Generally speaking the conditions so obtained depend upon the function x. In some applications, x is replaced by a random function (see Barlow and Proschan [1]). Inequalities are thus required which will hold for essentially all possible realizations of the random function, so that those obtained via a change of variables, like (1.2), are not useful.

In this paper, we consider only measures μ which are the difference between a measure ν and the measure which has unit mass concentrated *at the point* $\int x(t)d\nu(t)$. *Consequently, all the inequalities that we* obtain have either the form

$$\phi\left(\int x(t)d\nu(t)\right) \leqq \int \phi(x(t))d\nu(t) \tag{1.3}$$

or

$$\phi\left(\int x(t)d\nu(t)\right) \geqq \int \phi(x(t))d\nu(t) . \tag{1.4}$$

For the cone of the convex functions ϕ satisfying $\phi(0) = 0$ (§ 3) and the cone of starshaped functions (§ 4), conditions on ν are given for (1.3) or (1.4) which are *independent* of the function x.

For discrete measures and convex functions ϕ, sufficient conditions independent of the function x have been obtained by various authors (see § 5). For more general measures and convex ϕ, sufficient conditions have been obtained by Brunk [5]. The relation between our work and his is discussed in § 3. Related results in higher dimensions have also been obtained by Brunk [6].

In all of these cases, the direction of the inequality is as in (1.3), and of course this is also the direction of Jensen's inequality. Thus, results of the form (1.4) are more novel.

Observe that inequalities of the form (1.2) can be viewed as having the form of (1.1) where the original cone of functions is extended by increasing transformations of the variable. We have not found it convenient to adopt this point of view; instead, the results are obtained via inequalities of the form (1.1) for the original cones of functions. These preliminary inequalities are given in § 2.

Throughout this paper, we use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing". We consider functions defined on intervals $[a, b]$; although the endpoints a and b need not be finite, it should always be understood that they are to be included in an interval only if they are finite.

2. Preliminary inequalities. One of the earliest inequalities of the form (1.1) is the result of Hardy, Littlewood and Pólya [7] for convex functions. They observed that a convex function can be approximated by positive combinations of functions of the form

$$\phi(x) = x , \quad \phi(x) = -x , \quad \phi(x) = 1 , \quad \phi(x) = -1$$

and

$$\phi(x) = \begin{cases} x - u, & x > u \\ 0, & x \leqq u, \end{cases} \quad -\infty < u < \infty .$$

Their conclusion was that (1.1) holds for all convex ϕ if and only if it holds for these special convex ϕ. The idea of their proof can be used to obtain several related inequalities that we shall require. These inequalities all characterize the signed measures μ of bounded variation that satisfy

$$\int_{[a,b]} \phi(x) d\mu(x) \geqq 0$$

for all functions ϕ of a given class $\mathscr{C}$, $-\infty \leqq a < b \leqq \infty$. The results are obtained by finding a subclass $\mathscr{D} \subset \mathscr{C}$ with the property that for all $\phi \in \mathscr{C}$, there exists a sequence $\{\phi_i\}$ of *positive* combinations of elements of $\mathscr{D}$ such that $\lim_i \phi_i(x) = \phi(x)$. Furthermore the sequence has the property that $\phi_i(x)$ is increasing in i if $\phi(x) \geqq 0$ and $\phi_i(x)$ is decreasing in i if $\phi(x) \leqq 0$. From the Lebesgue monotone convergence theorem, we have that $\int_{[a,b]} \phi(x) d\mu(x) \geqq 0$ for all $\phi \in \mathscr{D}$ if and only if the inequality holds for all $\phi \in \mathscr{C}$. The following theorems can be easily proved in this way; in each case, the class $\mathscr{C}$ is stated in the hypotheses of the theorem and the class $\mathscr{D}$ can be reconstructed from the conditions given.

THEOREM 2.1. $\int_{[a,b]} \phi(x) d\mu(x) \geqq 0$ *for all* ϕ *convex on* $[a, b]$ *such that* $\phi(c) = 0, a \leqq c \leqq b$, *with* c *finite, if and only if*

$$\int_{[u,b]} (x - u) d\mu(x) \geqq 0 \quad \textit{for all} \quad u \in [c, b) , \tag{2.1}$$

$$\int_{[a,u]} (u - x) d\mu(x) \geqq 0 \quad \textit{for all} \quad u \in (a, c] , \tag{2.2}$$

and

$$\int_{[a,b]} (x - c) d\mu(x) = 0 . \tag{2.3}$$

If ϕ satisfies the conditions of Theorem 2.1 and $\int_{[a,b]} \phi(x) d\mu(x) = 0$ for all measures μ satisfying (2.1), (2.2) and (2.3), then $\phi(x) = \alpha(x - c)$. In fact, by considering measures μ of the form $\mu\{x_1\} = \theta$, $\mu\{x_2\} = 1 - \theta$, $\mu\{\theta x_1 + (1 - \theta)x_2\} = -1$ $(a \leqq x_1 \leqq c \leqq x_2 \leqq b,\ 0 < \theta < 1)$, we conclude from $\int_{[a,b]} \phi(x) d\mu(x) = 0$ that ϕ is linear because $\phi(\theta x_1 + (1 - \theta)x_2) = \theta\phi(x_1) + (1 - \theta)\phi(x_2)$.

We remark that without the condition $\phi(c) = 0$, the conditions (2.1) – (2.3) must be augmented by the requirement that $\int_{[a,b]} d\mu = 0$. This is essentially the result of Hardy, Littlewood and Pólya [7]. See also Karamata [8], Levin and Stečkin [12], Brunk [5], and Karlin and Novikoff [9].

THEOREM 2.2. $\int_{[a,b]} \phi(x) d\mu(x) \geqq 0$ *for all* ϕ *convex on* $[a, b]$ *such that* $\phi(c) = 0$, $a \leqq c \leqq b$ *with* c *finite, and* $\phi(x) \geqq 0$, $a \leqq x \leqq b$, *if and only if* (2.1) *and* (2.2) *hold.*

Equality $\int_{[a,b]} \phi(x)d\mu(x) = 0$ in Theorem 2.2 is obtained for all measures μ satisfying (2.1) and (2.2) if and only if $\phi(x) = 0$ when $a < c < b$, $\phi(x) = \alpha(x - c)$ when $c = a$ or $c = b$. This can be obtained in the same way as the case of equality in Theorem 2.1.

Without the condition $\phi(c) = 0$, (2.1) and (2.2) are no longer sufficient, but in this case, the appropriate conditions can be found as a special case of results due to Ziegler [17]. See also Karlin and Studden [10].

The remaining theorems of this section are concerned with starshaped functions. Perhaps the most natural domain for these functions is $[0, \infty)$, where they are considered by Bruckner and Ostrow [4]. Our original interest in these functions was also on this domain, where they arise in describing certain classes of probability distributions of importance in reliability theory and elsewhere (see Barlow and Proschan [1]). However, we consider more general domains here, and extend the definition in two ways.

DEFINITION 2.3. A real valued function ϕ on the interval I is said to be *starshaped* if $\phi(\alpha x) \leqq \alpha\phi(x)$ whenever $x \in I$, $\alpha x \in I$ and $0 \leqq \alpha \leqq 1$.

DEFINITION 2.4. A real valued function on the interval I is said to be a *supported starshaped* function if it is starshaped, and if, whenever 0 is an interior point of I, there exists a linear function l on I such that $l(0) = 0$ and $l(x) \leqq \phi(x)$ for all $x \in I$.

Unless 0 is an interior point of I, there is no distinction between starshaped and supported starshaped functions. If 0 is an interior point of I, then ϕ is starshaped on I if and only if

(i) $\phi(x)/x$ is increasing in $x < 0, x \in I$,

(ii) $\phi(x)/x$ is increasing in $x > 0, x \in I$,

(iii) $\phi(0) \leqq 0$.

On the other hand, if 0 is an interior point of I, ϕ is a supported starshaped function on I if and only if

(iv) $x_1 < x_2$ and $x_1 \neq 0 \neq x_2$ implies $\phi(x_1)/x_1 \leqq \phi(x_2)/x_2$,

(v) $\phi(0) = 0$.

When the interval I is of the form $[0, b]$, then a starshaped function ϕ is a generalized convex function in the sense defined by Karlin and Novikoff [9] and Ziegler [17]; in the notation of Karlin and Studden [10], $n = 0$ and $u_0(x) = x$. In this case, theorems similar to those below are obtainable as special cases of their results.

THEOREM 2.5. $\int_{[a,b]} \phi(x)d\mu(x) \geqq 0$ *for all starshaped* ϕ *on* $[a, b]$, $a \leqq 0 \leqq b$, *such that* $\phi(0) = 0$ *if and only if*

$$\int_{[a,0]} x d\mu(x) = \int_{[0,b]} x d\mu(x) = 0 \ , \tag{2.4}$$

$$\int_{[a,u]} x d\mu(x) \leqq 0 \quad \textit{for all } u \ , \quad a \leqq u < 0 \ , \tag{2.5}$$

$$\int_{[u,b]} x d\mu(x) \geqq 0 \quad \textit{for all } u \ , \quad 0 < u \leqq b \ . \tag{2.6}$$

THEOREM 2.6. *$\int_{[a,b]} \phi(x) d\mu(x) \geqq 0$ for all supported starshaped ϕ on $[a, b]$, $a \leqq 0 \leqq b$, such that $\phi(0) = 0$ if and only if* (2.5), (2.6), *and*

$$\int_{[a,b]} x d\mu(x) = 0 \ . \tag{2.7}$$

3. Inequalities for convex functions. In the following, G denotes a function of bounded variation on $[a, b]$, $-\infty \leqq a \leqq 0 \leqq b \leqq \infty$ (the endpoints of the interval are excluded when not finite). We assume that $G(u) = \int_{[a,u]} dG(x)$, and use the notation $\bar{G}(u) = \int_{(u,b]} dG(x)$. In addition, we assume without further mention that $\int_{[a,b]} x(t) dG(t) < \infty$. Occasionally, we find it convenient to use the letter G to denote the measure determined by G: i.e., we write $G\{A\} = \int_A dG(x)$.

THEOREM 3.1. *Let $t_0 \in [a, b]$ be fixed.*

$$\phi\left(\int_{[a,b]} x(t) dG(t)\right) \leqq \int_{[a,b]} \phi(x(t)) dG(t) \tag{3.1}$$

for all convex functions ϕ such that $\phi(0) = 0$ and all increasing functions x such that $x(t_0) = 0$ if and only if

$$0 \leqq G(t) \leqq 1, a \leqq t < t_0 \quad \textit{and} \quad 0 \leqq \bar{G}(t) \leqq 1, t_0 \leqq t < b \ . \tag{3.2}$$

REMARK. In this and the following theorems, ϕ need not be convex (or even defined) over all of the interval $(-\infty, \infty)$. But ϕ must be convex on an interval containing the point $\int_{[a,b]} x(t) dG(t)$ and the range of $x(t)$ for t in the support of G.

Proof. Suppose first that (3.2) holds. Let

$$G^*(z) = G\{t \colon a \leqq t \leqq b \text{ and } x(t) \leqq z\} \ ,$$

and let H^* be the probability distribution degenerate at $\mu = \int_{-\infty}^{\infty} x dG^*(z)$. Since G^* has no mass outside the interval $[x(a), x(b)]$, (3.1) can be rewritten as

$$\int_{-\infty}^{\infty}\phi(z)dH^*(z) \leqq \int_{-\infty}^{\infty}\phi(z)dG^*(z)\ . \tag{3.1$'$}$$

Observe that (3.2) can be rewritten as

$$0 \leqq G^*(z) \leqq 1, z \leqq 0 \quad \text{and} \quad 0 \leqq \bar{G}^*(z) \leqq 1, z \geqq 0\ . \tag{3.2$'$}$$

By definition of H^*, we have that $\int_{-\infty}^{\infty} zdG^*(z) = \int_{-\infty}^{\infty} zdH^*(z)$. Thus, by Theorem 2.1, we need only check that

$$\int_u^{\infty}\bar{G}^*(z)dz \geqq \int_u^{\infty}\bar{H}^*(z)dz,\ u \geqq 0$$

and

$$\int_{-\infty}^{u} G^*(z)dz \geqq \int_{-\infty}^{u} H^*(z)dz,\ u \leqq 0\ .$$

Case 1. $u \geqq 0$. If

$$u \leqq \mu,\ \int_u^{\infty}\bar{H}^*(z)dz = \mu - u = \int_0^{\infty}\bar{G}^*(z)dz - \int_{-\infty}^{0} G^*(z)dz - u$$

so that $\int_u^{\infty}\bar{G}^*(z)dz \geqq \int_u^{\infty}\bar{H}^*(z)dz$ is equivalent to

$$\int_{-\infty}^{0} G^*(z)dz + u \geqq \int_0^{u}\bar{G}^*(z)dz\ .$$

But this follows from $G^*(z) \geqq 0, z \leqq 0$, and $\bar{G}^*(z) \leqq 1, z \geqq 0$.

If $u \geqq \mu$, $\int_u^{\infty}\bar{H}^*(z)dz = 0$ so that $\int_u^{\infty}\bar{G}^*(z)dz \geqq \int_u^{\infty}\bar{H}^*(z)dz$ is equivalent to $\int_u^{\infty}\bar{G}^*(z)dz \geqq 0$. This follows from $\bar{G}^*(z) \geqq 0, z \geqq 0$.

Case 2. $u \leqq 0$. If $u \leqq \mu$, then $\int_{-\infty}^{u} G^*(z)dz \geqq \int_{-\infty}^{u} H^*(z)dz = 0$ since $G^*(z) \geqq 0, z \leqq 0$, while $H^*(z) = 0, z \leqq \mu$.

If $u > \mu$, $\int_{-\infty}^{u} H^*(z)dz = u - \mu = u - \int_0^{\infty}\bar{G}^*(z)dz + \int_{-\infty}^{0} G^*(z)dz$ and so $\int_{-\infty}^{u} G^*(z)dz \geqq \int_{-\infty}^{u} H^*(z)dz$ is equivalent to

$$\int_0^{\infty}\bar{G}^*(z)dz - u \geqq \int_u^{0} G^*(z)dz\ .$$

But this is a consequence of $\overline{G^*}(z) \geqq 0, z \geqq 0$, and $G^*(z) \leqq 1, z \leqq 0$.

It remains to show that (3.2) is necessary. Choose $t_1 \geqq t_0$. Let

$$x(t) = \begin{cases} 0, & t \leqq t_1 \\ 1, & t > t_1\ , \end{cases}$$

and let $\phi(z) = z^2$. Then (3.1) becomes

$$\int_{(t_1,b]} dG(t) \geqq \left(\int_{(t_1,b]} dG(t)\right)^2 ,$$

i.e., $\bar{G}(t_1) \geqq [\bar{G}(t_1)]^2$. This implies $0 \leqq \bar{G}(t_1) \leqq 1$.

Next, choose $t_2 < 0$.

Let

$$x(t) = \begin{cases} -1, & t \leqq t_2 \\ 0, & t > t_2 , \end{cases}$$

and again take $\phi(z) = z^2$. Then (3.1) becomes $G(t_2) \geqq [G(t_2)]^2$, hence $0 \leqq G(t_2) \leqq 1$.

If ϕ is convex, then $\psi(x) = \phi(x) - \phi(0)$ is convex and satisfies $\psi(0) = 0$, so that Theorem 3.1 can be restated without the hypothesis $\phi(0) = 0$ as follows:

THEOREM 3.1a. *Fix* $t_0 \in [a, b]$.

$$\phi\left(\int_{[a,b]} x(t)dG(t)\right) - \phi(0) \leqq \int_{[a,b]} [\phi(x(t)) - \phi(0)]dG(t) \tag{3.3}$$

for all convex functions ϕ *and increasing functions* x *such that* $x(t_0) = 0$ *if and only if* (3.2).

If the hypotheses of Theorem 3.1a are augmented by the condition $\phi(0) \leqq \phi(0)G(b)$, then we can replace (3.3) with (3.1). The result so obtained is a modification of Theorem 3.1 that admits a widened class of functions ϕ.

We remark that if G is a probability distribution on $[a, b]$, then (3.2) is satisfied and (3.3) reduces to (3.1). Thus we obtain the special case of Jensen's inequality. The next theorem gives conditions for the reverse inequality, and here the results are somewhat more novel.

THEOREM 3.2. *Fix* $t_0 \in [a, b]$.

$$\phi\left(\int_{[a,b]} x(t)dG(t)\right) \geqq \int_{[a,b]} \phi(x(t))dG(t) \tag{3.4}$$

for all convex functions ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(t_0) = 0$ *if and only if either*

$$\begin{array}{l} \textit{there exists } s \leqq t_0 \textit{ such that } G(t) \leqq 0, t < s, G(t) \geqq 1, \\ s \leqq t < t_0 \textit{ and } \bar{G}(t) \leqq 0, t \geqq t_0 , \end{array} \tag{3.5}$$

or

(3.6) *there exists* $s \geqq t_0$ *such that* $G(t) \leqq 0,\ t < t_0,\ \bar{G}(t) \geqq 1,$ $t_0 \leqq t < s,$ *and* $\bar{G}(t) \leqq 0,\ t \geqq s$.

Proof. Let $H^*(z) = G\{t\colon a \leqq t \leqq b \text{ and } x(t) \leqq z\}$, and let G^* be the probability distribution degenerate at $\mu = \int_{-\infty}^{\infty} z dH^*(z)$. Then (3.4) can be rewritten as

$$\int_{-\infty}^{\infty} \phi(z) dG^*(z) \geqq \int_{-\infty}^{\infty} \phi(z) dH^*(z) \ . \tag{3.4$'$}$$

Suppose that (3.6) holds. We may rewrite it in the following way:

(3.6′) There exists $y \geqq 0$ such that $H^*(z) \leqq 0,\ z < 0,\ \bar{H}^*(z) \geqq 1,$ $0 \leqq z < y,$ and $\bar{H}^*(z) \leqq 0,\ z \geqq y.$

Since $\int_{-\infty}^{\infty} z dG^*(z) = \int_{-\infty}^{\infty} z dH^*(z)$, we can apply Theorem 2.1 by checking that

$$\int_u^{\infty} \bar{G}^*(z)dz \geqq \int_u^{\infty} \bar{H}^*(z)dz,\ u\, H^*(z) \geqq 0, \text{ and } \int_{-\infty}^{u} G^*(z)dz \geqq \int_{-\infty}^{u} H^*(z)dz,\ u \leqq 0 \ .$$

Case 1a. $0 \leqq u \leqq \mu$. The condition $\int_u^{\infty} \bar{G}^*(z)dz \geqq \int_u^{\infty} \bar{H}^*(z)dz$ becomes $\mu - u \geqq \int_u^{\infty} \bar{H}^*(z)dz$, i.e.,

$$\int_0^{\infty} \bar{H}^*(z)dz - \int_{-\infty}^{0} H^*(z)dz - u \geqq \int_u^{\infty} \bar{H}^*(z)dz \ ,$$

or

$$\int_0^{u} \bar{H}^*(z)dz \geqq u + \int_{-\infty}^{0} H^*(z)dz \ .$$

If $u < y$, this follows from $\bar{H}^*(z) \geqq 1,\ 0 \leqq z \leqq u$ and $H^*(z) \leqq 0$, $z < 0$.

If $u > y$, we employ the condition $u \leqq \mu$ together with $\int_u^{\infty} \bar{H}^*(z)dz \leqq 0$ to conclude $\mu - u \geqq \int_u^{\infty} \bar{H}^*(z)dz$.

Case 1b. $u \geqq 0$ and $u \geqq \mu$. Then $\int_u^{\infty} \bar{G}^*(z)dz \geqq \int_u^{\infty} \bar{H}^*(z)dz$ is just $0 \geqq \int_u^{\infty} \bar{H}^*(z)dz$. If $u \geqq y$, this is a trivial consequence of $\bar{H}^*(z) \leqq 0$, $z \geqq y$. If $u \leqq y$, we observe that $u \geqq \mu$ is equivalent to

$$u \geqq -\int_{-\infty}^{0} H^*(z)dz + \int_0^{u} \bar{H}^*(z)dz + \int_u^{\infty} \bar{H}^*(z)dz \ .$$

Since $-\int_{-\infty}^{0} H^*(z)dz \geqq 0$ and $\int_0^{u} \bar{H}^*(z)dz \geqq u$, we conclude from this that $0 \geqq \int_u^{\infty} \bar{H}^*(z)dz$.

Case 2a. $\mu \leqq u \leqq 0$. The condition $\int_{-\infty}^{u} G^*(z)dz \geqq \int_{-\infty}^{u} H^*(z)dz$ becomes $u - \mu \geqq \int_{-\infty}^{u} H^*(z)dz$. But $u - \mu \geqq 0 \geqq \int_{-\infty}^{u} H^*(z)dz$.

Case 2b. $u < \mu$ and $u \leqq 0$. The condition

$$\int_{-\infty}^{u} G^*(z)dz \geqq \int_{-\infty}^{u} H^*(z)dz \quad \text{becomes} \quad 0 \geqq \int_{-\infty}^{u} H^*(z)dz$$

which follows from $H^*(z) \leqq 0, z \leqq 0$.

Now suppose that (3.5) holds. In this case it is possible to prove (3.4) in a manner analogous to the proof just given under the supposition that (3.6) holds. Alternatively, we can use the result that (3.6) implies (3.4): Let $G^\dagger(x) = \bar{G}(-x)$, $x^\dagger(t) = -x(-t)$, $\phi^\dagger(z) = \phi(-z)$, $a^\dagger = -b$, $b^\dagger = -a$, and $t_0^\dagger = -t_0$. Then (3.6) with $G^\dagger$ in place of G and $t_0^\dagger$ in place of t_0 is equivalent to (3.5) so it implies (3.4) with G, x, ϕ, a and b replaced by $G^\dagger$, $x^\dagger$, $\phi^\dagger$, $a^\dagger$ and $b^\dagger$ respectively. However, (3.4) with this replacement is equivalent to (3.4) without this replacement.

Next, suppose that (3.4) holds. Choose $t_1 < t_0$, let $x(t) = -1$, $t \leqq t_1$, $x(t) = 0$, $t > t_1$, and let $\phi(z) = z^2$. Then (3.4) becomes

$$\int_{[a,t_1]} dG(t) \leqq \left[\int_{[a,t_1]} (-1)dG(t)\right]^2 = \left[\int_{[a,t_1]} dG(t)\right]^2 ,$$

i.e., $G(t_1) \leqq [G(t_1)]^2$. Hence $G(t_1) \leqq 0$ or $G(t_1) \geqq 1, t_1 < t_0$. Similarly, by choosing $t_1 \geqq t_0, x(t) = 0, t \leqq t_1, x(t) = 1, t > t_1$, and $\phi(z) = z^2$, we conclude that $\bar{G}(t_1) \leqq 0$ or $\bar{G}(t_1) \geqq 1, t_1 \geqq t_0$.

Now let $t_0 \leqq t_1 < t_2 < b$ and suppose that $\bar{G}(t_1) \leqq 0$. Let

$$x(t) = \begin{cases} 0, & t \leqq t_1 \\ 1, & t_1 < t \leqq t_2 \\ 1 + \varepsilon, & t_2 < t \leqq b \,. \end{cases}$$

Then $\int_{[a,b]} x(t)dG(t) = \bar{G}(t_1) - \bar{G}(t_2) + (1+\varepsilon)\bar{G}(t_2) = \bar{G}(t_1) + \varepsilon\bar{G}(t_2)$. Since $\bar{G}(t_1) \leqq 0$, we can choose ε sufficiently small that $\bar{G}(t_1) + \varepsilon\bar{G}(t_2) < 1$. Let $\phi(z) = z - 1$ if $z \geqq 1$ and $\phi(z) = 0$ for $z < 1$. Then (3.4) becomes $\int_{[t_2,b]} \varepsilon dG(t) \leqq 0$, i.e., $\bar{G}(t_2) \leqq 0$. Similarly, if $a \leqq t_1 < t_2 < t_0$ and we let

$$x(t) = \begin{cases} -(1+\varepsilon), & a < t \leqq t_1 \\ -1, & t_1 < t \leqq t_2 \\ 0, & t > t_2 \,, \end{cases} \qquad \phi(z) = \begin{cases} -1 - z, & z \leqq -1 \\ 0, & z > -1 \end{cases}$$

then we can conclude that $G(t_2) \leqq 0$ implies $G(t_1) \leqq 0$.

Finally, suppose that $a < t_1 < t_0 \leqq t_2 < b$, and $G(t_1) \geqq 1$, $\bar{G}(t_2) \geqq 1$. Choose δ so small that $t_1 < t_0 - \delta < t_0$, and let $x(t) = -1/\{G(t_0 - \delta)\}$,

$t \leqq t_0 - \delta$, $x(t) = 0$, $t_0 - \delta < t \leqq t_0$, and $x(t) = 1/\{\bar{G}(t_0)\}$, $t > t_0$. Then

$$\int_{[a,b]} x(t)dG(t) = \int_{[a,t_0-\delta]} \frac{dG(t)}{-G(t_0 - \delta)} + \int_{(t_0,b]} \frac{dG(t)}{\bar{G}(t_0)} = -1 + 1 = 0 .$$

Inequality (3.4) becomes $\int_{[a,b]} \phi(x(t))dG(t) \leqq 0$. With $\phi(z) = |z|$, we obtain

$$\int_{[a,t_0-\delta]} \frac{dG(t)}{G(t_0)} + \int_{(t_0,b]} \frac{dG(t)}{\bar{G}(t_0)} = 2 \leqq 0 , \qquad \text{a contradiction} .$$

It follows that either (3.5) or (3.6) must be satisfied.

In the same manner as for Theorem 3.1, it is possible to restate Theorem 3.2 without the hypothesis $\phi(0) = 0$ as follows:

THEOREM 3.2a.

$$\phi\left(\int_{[a,b]} x(t)dG(t)\right) - \phi(0) \geqq \int_{[a,b]} [\phi(x(t)) - \phi(0)]dG(t) \tag{3.7}$$

for all convex functions ϕ and all increasing functions x such that $x(t_0) = 0$, $a \leqq t_0 \leqq b$, if and only if either (3.5) *or* (3.6).

Both Theorems 3.1 and 3.2 were obtained via Theorem 2.1. It follows from the conditions for equality there that equality holds in (3.1) (in 3.4) for all $x(t)$ such that $x(t_0) = 0$ and all G satisfying (3.2) (satisfying (3.5) or (3.6)) if and only if $\phi(x) = \alpha x$. In fact the same can be said if equality holds for $x(t) = t - t_0$ and all G satisfying the appropriate conditions. On the other hand, for certain specific $x(t)$, there may be other cases of equality.

We state several immediate but particularly interesting consequences of Theorems 3.1 and 3.2.

The condition that $\phi(0) = 0$ may be of special interest for functions ϕ on $[0, b]$, particularly when $t_0 = 0$. In this case, we obtain the following two special cases.

COROLLARY 3.3.

$$\phi\left(\int_{[0,b]} x(t)dG(t)\right) \leqq \int_{[0,b]} \phi(x(t))dG(t)$$

for all convex functions ϕ such that $\phi(0) = 0$ and all increasing functions x such that $x(0) = 0$ if and only if $0 \leqq \bar{G}(t) \leqq 1$, $0 \leqq t < b$.

COROLLARY 3.4.

$$\phi\left(\int_{[0,b]} x(t)dG(t)\right) \geqq \int_{[0,b]} \phi(x(t))dG(t)$$

for all convex functions ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(0) = 0$ *if and only if there exists* s, $0 < s < b$, *such that* $\bar{G}(t) \geqq 1$, $0 \leqq t < s$ *and* $\bar{G}(t) \leqq 0$, $s \leqq t \leqq b$.

In Theorems 3.1 and 3.2 it is assumed that a point t_0 is known such that $x(t_0) = 0$. The following corollaries are apparently more general than the theorems, because they assume only that an interval $[t_0, t_0']$ is known such that $x(t) = 0$ for some $t \in [t_0, t_0']$. This in essence requires that the inequalities hold for a wider class of functions $x(t)$, namely those for which $x(t) = 0$ for some $t \in [t_0, t_0']$. We obtain the conditions of the corollaries as the intersection of conditions of the theorems over all points where it may be that $x(t) = 0$.

COROLLARY 3.5. *Inequality* (3.1) *holds for all convex functions* ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(t) = 0$ *for some* $t \in [t_0, t_0']$, $a \leqq t_0 \leqq t_0' \leqq b$, *if and only if* $0 \leqq G(t) \leqq 1$, $a \leqq t < t_0'$ *and* $0 \leqq \bar{G}(t) \leqq 1$, $t_0 \leqq t \leqq b$.

COROLLARY 3.6. *Inequality* (3.4) *holds for all convex functions* ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(t) = 0$ *for some* $t \in [t_0, t_0']$, $a \leqq t_0 \leqq t_0' \leqq b$, *if and only if either there exists* $s \leqq t_0'$ *such that* $G(t) \leqq 0$, $t < s$, $G(t) \geqq 1$, $s \leqq t < t_0'$, *and* $\bar{G}(t) \leqq 0$, $t \geqq t_0$, *or there exists* $s \geqq t_0$ *such that* $G(t) \leqq 0$, $t < t_0'$, $G(t) \geqq 1$, $t_0 \leqq t < s$ *and* $\bar{G}(t) \leqq 0$, $t \geqq s$.

The special cases of these corollaries in which $t_0 = a$, $t_0' = b$ are particularly interesting, though we do not explicitly spell them out. This case of Corollary 3.5 bears comparison with Theorem 1 of Brunk [5]. However, Brunk's conditions are of a different nature than ours, because they depend upon the function x.

Both Theorem 3.1a and Theorem 3.2a yield interesting corollaries when the condition $G(b) = 1$ is imposed. In this case (3.3) reduces to (3.1) and (3.7) reduces to (3.4).

COROLLARY 3.7. (3.1) *holds for all convex functions* ϕ *and all increasing functions* x *such that* $x(t_0) = 0$ *if and only if* (3.2) *and* $G(b) = 1$.

COROLLARY 3.8. (3.4) *holds for all convex functions* ϕ *and all increasing functions* x *such that* $x(t_0) = 0$ *if and only if* $G(b) = 1$, *and* (3.5) *or* (3.6).

The sufficiency of these conditions follows from Theorems 3.1a and 3.2a. The necessity of $G(b) = 1$ is obtained with $\phi(x) \equiv 1$ and $\phi(x) \equiv$

-1 in (3.1) and (3.4).

With $t_0 = a$, the sufficiency of the conditions in Corollary 3.7 has been obtained by Brunk [5], Corollary 2.

We point out that in both Theorems 3.1 and 3.2, the necessity of the conditions was proved using only nonnegative functions ϕ. This means that the conditions for the inequalities to hold cannot be relaxed with the additional hypothesis that $\phi(x) \geqq 0$. It was with this fact in mind that we included Theorem 2.2, which is to be compared with Theorem 2.1.

Finally, we remark that results similar to those of this section can be obtained for functions ϕ concave-convex about the origin, $\phi(x) \leqq (\geqq) -\phi(-x)$ for $x \geqq 0$ and $\phi(0) = 0$. Discrete versions of such results have been obtained by Lawrence [11].

4. Inequalities for starshaped functions.

We retain the convention of § 3, that $-\infty \leqq a \leqq 0 \leqq b \leqq \infty$, and G is a function of bounded variation on $[a, b]$ such that $G(u) = \int_{[a,u]} dG(x)$. Also, we continue to require $\int_{[a,b]} x(t)dG(t) < \infty$.

THEOREM 4.1. *Fix $t_0 \in [a, b]$.*

$$\phi\left(\int_{[a,b]} x(t)dG(t)\right) \leqq \int_{[a,b]} \phi(x(t))dG(t) \tag{4.1}$$

for all supported starshaped functions ϕ such that $\phi(0) = 0$ and all increasing functions x such that $x(t_0) = 0$ if and only if there exists t_1 and t_2, $a \leqq t_1 \leqq t_0 \leqq t_2 \leqq b$, such that

$$\begin{aligned} &G(u) = 0, u < t_1; 0 \leqq G(u) \leqq 1 \text{ and } G(u) \text{ decreasing} \\ &\text{in } u, t_1 \leqq u < t_0 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &0 \leqq \bar{G}(u) \leqq 1 \text{ and } \bar{G}(u) \text{ increasing}, t_0 \leqq u < t_2 ; \\ &\bar{G}(u) = 0, u \geqq t_2 . \end{aligned} \tag{4.3}$$

Proof. Let $G^*(z) = G\{t: a \leqq t \leqq b \text{ and } x(t) \leqq z\}$, and let H^* be degenerate at $m = \int_{-\infty}^{\infty} z dG^*(z)$. Then (4.1) can be rewritten as

$$\int_{-\infty}^{\infty} \phi(z)dH^*(z) \leqq \int_{-\infty}^{\infty} \phi(z)dG^*(z) , \tag{4.1'}$$

and the conditions (4.2), (4.3) can be rewritten as follows: There exist z_1 and z_2, $z_1 \leqq 0 \leqq z_2$ such that

$$G^*(z) = 0,\ z < z_1;\ 0 \leqq G^*(z) \leqq 1 \text{ and } G^*(z) \text{ decreasing in } z,\ z_1 \leqq z \leqq 0 \tag{4.2'}$$

and

$$0 \leqq \bar{G}^*(z) \leqq 1 \text{ and } \bar{G}^*(z) \text{ increasing in } z,\ 0 \leqq z < z_2;\ \bar{G}^*(z) = 0,\ z \geqq z_2\ . \tag{4.3'}$$

We suppose that (4.2′) and (4.3′) hold, and verify the conditions of Theorem 2.6 with $\mu = G^* - H^*$.

By definition of G^* and H^*, we have (2.7).

To check (2.5), we first note that

$$m \equiv \int_{-\infty}^{\infty} xdG^*(x) = \int_0^\infty \bar{G}^*(x)dx - \int_{-\infty}^0 G^*(x)dx = \int_0^{z_2} \bar{G}^*(x)dx - \int_{z_1}^0 G^*(x)dx\ .$$

Consequently

$$z_1 \leqq m \leqq z_2\ . \tag{4.4}$$

Further, we note that integration by parts yields

$$\int_{(-\infty,u]} xdG^*(x) = uG^*(u) - \int_{(-\infty,u]} G^*(x)dx\ .$$

To check (2.5), we must show that

$$\int_{(-\infty,u]} xdG^*(x) \leqq \int_{(-\infty,u]} xdH^*(x),\ u \leqq 0\ .$$

(a) If $u < z_1$, then by (4.4), $u < m$ so that

$$0 = \int_{(-\infty,u]} xdG^*(x) = \int_{(-\infty,u]} xdH^*(x)\ .$$

(b) If $z_1 < u < m \leqq 0$,

$$\begin{aligned}\int_{(-\infty,u]} xdG^*(x) - \int_{(-\infty,u]} xdH^*(x) &= \int_{(-\infty,u]} xdG^*(x) \\ &= uG^*(u) - \int_{(-\infty,u]} G^*(x)dx \leqq uG^*(x) \leqq 0 \text{ by (4.2')}\ .\end{aligned}$$

(c) If $z_1 < m < u < 0$,

$$\begin{aligned}&\int_{(-\infty,u]} xdG^*(x) - \int_{(-\infty,u]} xdH^*(x) \\ &\quad = uG^*(u) - \int_{z_1}^u G^*(x)dx + \int_{z_1}^0 G^*(x)dx - \int_0^{z_2} \bar{G}^*(x)dx \\ &\quad = uG^*(u) + \int_u^0 G^*(x)dx - \int_0^{z_2} \bar{G}^*(x)dx \\ &\quad \leqq uG^*(u) + \int_u^0 G^*(x)dx \leqq uG^*(u) + G^*(u)\int_u^0 dx = 0\ .\end{aligned}$$

Next, we must check (2.6), i.e., $\int_{[u,\infty)} x dG^*(x) \geqq \int_{[u,\infty)} x dH^*(x)$, $u \geqq 0$.
(a) If $0 \leqq u \leqq m \leqq z_2$, then

$$\begin{aligned}\int_{[u,\infty)} x dG^*(x) &- \int_{[u,\infty)} x dH^*(x) \\ &= u\bar{G}^*(u^-) + \int_u^{z_2} \bar{G}^*(x)dx - \int_0^{z_2} \bar{G}^*(x)dx + \int_{z_1}^0 G^*(x)dx \\ &\geqq u\bar{G}^*(u^-) - \int_0^u \bar{G}^*(x)dx \geqq u\bar{G}^*(u^-) - \bar{G}^*(u^-)\int_0^u dx = 0 .\end{aligned}$$

(b) If $m < u < z_2$ and $u \geqq 0$,

$$\begin{aligned}\int_{[u,\infty)} x dG^*(x) &- \int_{[u,\infty)} x dH^*(x) \\ &= \int_{[u,\infty)} x dG^*(x) = u\bar{G}^*(u^-) + \int_u^{z_2} \bar{G}^*(x)dx \geqq 0 .\end{aligned}$$

(c) If $u \geqq z_2$, then

$$\int_{[u,\infty)} x dG^*(x) = \int_{[u,\infty)} x dH^*(x) = 0 .$$

This concludes the proof that (4.2) and (4.3) imply (4.1). It remains to show the converse.

Following the proof of Theorem 3.1, we conclude that $s_2 > t_0$ implies $0 \leqq \bar{G}(s_2) \leqq 1$, and $s_1 < t_0$ implies $0 \leqq G(s_1) \leqq 1$.

Next, suppose that $t_0 < s_2 < s_2 + \delta$ and that $\bar{G}(s_2 + \delta) > 0$.

Let $x(t) = \begin{cases} 0, & t \leqq s_2 \\ 1, & s_2 < t \leqq s_2 + \delta \\ 1 + \varepsilon, & t > s_2 + \delta , \end{cases}$ and let $\phi(z) = \begin{cases} 0, & z \leqq 1 \\ z, & z > 1 . \end{cases}$

Then for sufficiently large ε,

$$\begin{aligned}\int x(t)dG(t) &= \bar{G}(s_2) - \bar{G}(s_2 + \delta) + (1 + \varepsilon)\bar{G}(s_2 + \delta) \\ &= \bar{G}(s_2) + \varepsilon\bar{G}(s_2 + \delta) > 1 ,\end{aligned}$$

and (4.1) becomes

$$(1 + \varepsilon)\bar{G}(s_2 + \delta) \geqq \bar{G}(s_2) + \varepsilon\bar{G}(s_2 + \delta) ,$$

that is

$$\bar{G}(s_2 + \delta) \geqq \bar{G}(s_2) .$$

This proves (4.3). Condition (4.2) follows similarly with

$$x(t) = \begin{cases} -(1+\varepsilon), & t \leqq s_1 \\ -1, & s_1 < t \leqq s_1 + \delta \\ 0, & t > s_1 + \delta , \end{cases} \quad \text{and} \quad \phi(z) = \begin{cases} -z, & z < -1 \\ 0, & z \geqq -1 , \end{cases}$$

where $s_1 < s_1 + \delta < t_0$.

THEOREM 4.2. *Fix* $t_0 \in [a, b]$.

$$\phi\left(\int_{[a,b]} x(t)dG(t)\right) \leqq \int_{[a,b]} \phi(x(t))dG(t) \tag{4.5}$$

for all starshaped functions ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(t_0) = 0$ *if and only if either*

(4.6) *there exists* $t_1, a \leqq t_1 \leqq t_0$ *such that* $G(u) = 0, u < t_1$; $0 \leqq G(u) \leqq 1$ *and* $G(u)$ *decreasing in* $u, t_1 \leqq u < t_0$, *and* $\bar{G}(u) = 0, u \geqq t_0$,

or

(4.7) *there exists* $t_2, t_0 \leqq t_2 \leqq b$, *such that* $G(u) = 0, u < t_0$, $0 \leqq \bar{G}(u) \leqq 1$ *and* $\bar{G}(u)$ *increasing*, $t_0 < u < t_2, \bar{G}(u) = 0, u \geqq t_2$.

Proof. Though (4.5) can be obtained from Theorem 2.5, we use Theorem 4.1. If (4.6) or (4.7), then (4.2) and (4.3) and in addition, G has no mass to the right of t_0, or no mass to the left of t_0. Thus, the integrals of the inequality can be extended over $[a, t_0]$ (where $x(t) \leqq 0$) or over $[t_0, b]$ (where $x(t) \geqq 0$). But on $(-\infty, 0]$ or $[0, \infty)$, starshaped functions ϕ are supported starshaped functions and the inequality follows from Theorem 4.1.

$$\text{Let } x(t) = \begin{cases} -1, & t \leqq t_1 \\ 0, & t_1 < t \leqq t_2 \\ \alpha, & t > t_2 , \end{cases} \quad \text{where} \quad t_1 < t_0 < t_2 ,$$

and suppose $G(t_1) > 0, \bar{G}(t_2) > 0$. Then $m = \int_a^b x(t)dG(t) = \alpha\bar{G}(t_2) - G(t_1)$ is strictly positive for some α and strictly negative for some α. According to Theorem 2.5, we must have $\int_{[a,t_0)} x(t)d\mu(t) = 0$, where $\mu = G - H$ and H is a probability distribution degenerate at m. This is impossible, since m can be strictly positive or strictly negative while $\int_{[a,t_0)} x(t)dG(t)$ is unchanged.

THEOREM 4.3. *Fix* $t_0 \in [a, b]$.

$$\phi\Big(\int_{[a,b]} x(t)dG(t)\Big) \geqq \int_{[a,b]} \phi(x(t))dG(t) \tag{4.8}$$

for all supported starshaped functions ϕ such that $\phi(0) = 0$ and all increasing functions x such that $x(t_0) = 0$ if and only if either

(4.9) *there exists t_1, $a \leqq t_1 \leqq t_0$, such that $G(t) = 0$, $a \leqq t < t_1$; $G(t) \geqq 1$ and $G(t)$ is increasing, $t_1 \leqq t < t_0$; $\bar{G}(t) \leqq 0$, $t_0 \leqq t \leqq b$,*

or

(4.10) *there exists t_2, $t_0 \leqq t_2 \leqq b$, such that $G(t) \leqq 0$, $a \leqq t < t_0$; $\bar{G}(t) \geqq 1$ and $\bar{G}(t)$ is decreasing in t, $t_0 \leqq t < t_2$; $\bar{G}(t) = 0$, $t_2 \leqq t \leqq b$.*

Proof. Let $H^*(z) = G\{t: a \leqq t \leqq b \text{ and } x(t) \leqq z\}$, and let G^* be the probability distribution degenerate at $\int_{-\infty}^{\infty} zdH^*(z)$. Then (4.8) can be rewritten as

$$\int_{-\infty}^{\infty} \phi(z)dG^*(z) \geqq \int_{-\infty}^{\infty} \phi(z)dH^*(z) \tag{4.8$'$}$$

and (4.10) can be rewritten as

(4.10′) there exists $z_2 > 0$ such that $H^*(z) \leqq 0$, $z < 0$; $\bar{H}^*(z) \geqq 1$ and $\bar{H}^*(z)$ is decreasing in z, $0 \leqq z < z_2$; $\bar{H}^*(z) = 0$, $z \geqq z_2$.

We show that (4.10′) implies (4.8′) by applying Theorem 2.6 with $\mu = G^* - H^*$. First, note that

$$\begin{aligned} m = \int_{-\infty}^{\infty} zdH^*(z) &= \int_0^{\infty} \bar{H}^*(z)dz \\ &- \int_{-\infty}^{0} H^*(z)dz \geqq \int_0^{z_2} \bar{H}^*(z)dz \geqq z_2 \ , \end{aligned} \tag{4.11}$$

since $\bar{H}^*(z) \geqq 0$, $z \geqq 0$, $\bar{H}^*(z) \geqq 1$, $0 < z < z_2$, and $H^*(z) \leqq 0$, $z \leqq 0$.

Clearly (2.7) is satisfied. To verify (2.5), we note by (4.11) that $m \geqq 0$ so (2.5) is $\int_{(-\infty,u]} zdH^*(z) \geqq 0$, $u \leqq 0$. But

$$\int_{(-\infty,u]} zdH^*(z) = uH^*(u) - \int_{-\infty}^{u} H^*(z)dz \geqq 0$$

because $H^*(z) < 0$, $z \leqq u$, $H^*(u) < 0$ and $u \leqq 0$.

To verify (2.6), suppose first that $u > z_2$. Then (2.6) becomes $\int xdG^*(x) \geqq 0$ which follows from (4.11). If $u \leqq z_2$ (then $u \leqq m$ by

(4.11)), (2.6) becomes

$$\int_{[u,\infty)} zdG^*(z) \geqq \int_{[u,z_2)} zdH^*(z) \ , \quad \text{i.e.,} \quad \int_{-\infty}^{\infty} zdH^*(z) \geqq \int_{[u,z_2)} zdH^*(z) \ ,$$

or

$$\int_{(-\infty,0]} zdH^*(z) + \int_{(0,u]} zdH^*(z) \geqq 0 \ .$$

Now

$$\int_{(-\infty,0]} zdH^*(z) = -\int_{-\infty}^{0} H^*(z)dz \geqq 0$$

and

$$\int_{(0,u]} zdH^*(z) = -u\bar{H}^*(u) + \int_0^u \bar{H}^*(z)dz \geqq 0$$

because $\bar{H}^*(z)$ is decreasing in z, $0 < z < z_2$.

The proof that (4.9) implies (4.8) is analogous. Consequently, we turn to the problem of showing that (4.9) or (4.10) is necessary. Suppose that (4.8) holds.

A. $\bar{G}(s_2) \geqq 1$ or $\bar{G}(s_2) \leqq 0$, $s_2 > t_0$; $G(s_1) \geqq 1$ or $G(s_1) \leqq 0$, $s_1 < t_0$.

To see this, let $\phi(z) = z^2$; first let

$$x(t) = \begin{cases} 0, \, t \leqq s_2 \\ 1, \, t > s_2 \end{cases} , \quad \text{then let} \quad x(t) = \begin{cases} -1, \, t \leqq s_1 \\ 0, \, t > s_1 \end{cases} .$$

Apply (4.8) to obtain $[\bar{G}(s_2)]^2 \geqq \bar{G}(s_2)$ and $[G(s_1)]^2 \geqq G(s_1)$.

B. $G(s_1)\bar{G}(s_2) \leqq 0$ if $s_1 < t_0 < s_2$.

To see this, suppose the contrary, let

$$x(t) = \begin{cases} -1/|\,G(s_1)\,|, \, t \leqq s \\ 0, \, s_1 < \quad t \leqq s_2 \\ 1/|\,\bar{G}(s_2)\,|, \, t > s_2 \ , \end{cases}$$

and let

$$\phi_1(z) = \begin{cases} z, \, z \leqq 0 \\ 0, \, z \geqq 0 \ , \end{cases} \qquad \phi_2(z) = \begin{cases} 0, \, z \leqq 0 \\ z, \, z \geqq 0 \ . \end{cases}$$

Then $\int_{[a,b]} x(t)dG(t) = 0$. With $\phi \equiv \phi_1$, (4.8) becomes

$$0 \geqq -G(s_1)/|\,G(s_1)\,| \ , \quad \text{a contradiction if} \quad G(s_1) < 0 \ ;$$

with $\phi \equiv \phi_2$, (4.8) becomes

$$0 \geqq \bar{G}(s_2)/|\,\bar{G}(s_2)\,| \ , \quad \text{a contradiction if} \quad \bar{G}(s_2) > 0 \ .$$

C. If $t_0 < s_2 < s_2 + \delta \leqq b$ and $\bar{G}(s_2 + \delta) \geqq 1$, then $\bar{G}(s_2) \geqq \bar{G}(s_2 + \delta)$. To see this, suppose the contrary, that $\bar{G}(s_2) < \bar{G}(s_2 + \delta)$.

$$\text{Let} \quad x(t) = \begin{cases} 0, & t \leqq s_2 \\ 1, & s_2 < t \leqq s_2 + \delta \\ 1 + \varepsilon, & t > s_2 + \delta \,, \end{cases} \qquad \phi(x) = \begin{cases} 0, & x < 1 \\ x, & x \geqq 1 \,. \end{cases}$$

Then

$$\begin{aligned} m &= \int_a^b x(t)dG(t) = [\bar{G}(s_2) - \bar{G}(s_2 + \delta)] + (1 + \varepsilon)\bar{G}(s_2 + \delta) \\ &= \varepsilon\bar{G}(s_2 + \delta) + \bar{G}(s_2) \,, \end{aligned} \tag{4.12}$$

and $m < \bar{G}(S_2 + \delta)$ for sufficiently small $\varepsilon > 0$. From (4.8), we conclude that

$$0 \geqq \int_{(s_2+\delta, b]} \phi(1 + \varepsilon)dG(t) = (1 + \varepsilon)\bar{G}(s_2 + \delta)$$

contradicting $\bar{G}(s_2 + \delta) \geqq 1$.

C′. If $a \leqq s_1 < s_1 + \delta < t_0$, and $G(s_1) \geqq 1$, then $G(s_1 + \delta) \geqq G(s_1)$. The proof of this is analogous to C above.

D. If $t_0 < s_2 < s_2 + \delta \leqq b$, $\bar{G}(s_2) \geqq 1$, and $\bar{G}(s_2 + \delta) \leqq 0$, then $\bar{G}(s_2 + \delta) = 0$.

If we assume the contrary and take $x(t)$, $\phi(x)$ as in C, we can choose $\varepsilon > 0$ so that $0 < m < 1$. Then (4.8) becomes

$$0 \geqq [\bar{G}(s_2) - \bar{G}(s_2 + \delta)] + (1 + \varepsilon)\bar{G}(t_2 + \delta) = \bar{G}(s_2) + \varepsilon\bar{G}(s_2 + \delta) = m \,,$$

contradicting $m > 0$.

D′. If $a \leqq s_1 < s_1 + \delta < t_0$, $G(s_1 + \delta) \geqq 1$ and $\bar{G}(s_1) \leqq 0$, then $\bar{G}(s_1) = 0$. The proof is analogous to D.

THEOREM 4.4. *Fix* $t_0 \in [a, b]$.

$$\phi\Big(\int_{[a,b]} x(t)dG(t)\Big) \geqq \int_{[a,b]} \phi(x(t))dG(t) \tag{4.13}$$

for all starshaped functions ϕ *such that* $\phi(0)$ *and all increasing functions* x *such that* $x(t_0) = 0$ *if and only if either*

(4.14) *there exists* $t_1 \leqq t_0$ *such that* $G(t) = 0, t < t_1; G(t) \geqq 1$ *and* $G(t)$ *is increasing*, $t_1 \leqq t < t_0; \bar{G}(t) = 0, t \geqq t_0$,

or

(4.15) *there exists* $t_2 \geqq t_0$ *such that* $\bar{G}(t) \geqq 1$ *and* $\bar{G}(t)$ *is decreasing in* $t, t_0 \leqq t < t_2; \bar{G}(t) = 0, t > t_2; G(t) = 0, t < t_0$.

Proof. Since every supported starshaped function is starshaped, the conditions (4.9) or (4.10) are necessary for (4.13). Suppose that (4.10) holds. Then, in order to satisfy (2.5) of Theorem 2.5 we must in addition guarantee that $\int_{-\infty}^{0} x dH^*(x) = 0$, because, by (4.11), $m \geqq 0$. But $\int_{-\infty}^{0} x dH^*(x) = -\int_{-\infty}^{0} H^*(x) dx = 0$ together with $H^*(x) \leqq 0$, $x < 0$, implies $H^*(x) = 0, x < 0$. Thus (4.14) or (4.15) is necessary; it is sufficient by Theorem 4.3 since, on $[0, \infty)$ or $(-\infty, 0]$, starshaped functions ϕ are supported starshaped functions. If (4.14) or (4.15), the interval $[a, b]$ is effectively replaced by $[a, t_0]$ or $[t_0, b]$.

Only minor modifications of Theorems 4.2 and 4.4 are required to eliminate the condition $\phi(0) = 0$, provided that $x(t) \neq 0$ for $t \neq t_0$. In this case, the inequalities depend upon $\phi(0)$ only through $G\{t_0\}$. Since starshaped functions ϕ satisfy $\phi(0) \leqq 0$, conditions can be imposed on $G\{t_0\}$ so that the inequalities do not become false even when $-\phi(0)$ is arbitrarily large. In the case of Theorem 4.2, the condition $G\{t_0\} \leqq 0$ must be added; in the case of Theorem 4.4, the condition $G\{t_0\} \geqq 0$ is required.

The most natural domain for starshaped functions is $[0, b]$; on this domain, a starshaped function is a supported starshaped function. From Theorem 4.1 or 4.2 we obtain the special case of

COROLLARY 4.5.

$$\phi\left(\int_{[0,b]} x(t) dG(t)\right) \leqq \int_{[0,b]} \phi(x(t)) dG(t)$$

for all starshaped functions ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(0) = 0$ *if and only if there exists* t_2, $0 \leqq t_2 \leqq b$, *such that* $0 \leqq \bar{G}(u) \leqq 1$ *and* $\bar{G}(u)$ *is increasing,* $0 \leqq u < t_2$, *and* $\bar{G}(u) = 0$, $u \geqq t_2$.

Similarly from Theorem 4.3 or 4.4 we obtain

COROLLARY 4.6.

$$\phi\left(\int_{[0,b]} x(t) dG(t)\right) \geqq \int_{[0,b]} \phi(x(t)) dG(t)$$

for all starshaped functions ϕ *such that* $\phi(0) = 0$ *and all increasing functions* x *such that* $x(0) = 0$ *if and only if there exists* t_2, $0 \leqq t_2 \leqq b$, *such that* $\bar{G}(t) \geqq 1$ *and* $\bar{G}(t)$ *is decreasing in* t, $0 \leqq t < t_2$; $\bar{G}(t) = 0$, $t \geqq t_2$.

In the same spirit as Corollaries 3.5 and 3.6, and via the same argunments, we obtain from Theorems 4.1 and 4.3 the following corollaries:

COROLLARY 4.7. (4.1) *holds for all supported starshaped functions* ϕ *and all increasing functions* x *such that* $x(t) = 0$ *for some* $t \in [t_0, t_0']$, $a \leqq t_0 \leqq t_0' \leqq b$, *if and only if there exists* t_1 *and* t_2, $t_1 \leqq t_2$, $a \leqq t_1 \leqq t_0'$, $t_0 \leqq t_2 \leqq b$, *such that* $G(u) = 0, a \leqq u < t_1$; $0 \leqq G(u) \leqq 1$ *and* $G(u)$ *decreasing in* $u, t_1 \leqq u < t_0'$; $0 \leqq \bar{G}(u) \leqq 1$ *and* $\bar{G}(u)$ *increasing*, $t_0 \leqq u < t_2$; $\bar{G}(u) = 0, t_2 \leqq u \leqq b$.

COROLLARY 4.8. (4.8) *holds for all supported starshaped functions and all increasing functions* x *such that* $x(t) = 0$ *for some* $t \in [t_0, t_0']$, $a \leqq t_0 \leqq t_0' \leqq b$ *if and only if either* (i) *there exists* $t_1, a \leqq t_1 \leqq t_0'$, *such that* $G(t) = 0, a \leqq t < t_1$; $G(t) \geqq 1$ *and* $G(t)$ *increasing*, $t_1 \leqq t < t_0'$; $\bar{G}(t) \leqq 0, t_0 \leqq t \leqq b$, *or* (ii) *there exists* $t_2, t_0 \leqq t_2 \leqq b$, *such that* $\bar{G}(t) \geqq 1$ *and* $\bar{G}(t)$ *is decreasing*, $t_0 \leqq t < t_2$; $\bar{G}(t) = 0, t_2 \leqq t \leqq b$; $G(t) \leqq 0, a \leqq t < t_0'$, *or* (iii) G *is a point mass of at least one at some point* $s \in [t_0, t_0']$.

We remark that (i) guarantees that (4.9) holds and (ii) guarantees (4.10) holds no matter what $t \in [t_0, t_0']$ satisfies $x(t) = 0$. However (iii) guarantees (4.9) when $t > s$ and (4.10) when $t < s$.

Again, the special cases $t_0 = a, t_0' = b$ are of particular interest, and may be easily written out. This case of Corollary 4.8 takes a particularly simple form because only (iii) is possible.

5. **Discrete versions.** Some results similar to those of § 3 are known in the discrete case. One such inequality is due to Szëgo [14]; it states that if $0 < x_1 \leqq x_2 \leqq \cdots \leqq x_{2m-1}$ and ϕ is convex on $[0, x_{2m-1}]$, then

$$\phi\left[\sum_{j=1}^{2m-1} (-1)^{j-1} x_j\right] \leqq \sum_{j=1}^{2m-1} (-1)^{j-1} \phi(x_j) \ . \tag{5.1}$$

Closely related results were obtained by Weinberger [15], Bellman [3], and Wright [16]. These results were further generalized by Brunk [5] and Olkin [13], who proved that if $0 \leqq b_1 \leqq \cdots \leqq b_n \leqq 1$, if $0 \leqq x_1 \leqq \cdots \leqq x_n$, and if ϕ is convex, then

$$\phi\left[\sum_{j=1}^{n} (-1)^{n-j} b_j x_j\right] - \phi(0) \leqq \sum_{j=1}^{n} (-1)^{n-j} b_j [\phi(x_j) - \phi(0)] \ . \tag{5.2}$$

Notice that if $n = 2m - 1$ and $b_j \equiv 1$, then (5.2) reduces to (5.1).

Theorem 3.1a yields necessary and sufficient conditions on $a_1, a_2, \cdots, a_n$ in order that

$$\phi\left(\sum_{j=1}^{n} a_j x_j\right) - \phi(0) \leqq \sum_{j=1}^{n} a_j [\phi(x_j) - \phi(0)] \tag{5.3}$$

for all convex functions ϕ and $0 \leqq x_1 \leqq \cdots \leqq x_n$. In addition, discrete versions of various other results are of interest.

In the following, we assume that $\phi(0) = 0$, so that (5.3) becomes

$$\phi\left(\sum_{j=1}^{n} a_j x_j\right) \leqq \sum_{j=1}^{n} a_j \phi(x_j) \ . \tag{5.4}$$

For various conditions on the x_j, below are listed necessary and sufficient conditions on the a_j in order that (5.4) holds for all convex functions ϕ such that $\phi(0) = 0$. Similarly, conditions are listed for its reversal,

$$\phi\left(\sum_{1}^{n} a_j x_j\right) \geqq \sum_{1}^{n} a_j \phi(x_j) \ . \tag{5.5}$$

We use the notation $A_i = \sum_1^i a_j$ and $\bar{A}_i = \sum_i^n a_j$.

A. $0 \leqq x_1 \leqq \cdots \leqq x_n$:

(5.4) if and only if $0 \leqq \bar{A}_i \leqq 1, 1 \leqq i \leqq n$;

(5.5) if and only if there exists $j, 0 \leqq j \leqq n$ such that $\bar{A}_i \geqq 1$, $1 \leqq i \leqq j, \bar{A}_i \leqq 0, j+1 \leqq i \leqq n$.

B. $x_1 \leqq \cdots \leqq x_n$:

(5.4) if and only if $0 \leqq A_i \leqq 1$ and $0 \leqq \bar{A}_i \leqq 1, 1 \leqq i \leqq n$;

(5.5) if and only if $A_i \leqq 0$ and $\bar{A}_i \leqq 0, i = 1, 2, \cdots, n$, *or* for some $j, 0 \leqq j \leqq n, \bar{A}_1 \geqq 1, \cdots, \bar{A}_j \geqq 1, \bar{A}_{j+1} \leqq 0, \cdots, \bar{A}_n \leqq 0$, and $\bar{A}_i \geqq \bar{A}_1, 1 \leqq i \leqq j$. Note that this last condition is equivalent to $A_1 \leqq 0, \cdots, A_{k-1} \leqq 0, A_{k+1} \geqq 1, \cdots, A_n \geqq 1$, and $A_i \geqq A_k, k \leqq i \leqq n$, for some $k, (1 \leqq k \leqq n)$.

C. $x_1 \leqq \cdots \leqq x_k \leqq 0 \leqq x_{k+1} \leqq \cdots \leqq x_n$:

(5.4) if and only if $0 \leqq A_i \leqq 1, 1 \leqq i \leqq k$, and $0 \leqq \bar{A}_i \leqq 1$, $k+1 \leqq i \leqq n$;

(5.5) if and only if there exists $j \leqq k$ such that $A_i \leqq 0, i < j$; $A_i \geqq 1, j \leqq i \leqq k$; $\bar{A}_i \leqq 0, i \geqq k+1$, *or* there exists $j \geqq k$ such that $A_i \leqq 0, i \leqq k$; $\bar{A}_i \geqq 1, k+1 \leqq i \leqq j$; $\bar{A}_i \leqq 0, i > j$.

D. $x_1 \leqq \cdots \leqq x_n$ and $x_k \leqq 0 \leqq x_l (k \leqq l)$:

(5.4) if and only if $0 \leqq A_i \leqq 1, i < l$, and $0 \leqq \bar{A}_i \leqq 1, i > k$;

(5.5) if and only if there exists $1 \leqq j \leqq l$ such that $A_i \leqq 0$, $i < j$; $A_i \geqq 1, j \leqq i < l$; $\bar{A}_i \leqq 0, i \geqq k+1$, *or* there exists $k \leqq j \leqq n$ such that $A_i \leqq 0, i < l, \bar{A}_i \geqq 1, k+1 \leqq i \leqq j$; $\bar{A}_i \leqq 0, i > j$.

We turn now to the case that ϕ is a supported starshaped function such that $\phi(0) = 0$; below are necessary and sufficient conditions for (5.4) and for (5.5) to hold for all such functions with various conditions on the x_i.

A. $0 \leqq x_1 \leqq \cdots \leqq x_n$:

(5.4) if and only if for some $j, 0 \leqq j \leqq n, 0 \leqq \bar{A}_1 \leqq \bar{A}_2 \leqq \cdots \leqq$

$\bar{A}_j$; $\bar{A}_{j+1} = \cdots = \bar{A}_n = 0$.

(5.5) if and only if for some $j, 0 \leqq j \leqq n, \bar{A}_1 \geqq \bar{A}_2 \geqq \cdots \geqq \bar{A}_j \geqq 1, \bar{A}_{j+1} = \cdots = \bar{A}_n = 0$.

B. $x_1 \leqq \cdots \leqq x_n$:

(5.4) if and only if for some j, $0 \leqq j \leqq n$, $0 \leqq a_j \leqq 1$ and $a_i = 0$, $i \neq j$.

(5.5) if and only if for some j, $0 \leqq j \leqq n$, $a_j \geqq 1$ and $a_i = 0$, $i \neq j$. This condition arises from (iii) of Theorem 4.8.

C. $x_1 \leqq \cdots \leqq x_k \leqq 0 \leqq x_{k+1} \leqq \cdots \leqq x_n$:

(5.4) if and only if there exists j_1 and $j_2, j_1 \leqq j_2, 1 \leqq j_1 \leqq k+1, k \leqq j_2 \leqq n$, such that $A_1 = \cdots = A_{j_1-1} = 0, 1 \geqq A_{j_1} \geqq \cdots \geqq A_k \geqq 0, 0 \leqq \bar{A}_{k+1} \leqq \cdots \leqq \bar{A}_{j_2} \leqq 1, \bar{A}_{j_2+1} = \cdots = \bar{A}_n = 0$.

(5.5) if and only if either

(i) there exists $j_1, 1 \leqq j_1 \leqq k+1$ such that

$$A_1 = \cdots = A_{j_1-1} = 0, 1 \leqq A_{j_1} \leqq \cdots \leqq A_k, \bar{A}_{k+1} = \cdots = \bar{A}_n = 0 ,$$

or

(ii) there exists $j_2, k \leqq j_2 \leqq n$ such that

$$A_i \leqq 0, 1 \leqq i \leqq k, \bar{A}_{k+1} \geqq \cdots \geqq \bar{A}_{j_2} \geqq 1, \bar{A}_{j_2+1} = \cdots = \bar{A}_n = 0 ,$$

or

(iii) $a_j \geqq 1$ for $j = k$ or $k+1$ and $a_i = 0$, $i \neq j$.

D. $x_1 \leqq \cdots \leqq x_n$ and $x_k \leqq 0 \leqq x_l (k \leqq l)$:

(5.4) if and only if there exists $j_1 \leqq j_2, 1 \leqq j_1 \leqq l, k \leqq j_2 \leqq n$, such that $A_1 = \cdots = A_{j_1-1} = 0, 1 \geqq A_{j_1} \geqq \cdots \geqq A_{l-1} \geqq 0, 0 \leqq \bar{A}_{k+1} \leqq \cdots \leqq \bar{A}_{j_2} \leqq 1, \bar{A}_{j_2+1} = \cdots = \bar{A}_n = 0$.

(5.5) if and only if either

(i) there exists $j_1, 1 \leqq j_1 \leqq l$, such that

$$A_1 = \cdots = A_{j_1-1} = 0, 1 \leqq A_{j_1} \leqq \cdots \leqq A_{l-1}, \bar{A}_{k+1} = \cdots = \bar{A}_n \leqq 0 ,$$

or

(ii) there exists $j_2, k \leqq j_2 \leqq n$, such that

$$A_i \leqq 0, 1 \leqq i < l, \bar{A}_{k+1} \geqq \cdots \geqq \bar{A}_{j_2} \geqq 1, \bar{A}_{j_2+1} = \cdots = \bar{A}_n = 0 ,$$

or

(iii) there exists $j, k \leqq j \leqq l$, such that $a_j \geqq 1$ and $a_i = 0, i \neq j$.

6. **Applications.** Our interest in the foregoing results arose from the study of statistical problems for certain restricted families of probability distributions of importance in reliability theory. We say that a distribution F is convex (starshaped) with respect to H if $F(0) = H(0) = 0$ and $H^{-1}F$ is convex (starshaped) on the support of

F. Of particular interest is the case that H is the exponential distribution, i.e., $\bar{H}(x) = e^{-x}$ (other cases are interesting, e.g., $H(x) = x$, $0 \leqq x \leqq 1$).

Distributions F convex with respect to the exponential distribution are called *increasing hazard rate* (IHR) distributions, because those that are absolutely continuous are characterized by having an increasing hazard rate $r(x) = [dF(x)/dx]/\bar{F}(x)$. Distributions F starshaped with respect to the exponential distribution are called *increasing hazard rate average* (IHRA) distributions, because those that are absolutely continuous are characterized by the property that $(1/t)\int_0^t r(x)dx$ is increasing. The properties of IHR and IHRA arise in formulating descriptions of "wearout" in reliability theory.

If $0 \leqq X_1 \leqq \cdots \leqq X_n$ are order statistics from F, then $H^{-1}F(X_i)$ are distributed as order statistics Y_i from H. With $\phi(x) = H^{-1}F(x)$, where ϕ is convex or starshaped, we obtain from the foregoing inequalities necessary and sufficient conditions on the a_i in order that

$$F(\sum a_i X_i) \overset{st}{\leqq} H(\sum a_i Y_i) \quad \text{or} \quad H(\sum a_i Y_i) \overset{st}{\leqq} F(\sum a_i X_i) \ ,$$

where $\overset{st}{\leqq}$ denotes "stochastically less than." Such inequalities are used by Barlow and Proschan [2] to construct conservative tolerance limits for IHR or IHRA distributions.

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