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SEMIGROUPS SATISFYING IDENTITY $xy = f(x, y)$

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Dedicated to Professor Keizo Asano on his Sixtieth Birthday

Let $f(x, y)$ be a word of length greater than 2 starting in y and ending in x. The purpose of this paper is to prove that a semigroup satisfies an identity $xy = f(x, y)$ if and only if it is an inflation of a semilattice of groups satisfying the same identity. As its consequence we find a necessary and sufficient condition for $xy = f(x, y)$ to imply commutativity.

Recently E. J. Tully has proved [7] that if a semigroup S satisfies an identity of the form $xy = y^m x^n$ then S is an inflation of a semilattice of abelian groups G_α's satisfying $x^k = 1$ for all $x \in G_\alpha$ where k is the greatest common divisor of $m - 1$ and $n - 1$; hence $xy = y^m x^n$ implies commutativity. This paper is to consider the general case of the right side of $xy = y^m x^n$ with the left side unchanged.

Let $f(x, y)$ denote a word involving both x and y, and $|f(x, y)|$ be the length of the word $f(x, y)$: $|x|_f$ be the number of x's which appear in $f(x, y)$; $|y|_f$ be also defined for y. For example if $f(x, y) = x^3y^2xy$, $|f(x, y)| = 7$, $|x|_f = 4$, $|y|_f = 3$. Throughout this paper we assume $|f(x, y)| > 2$, equivalently $|x|_f > 1$ or $|y|_f > 1$ or both.

Consider an identity of the form

$$xy = f(x, y) \tag{1}$$

in semigroups. A question is raised: Under what condition on $f(x, y)$ does (1) imply commutativity $xy = yx$? What we can say immediately is that $f(x, y)$ has to start in y. Because if $f(x, y)$ starts in x, then left zero semigroups of order > 1 satisfy (1) but are not commutative. For the similar reason $f(x, y)$ must end in x. From now on we assume $f(x, y)$ in (1) has the form:

$$\begin{cases} f(x, y) = y^{m_1}x^{n_1} \cdots y^{m_s}x^{n_s},\ m_i > 0,\ n_i > 0,\ i = 1, \cdots, s\ , \\ \text{and } |f(x, y)| > 2\ . \end{cases} \tag{2}$$

A semigroup D is called an inflation of a semigroup T if T is a subsemigroup of D and there is a mapping φ of D into T such that

$$\varphi(x) = x \qquad \text{for } x \in T$$

and

$$xy = \varphi(x)\varphi(y) \qquad \text{for } x, y \in D\ .$$

Let L be a semilattice. A semigroup S is called a semilattice L of semigroups S_α, $\alpha \in L$, if S is a disjoint union of $\{S_\alpha; \alpha \in L\}$ and

$$S_\alpha S_\beta \subseteqq S_{\alpha\beta} \qquad \text{for all } \alpha, \beta \in L\,.$$

Needless to say an identity is preserved by homomorphic images and by subsemigroups; in particular the identity $xy = f(x, y)$ with (2) is satisfied by any semilattice.

THEOREM. *A semigroup S satisfies an identity $xy = f(x, y)$ with (2) if and only if S is an inflation of a semilattice of groups satisfying the same identity.*

COROLLARY. *Let $f(x, y)$ be a word involving both x and y, and let $|f(x, y)| > 2$. $xy = f(x, y)$ implies $xy = yx$ in semigroups if and only if*

(3.1) *$f(x, y)$ starts in y and ends in x, and*

(3.2) *$xy = f(x, y)$ implies $xy = yx$ in groups.*

The statement in the theorem can be replaced by another statement. In the following proposition we do not assume $xy = f(x, y)$.

PROPOSITION. *The following three statements are equivalent:*

(4.1) *S^2 is a semilattice L of groups G_α, $\alpha \in L$.*

(4.2) *S is a semilattice L of semigroups S_α, $\alpha \in L$, each of which is an inflation of a group G_α and*

$$x_\alpha y_\beta \in G_{\alpha\beta} \qquad \text{for } x_\alpha \in S_\alpha,\ y_\beta \in S_\beta\,.$$

(4.3) *S is an inflation of a semilattice L of groups G_α, $\alpha \in L$.*

Let $a_1, \cdots, a_k$ be a finite number of elements of a semigroup S. All the elements x of S each of which is the product of all of $a_1, \cdots, a_k$ (admitting repeated use) form a subsemigroup of S. It is called the content (of $a_1, \cdots, a_k$) in S and denoted by $C(a_1, \cdots, a_k)$ or C. The elements $a_1, \cdots, a_k$ are not required to be distinct. For example $C(a, a) = \{a^i; i > 1\}$ but $C(a) = \{a^i; i \geqq 1\}$. The number k is called the rank of $C(a_1, \cdots, a_k)$.

Any semigroup S has a smallest semilattice-congruence ($\mathscr{S}$-congruence) ρ_0, that is, S/ρ_0 is a semilattice and if S/ρ is a semilattice, then $\rho_0 \subseteqq \rho$. The decomposition of S induced by ρ_0 is called the greatest $\mathscr{S}$-decomposition of S. If $\rho_0 = S \times S$, S is called $\mathscr{S}$-indecomposable. An $\mathscr{S}$-decomposition $S = \bigcup_{\alpha \in L} S_\alpha$ of S is greatest if and only if each S_α is $\mathscr{S}$-indecomposable [4], [5], [6].

LEMMA 1. *A content is $\mathscr{S}$-indecomposable.* (See [6].)

LEMMA 2. *$a\rho_0 b$ if and only if there is a finite sequence of contents*

$C_1, \cdots, C_t$ *such that*

$$(5) \qquad a \in C_1, \quad C_i \cap C_{i+1} \neq \emptyset \ (i = 1, \cdots, t-1), \quad b \in C_t \,.$$

Proof. Define ρ_1 as follows: $a\rho_1 b$ if and only if there is a finite sequence of contents $C_1, \cdots, C_t$ satisfying (5). We will prove $\rho_0 = \rho_1$. It is easily shown that ρ_1 is an equivalence relation. To prove compatibility, suppose $a\rho_1 b$. There exists a sequence of contents satisfying (5), more specifically

$$(5') \qquad \begin{cases} C_i = C(a_{i1}, \cdots, a_{ik_i}) & i = 1, \cdots, t \\ a \in C_1, \quad d_i \in C_i \cap C_{i+1} & (i = 1, \cdots, t-1), \quad b \in C_t \,. \end{cases}$$

Let $c \in S$. Consider a sequence of contents

$$C_i' = C(c, a_{i1}, \cdots, a_{ik_i}) \ (i = 1, \cdots, t-1) \,.$$

Then $ca \in C_1'$, $cd_i \in C_i' \cap C_{i+1}'(i = 1, \cdots, t-1)$, $cb \in C_t'$. Hence $ca\rho_1 cb$. Likewise $ac\rho_1 bc$. Thus ρ_1 is a congruence. Since $a, a^2 \in C(a)$ and ab, $ba \in C(a, b)$, we see $a\rho_1 a^2$, $ab\rho_1 ba$ for all $a, b \in S$, that is, ρ_1 is an $\mathscr{S}$-congruence on S. Let ρ be an $\mathscr{S}$-congruence on S. We will prove $\rho_1 \subseteqq \rho$. Let $a\rho_1 b$. There is a sequence $C_1, \cdots, C_t$ described in (5'). Since C_i is $\mathscr{S}$-indecomposable $(i = 1, \cdots, t)$ by Lemma 1, we have $a\rho d_1$, $d_1\rho d_2, \cdots, d_{t-1}\rho b$, hence $a\rho b$. Since ρ_1 is the smallest $\mathscr{S}$-congruence, we have $\rho_0 = \rho_1$.

Let

$$S = \bigcup_{\alpha \in L} S_\alpha, \quad S_\alpha S_\beta \subseteqq S_{\alpha\beta}, \quad S_\alpha \cap S_\beta = \emptyset, \quad \alpha \neq \beta$$

be the greatest $\mathscr{S}$-decomposition of S. We notice that if a and b are in S_α then the contents $C_i (i = 1, \cdots, t)$ described in (5) are contained in S_α.

LEMMA 3. *An ideal of an $\mathscr{S}$-indecomposable semigroup is $\mathscr{S}$-indecomposable.*

Proof. An equivalent statement is proved in [4]. However, we will prove this by using Lemma 2. Let I be an ideal of an $\mathscr{S}$-indecomposable semigroup S. Let $a, b \in I$, $a \neq b$. By Lemma 2, there is a sequence of contents $C_i = C(a_{i1}, \cdots, a_{ik_i}) \subseteqq S (i = 1, \cdots, t)$ such that $a \in C_1$, $d_i \in C_i \cap C_{i+1} (i = 1, \cdots, t-1)$, $b \in C_t$. Consider a sequence of contents:

$$C_0' = C(a), C_i' = C(a, a_{i1}, \cdots, a_{ik_i}) \qquad (i = 1, \cdots, t)$$
$$C_{2t+1}' = C(b), C_{t+i}' = C(b, a_{i1}, \cdots, a_{ik_i}) \qquad (i = 1, \cdots, t) \,.$$

Then

$$a \in C'_0, \quad a^2 \in C'_0 \cap C'_1, \quad ad_i \in C'_i \cap C'_{i+1} \qquad (i = 1, \cdots, t-1)$$
$$ab \in C'_t \cap C'_{t+1}, \quad d_i b \in C'_{t+i} \cap C'_{t+i+1} \qquad (i = 1, \cdots, t-1)$$
$$b^2 \in C'_{2t} \cap C'_{2t+1}, \quad b \in C'_{2t+1}$$

and all $C'_j (j = 0, \cdots, 2t+1)$ are in I. By Lemma 2, we have proved that I is $\mathscr{S}$-indecomposable.

Proof of proposition. The proof of $(4.2) \Rightarrow (4.1)$ and $(4.3) \Rightarrow (4.1)$ is immediate. We will prove only $(4.1) \Rightarrow (4.2)$ and (4.3). Assume $S^2 = \bigcup_{\alpha \in L} G_\alpha$ where G_α is a group for each α. This is the greatest $\mathscr{S}$-decomposition of S^2 because the groups G_α are $\mathscr{S}$-indecomposable. Let

$$(6) \qquad S = \bigcup_{\xi \in L'} S_\xi$$

be the greatest $\mathscr{S}$-decomposition of S. Since S_ξ^2 is an ideal of the $\mathscr{S}$-indecomposable semigroup S_ξ, S_ξ^2 is also $\mathscr{S}$-indecomposable by Lemma 3. For each $\alpha \in L$ there is a unique α' in L' such that $G_\alpha \subseteqq S_{\alpha'}$; for each $\eta \in L'$ there is a unique η'' in L such that $S_\eta^2 \subseteqq G_{\eta''}$. We define two mappings f and g, $f: L \to L'$, $g: L' \to L$ by $\alpha' = f(\alpha)$ and $\eta'' = g(\eta)$, respectively; in other words

$$G_\alpha \subseteqq S_{f(\alpha)}, \quad S_\eta^2 \subseteqq G_{g(\eta)} .$$

This implies $G_\alpha \subseteqq S^2_{f(\alpha)} \subseteqq G_{gf(\alpha)}$. It follows that $gf(\alpha) = \alpha$; thus gf is the identity mapping on L. Likewise fg is the identity mapping on L'. Hence f and g are one-to-one and onto, $g = f^{-1}$.

Identifying α with $f(\alpha)$, and L with L', we have

$$(7) \qquad S = \bigcup_{\alpha \in L} S_\alpha$$

$$(8) \qquad S^2 = \bigcup_{\alpha \in L} S_\alpha^2, \quad S_\alpha^2 = G_\alpha .$$

We notice that $G_\alpha \subseteqq S_\alpha$, hence $S^2 \cap S_\alpha = S_\alpha^2 = G_\alpha$. By the assumption on (6), (7) is the greatest $\mathscr{S}$-decomposition of S.

Let e_α be an identity element of G_α. Since $S_\alpha^2 = G_\alpha$, e_α is a unique idempotent of S_α. We will prove (9) through (12) below:

$$(9) \qquad e_{\alpha\beta} e_\alpha = e_{\alpha\beta} = e_\alpha e_{\alpha\beta} \qquad \text{for all } \alpha, \beta \in L .$$

Noticing that $e_{\alpha\beta} e_\alpha \in G_{\alpha\beta}$,

$$(e_{\alpha\beta} e_\alpha)(e_{\alpha\beta} e_\alpha) = (e_{\alpha\beta} e_\alpha e_{\alpha\beta}) e_\alpha = (e_{\alpha\beta} e_{\alpha\beta} e_\alpha) e_\alpha = e_{\alpha\beta} e_\alpha .$$

Thus $e_{\alpha\beta} e_\alpha$ is an idempotent, hence $e_{\alpha\beta} e_\alpha = e_{\alpha\beta}$. The proof of the remaining part is done in the same way.

(10) $x_\alpha e_\beta = e_\beta x_\alpha$ for all $x_\alpha \in S_\alpha$, all $\alpha, \beta \in L$.

By using (9),

$$x_\alpha e_\beta = (x_\alpha e_\beta)e_{\alpha\beta} = x_\alpha(e_\beta e_{\alpha\beta}) = x_\alpha e_{\alpha\beta} = e_{\alpha\beta}x_\alpha e_{\alpha\beta}$$
$$= e_{\alpha\beta}(e_{\alpha\beta}x_\alpha) = e_{\alpha\beta}x_\alpha = (e_{\alpha\beta}e_\beta)x_\alpha = e_{\alpha\beta}(e_\beta x_\alpha) = e_\beta x_\alpha .$$

(11) $e_\alpha e_\beta = e_{\alpha\beta} = e_\beta e_\alpha$ for all $\alpha, \beta \in L$.

We have $e_\alpha e_\beta = e_\beta e_\alpha$ by (10). It can be easily proved that $e_\alpha e_\beta$ is an idempotent. Therefore $e_\alpha e_\beta = e_{\alpha\beta}$.

(12) $x_\alpha y_\beta = (x_\alpha e_\alpha)(y_\beta e_\beta)$ for all $x_\alpha \in S_\alpha$, $y_\beta \in S_\beta$, all $\alpha, \beta \in L$.

By (10) and (11) we have

$$(x_\alpha e_\alpha)(y_\beta e_\beta) = x_\alpha(e_\alpha y_\beta)e_\beta = x_\alpha(y_\beta e_\alpha)e_\beta = (x_\alpha y_\beta)(e_\alpha e_\beta) = (x_\alpha y_\beta)e_{\alpha\beta} = x_\alpha y_\beta .$$

Let mappings $\varphi_\alpha: S_\alpha \to G_\alpha (\alpha \in L)$ be defined by

$$\varphi_\alpha(x_\alpha) = x_\alpha e_\alpha = e_\alpha x_\alpha .$$

Then each φ_α is a homomorphism of S_α onto G_α such that $\varphi_\alpha(x_\alpha) = x_\alpha$ for all $x_\alpha \in G_\alpha$ and

(13) $x_\alpha y_\beta = \varphi_\alpha(x_\alpha)\varphi_\beta(x_\beta), \quad x_\alpha \in S_\alpha, \quad y_\beta \in S_\beta$.

Consequently S is an inflation of $G = \bigcup_{\alpha \in L} G_\alpha$ and S_α is an inflation of G_α. Thus we have proved that (4.1) $\Rightarrow$ (4.2) and (4.3).

REMARK. In the proof of the proposition, if we use the fact that a semilattice of groups is an inverse semigroup, we immediately have (11), hence (9). However, we proved these directly without using the property of inverse semigroups.

Proof of theorem. We will need two lemmas to prove the theorem.

LEMMA 4. *If S satisfies $xy = f(x, y)$ with* (2), *then every content of rank greater than* 1 *is a group.*

Proof. Let $C = C(a_1, \cdots, a_k)$, $k > 1$. Each element of C is expressed as a word involving all the letters $a_1, \cdots, a_k$. Let $w \in C$ be decomposed into the product of two words w_1, w_2 of a_i's, namely elements w_1, w_2 of S: $w = w_1 w_2$. Replacing w by $f(w_1, w_2)$ repeatedly we can arrange w such that $|a_i|_w > 1$ for all i. First we will prove that each element w has a form $a_i v'$, $v' \in C$. If a_i is the initial letter of w, then we have already the form since $|a_i|_w > 1$ and hence $v' \in C$. Suppose

$$w = w_1 a_i w_2, \ w_1, w_2 \in S .$$

Replacing $w_1(a_i w_2)$ by $f(w_1, a_i w_2)$ we have a form $w = a_i v'$. Since $|a_j|_w > 1$ for all j, $|a_i|_{v'} \geqq 1$ and $|a_j|_{v'} > 1$ for $j \neq i$. Hence $v' \in C$. Again we can arrange v' such that $|a_j|_{v'} > 1$ for all j. Now let w' be any element of C:

$$w' = a_{i_1} a_{i_2} \cdots a_{i_l} .$$

By repeating the same procedure

$$w = a_{i_1} w_{i_1} = a_{i_1} a_{i_2} w_{i_2} = \cdots = a_{i_1} a_{i_2} \cdots a_{i_l} w_{i_l} = w' w_{i_l}$$

where $w_{i_j} \in C (j = 1, \cdots, l)$. In the same way we have $w = w'_{i_l} w'$ for some $w'_{i_l} \in C$. Thus we have proved the right and left divisibility; therefore $C(a_1, \cdots, a_k)$ is a group.

LEMMA 5. *Let* $S = \bigcup_{\alpha \in L} S_\alpha$ *be greatest* $\mathscr{S}$*-decomposition of a semigroup* S. *If* S *satisfies* $xy = f(x, y)$ *with* (2), *then* S_α^2 *is a group and* $S^2 = \bigcup_{\alpha \in L} S_\alpha^2$.

Proof. Let $a \in S_\alpha^2$, $a = xy$ for some $x, y \in S_\alpha$. Then a is in the content $C(x, y)$ which is a subgroup of S_α^2 by Lemma 4. Thus S_α^2 is a union of subgroups, hence a disjoint union of maximal subgroups of S_α. We will prove that for two distinct arbitrary elements a and b of S_α^2 there are subgroups G_a and G_b which contain a and b, respectively, such that $G_a \cap G_b \neq \varnothing$. Then S_α^2 will be a group. Since a, $b \in S_\alpha$ there is a finite sequence of contents $C_1, C_2, \cdots, C_t$ in S such that

$$a \in C_1, \quad C_i \cap C_{i+1} \neq \varnothing \ (i = 1, \cdots t-1), \quad b \in C_t .$$

As remarked in § 1, $C_i \subseteqq S_\alpha \ (i = 1, \cdots, t)$. Since $a, b \in S_\alpha^2$, we may assume C_1 and C_t are of rank > 1. Also we may assume $C_2, \cdots, C_{t-1}$ are of rank > 1 for the following reason. Suppose C_i, $1 < i < t$, has rank 1, $C_i = [x]$, the cyclic subsemigroup generated by x. If x is in either $C_i \cap C_{i-1}$ or $C_i \cap C_{i+1}$, then $C_i \subseteqq C_{i-1}$ or $C_i \subseteqq C_{i+1}$, respectively; so C_i can be excluded from the sequence. If x is in C_i but not in $(C_i \cap C_{i-1}) \cup (C_i \cap C_{i+1})$, then we can replace C_i by $\{x^i; i > 1\}$ of rank > 1. By Lemma 4 all the $C_i (i = 1, \cdots, t)$ are subgroups of S_α. It is easy to prove that $C_i \cap C_{i+1}$ and $C_{i+1} \cap C_{i+2}$ are also subgroups; hence $C_i \cap C_{i+2} \neq \varnothing$ since the identity element of the group C_{i+1} has to lie in C_i and C_{i+2}. Continuing this procedure we have $C_1 \cap C_t \neq \varnothing$ as desired. Thus it has been proved that S_α^2 is a group. Let $G_\alpha = S_\alpha^2$ for each $\alpha \in L$. G_α is the greatest subgroup in S_α, a maximal subgroup in S. Now let $z \in S^2$, $z = xy$ for some $x \in S_\alpha$, $y \in S_\beta$, so $z \in S_{\alpha\beta}$. The element z is in a subgroup $C(x, y) \subset S_{\alpha\beta}^2$, hence $z \in G_{\alpha\beta}$. Thus we have

$$S^2 = \bigcup_{\alpha \in L} G_\alpha, \quad G_\alpha = S_\alpha^2 .$$

Proof of theorem. By Lemma 5 and the proposition, it has been proved that if S satisfies $xy = f(x, y)$ with (2) then S is an inflation of $\bigcup_{\alpha \in L} G_\alpha$, a semilattice L of groups G_α, in which G_α satisfies the identity $xy = f(x, y)$. We will prove the converse of the theorem. Suppose that S is an inflation of $\bigcup_{\alpha \in L} G_\alpha$, where each G_α is a group satisfying $xy = f(x, y)$ with (2). Let $x_\alpha \in S_\alpha$, $y_\beta \in S_\beta$. By proposition $x_\alpha y_\beta \in G_{\alpha\beta}$. By using (10) and by recalling the form of $f(x, y)$ we have

$$x_\alpha y_\beta = x_\alpha y_\beta e_{\alpha\beta} = (x_\alpha e_{\alpha\beta})(y_\lambda e_{\alpha\beta}) = f(x_\alpha e_{\alpha\beta}, y_\beta e_{\alpha\beta}) = f(x_\alpha, y_\beta) e_{\alpha\beta} = f(x_\alpha, y_\beta) .$$

This proves that S satisfies the same identity. Thus the proof of the theorem has been completed.

Proof of corollary. Suppose $xy = f(x, y)$ implies $xy = yx$ in semigroups. Then (3.2) is obvious and the necessity of (3.1) is already proved in § 1. It remains to prove the sufficiency of (3.1) and (3.2). The theorem describes the structure of S satisfying the identity $xy = f(x, y)$ in which $|f(x, y)| > 2$ and $f(x, y)$ satisfies (3.1), that is, $f(x, y)$ has the form (2). Now additionally assume (3.2). By using (10)

$$x_\alpha y_\beta = (x_\alpha e_{\alpha\beta})(y_\beta e_{\alpha\beta}) = (y_\beta e_{\alpha\beta})(x_\alpha e_{\alpha\beta}) = y_\beta x_\alpha .$$

Examples and problems. As the application of corollary we give a few examples below:

EXAMPLE 1. (Tully). $xy = y^m x^n$, $m \geqq 1$, $n \geqq 1$,

EXAMPLE 2. $xy = (yx)^m$, $m \geqq 1$,

EXAMPLE 3. $xy = y^{m_1} x^{n_1} \cdots y^{m_h} x^{n_h}$, $m = \sum_{i=1}^h m_i$, $n = \sum_{i=1}^h n_i$, the greatest common divisor of $m - 1$ and $n - 1$ is 2 or 1.

Each identity of Examples 1, 2 and 3 implies $xy = yx$ in semigroups because it can be easily proved that commutativity follows in groups.

EXAMPLE 4. $xy = (y^2 x^2)^2$.

In groups this identity is equivalent to the identity

$$x^3 = e, \ e \text{ the identity element} .$$

We know that the free group G generated by a and b subject to $x^3 = e$ is a finite noncommutative group [2] [3]. Therefore commutativity does not follow.

Finally we give examples which satisfy (3.2) but not (3.1).

EXAMPLE 5. $xy = x^2y^3x$,

EXAMPLE 6. $xy = yx^3y^2$,

EXAMPLE 7. $xy = xy^3$.

Each of Examples 5, 6, and 7 implies commutativity in groups. Accordingly we can say:

The conditions (3.1) and (3.2) of the corollary are independent.

ADDENDA. 1. Let $f(x, y)$ and $g(x, y)$ be words involving both x and y and let $|f(x, y)| \geqq 2$ and $|g(x, y)| \geqq 2$.

If $g(x, y) = f(x, y)$ implies $xy = yx$ in semigroups[1], then one of $g(x, y)$ and $f(x, y)$ is xy and the other satisfies the condition (3.1); hence this case is reduced to that of the corollary.

Suppose $|g(x, y)| > 2$ and $|f(x, y)| > 2$. Let F be the free semigroup generated by the two letters a and b, and I be the ideal of F consisting of all words with length more than 2. Let $S = F/I$. We see $S = \{0, a, b, a^2, b^2, ab, ba\}$. S satisfies $g(x, y) = f(x, y)$ but it is not commutative as $ab \neq ba$. We may assume the identity is $xy = f(x, y)$ which is the condition in the corollary.

2. Let $f(x, y)$ be a word involving both x and y.

$$x^2 = f(x, y) \qquad \text{does not imply } xy = yx \text{ in general}.$$

For the reason used for (3.1), we may assume $f(x, y)$ starts in y and ends in y. Let F be the free semigroup generated by a and b, and I be the set of all words which involve either at least two a's or at least two b's. I is an ideal. $S = F/I = \{0, a, b, ab, ba\}$. S satisfies $x^2 = f(x, y)$, but is not commutative.

PROBLEMS. 1. Let $f(x, y) = y^{m_1}x^{n_1} \cdots y^{m_h}x^{n_h}$ with (2). How can we describe explicitly (3.2) in terms of $m_1, n_1, \cdots, m_h, n_h$?

2. Determine the structure of semigroups satisfying an identity of the form $xy = y^{m_1}x^{n_1} \cdots y^{m_{h-1}}x^{n_{h-1}}y^{m_h}$, $m_1 \neq 0$, $m_h \neq 0$, $h \geqq 2$.

3. Determine the structure of semigroups satisfying an identity of the form $xy = x^{m_1}y^{n_1} \cdots x^{m_{h-1}}y^{n_{h-1}}$, $m_1 \neq 0$, $n_{h-1} \neq 0$.

[1] The author owes this result to Dr. D. G. Mead's helpful suggestion.

4. Let $f(x, y) = y^{m_1}x^{n_1} \cdots y^{m_h}x^{n_h}y^{m_{h+1}}$, $m_1, m_{h+1} \neq 0$, $h \neq 0$. Under what condition on $f(x, y)$ does the identity $x = f(x, y)$ imply $xy = yx$?

A semigroup S satisfying the identity $x = f(x, y)$ is a group.

In fact S is a union of groups. By [1] S is a semilattice L of completely simple semigroups $S_\alpha(\alpha \in L)$

$$S = \bigcup_{\alpha \in L} S_\alpha .$$

We can easily prove that $|L| = 1$; S is a completely simple semigroup, that is, a rectangular band B of groups. However we can prove that $|B| = 1$.

A partial answer follows:

Let $$\text{g.c.d.}\left(\sum_{i=1}^{h+1} m_i, \sum_{i=1}^{h} n_i - 1\right) = g .$$

If $g = 2$, the answer is affirmative.

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