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ON PRIME DIVISORS OF THE BINOMIAL COEFFICIENT

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A classical theorem discovered independently by J. Sylvester and I. Schur states that in a set of k consecutive integers, each of which is greater than k, there is a number having a prime divisor greater than k. In giving an elementary proof, P. Erdös expressed the theorem in the following form:

If $n \geqq 2k$, then $\binom{n}{k}$ has a prime divisor $p > k$.

Recently, P. Erdös suggested a problem of a complementary nature:

If $n \geqq 2k$, then $\binom{n}{k}$ has a prime divisor $p \leqq \frac{n}{2}$

The problem is solved by the following

THEOREM. If $n \geqq 2k$, then $\binom{n}{k}$ has a prime divisor $p \leqq \max\left\{\frac{n}{k}, \frac{n}{2}\right\}$, with the exception $\binom{7}{3}$.

Throughout the paper, p denotes a prime. J. Rosser and L. Schoenfeld [2] have obtained fairly precise estimates for $\theta(x) = \sum_{p \leqq x} \log(p)$, and $\pi(x) = \sum_{p \leqq x} 1$.

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) < \pi(x) \qquad \text{for } x \geqq 59 . \tag{1}$$

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right) \qquad \text{for } x > 1 . \tag{2}$$

$$\pi(x) < \frac{1.25506x}{\log x} \qquad \text{for } x > 1 . \tag{3}$$

$$\theta(x) < 1.01624x \qquad \text{for } x > 0 . \tag{4}$$

$$x - 2.05282\sqrt{x} < \theta(x) < x \qquad \text{for } 0 < x \leqq 10^8 . \tag{5}$$

Using these results, we are able to prove the theorem.

First we establish the following lemmas.

LEMMA 1. *If $\binom{n}{k}$ has no prime divisors $p \leqq n/2$, then*

$$\binom{n}{k} \leqq e^{\theta(n)-\theta(n-k)} \leqq n^{\pi(n)-\pi(n-k)} . \tag{6}$$

LEMMA 2. *For $k \geqq 59$,*

(7) $$n^{\pi(n)-\pi(n-k)} < e^{(n/\log n + k + k/2\log n)} .$$

LEMMA 3.

(8) $$\frac{2^{(n+1)k-1}}{\sqrt{k}} \leqq \binom{2^n k}{k} .$$

Proof of Lemma 1. $\binom{n}{k} \leqq \prod_{n-k<p\leqq n} p \leqq \prod_{n-k<p\leqq n} n$. Hence

$$\binom{n}{k} \leqq e^{\theta(n)-\theta(n-k)} \leqq n^{\pi(n)-\pi(n-k)} .$$

Proof of Lemma 2. From (1) and (2), we have

$$\begin{aligned} n^{\pi(n)-\pi(n-k)} &< n^{\{n/\log n[1+3/(2\log n)]-(n-k)/\log(n-k)[1+1/(2\log(n-k))]\}} \\ &< n^{\{n/\log n[1+3/2\log n]-(n-k)/\log n[1+1/2\log n]\}} \\ &< e^{\{n[1+3/2\log n]-(n-k)[1+1/2\log n]\}} \\ &< e^{(n//\log n+k+k/2\log n)} . \end{aligned}$$

Lemma 3 is proved by induction on n for all values of k.

The proof of the theorem is by contradiction. Three cases are considered. The general case is a Sylvester-Schur type argument. The other cases involve deducing contradictions from appropriate upper and lower bounds on the inequalities, (6), of Lemma 1.

Proof of the theorem. Assume $\binom{n}{k}$ has no prime divisors

$$p \leqq \max\left\{\frac{n}{k}, \frac{n}{2}\right\} .$$

1. $k < n^{2/3}$. $\binom{n}{k} = \dfrac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots 1} \geqq \left(\dfrac{n}{k}\right)^k .$

By sieving all multiples of 2, and 3, we have

$$\pi(n) - \pi(n-k) \leqq \frac{k}{2} \qquad \text{for } k \geqq 4 .$$

Therefore from (6), we have $(n/k)^k \leqq n^{k/2}$. Thus the assumption is false if $4 \leqq k < n^{1/2}$. By sieving all multiples of 2, 3, and 5, we have

$$\pi(n) - \pi(n-k) \leqq \frac{k}{3} \qquad \text{for } k \geqq 60 .$$

Thus from (6), we have $(n/k)^k \leqq n^{k/3}$. Hence the assumption is false if $60 \leqq k < n^{2/3}$.

2. $n^{2/3} \leqq k \leqq n/16$. Let $\tilde{n} = [n/2]$, and $\tilde{k} = [k/2]$; where $[x]$ denotes the integral part of x. If $p > k$ and p divides $\binom{\tilde{n}}{\tilde{k}}$, then p divides $\binom{n}{k}$ and $p \leqq n/2$. By assumption, there are no such primes. Therefore, $\binom{\tilde{n}}{\tilde{k}}$ has no prime divisors $p > 2\tilde{k} + 1$. Thus $\binom{\tilde{n}}{\tilde{k}} < \tilde{n}^{\pi(\sqrt{\tilde{n}})} \cdot e^{\theta(2\tilde{k}+1)}$ (see paper of M. Faulkner [1]). From (3), (4), and (8), we have

$$\frac{2^{5\tilde{k}-1}}{\sqrt{\tilde{k}}} < \tilde{n}^{(1.26\sqrt{\tilde{n}}/\log\sqrt{\tilde{n}})} \cdot e^{1.02(2\tilde{k}+1)} .$$

Taking logarithms, we obtain

$$3.45\tilde{k} - 0.70 - \frac{1}{2}\log(\tilde{k}) < 2.52\sqrt{\tilde{n}} + 1.02(2\tilde{k} + 1) ,$$

which is a contradiction for $\tilde{k} > 32$. Therefore the assumption is false if $n^{2/3} \leqq k \leqq n/16$ when $k \geqq 65$.

3. $n/16 < k \leqq n/2$. Consider $n/16 < k \leqq n/8$. By (6), (7) and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(n/\log n + k + k/2\log n)} .$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2}\log(k) < \frac{n}{\log n} + k + \frac{k}{2\log n} ;$$

which is false for $k \geqq 1901$. By (5), (6), and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(k+2.06\sqrt{15k})} .$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2}\log(k) < k + 2.6\sqrt{15k} ;$$

which is false for $k \geqq 25$. Thus the assumption is false if $n/16 < k \leqq n/8$ when $k \geqq 25$. By similar arguments, we show the assumption is false is $n/8 < k \leqq n/4$ when $k \geqq 32$; and if $n/4 < k \leqq n/2$ when $k > 105$.

We have proved the theorem for $k \geqq 4$ with the exception of a finite number of cases. The cases $k = 1, 2$, and 3, are easily resolved; and the remaining cases have been checked with the aid of an IBM 1620 computer in the following manner:

The values which were checked are $4 \leqq k \leqq 60$ with $2k \leqq n \leqq k^2$, and $61 \leqq k \leqq 105$ with $2k \leqq n \leqq 4k$.

For the i-th prime, p_i, the exponent to which p_i occurred in the "numerator", $n(n-1)\cdots(n-k+1)$, and in the "denominator", $k!$, of $\binom{n}{k}$, α_i and β_i respectively, were determined; and the values of p_i, n, and k, were reported if the difference, $\alpha_i - \beta_i$, was positive. Cross-checking was done manually. The first ten primes proved sufficient to verify the theorem in these cases.

This concludes the proof of the theorem.

In closing, I would like to thank Professor M. Faulkner for her gracious assistance.

REFERENCES

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