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DILATIONS OF RAPIDLY DECREASING FUNCTIONS

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Let S be the space of rapidly decreasing indefinitely differentiable functions on the n-dimensional Euclidean space R^n, and suppose that $\phi \in S$. We attempt to characterize the closed vector subspace of S which is generated by the set of all functions of the form

$$(x_1, \cdots, x_n) \rightarrow \phi(a_1x_1 + b_1, \cdots, a_nx_n + b_n)$$

where $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ are real numbers, with $a_1, \cdots, a_n$ nonzero. We also consider an analogous approximation problem in the space S' of temperate distribution on R^n.

1. Notation. We write R^n for the n-dimensional Euclidean space. Addition and multiplication in R^n are defined component-wise. If $k \leqq n$ is a positive integer and $x \in R^n$, then we write x_k for the k-th component of x. The set $R^\#$ is defined by $R^\# = \{x \in R^n : x_k \neq 0 \text{ for all } k\}$.

The Pontryagin character group of R^n is identified with R^n. We denote typical elements of R^n by $x, y, \cdots$, or by $\chi, \xi, \cdots$ if we are thinking of R^n as its own character group. If $\chi \in R^n$ then the bounded continuous character determined by χ is defined by

$$x \longrightarrow \exp(-2\pi i \chi \circ x)(x \in R^n) .$$

If W is an open subset of R^n, then $C^\infty(W)$ will denote the set of functions which are defined in W and are indefinitely differentiable there. $\boldsymbol{D}(W)$ will denote the set of indefinitely differentiable functions with compact support in W. $D'(W)$ will be written for the space of distributions which have support in W. The space of rapidly decreasing indefinitely differentiable functions on R^n and the space of temperate distributions on R^n will be designated by S and S', respectively. We shall always assume that S' is equipped with the strong topology $\beta(S', S)$. For details of these spaces, see Schwartz [4] and [5].

If f is a continuous function on R^n, then $Z(f)$ will denote the set of zeros of f.

Throughout, we use the standard notations of the calculus of n variables; see, for example, Hörmander [3, p. 4]. If $k \leqq n$ is a positive integer, then j_k will denote the projection of R^n onto its k-th factor: $j_k(x) = x_k$ for all $x \in R^n$. In general, if α is a multi-index then j^α will be written for the function on R^n which is defined by $j^\alpha(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for all $x \in R^n$.

Suppose that $\phi \in \boldsymbol{D}(R^n)$ and that $a \in R^\#$. Then the function $\phi^a \in \boldsymbol{D}(R^n)$ which is defined by $\phi^a(x) = \phi(ax)(x \in R^n)$ is called a *dilation* of ϕ. If $b \in R^n$ then the function $\phi_b \in \boldsymbol{D}(R_n)$ which is defined by $\phi_b(x) = \phi(x + b)$ is called a *translate* of ϕ. The dilation u^a and translate u_b of an arbitrary distribution $u \in \boldsymbol{D}'(R^n)$ are defined as follows: we set $u^a(\phi) = |1/j(a)|\, u(\phi^{a^{-1}})$ and $u_b(\phi) = u(\phi_b)$ for all $\phi \in \boldsymbol{D}(R^n)$. It is easy to see that for any distribution u, $\operatorname{supp} u^a = a^{-1}.\operatorname{supp} u$; and that if $u \in \boldsymbol{S}'$ then $u^a \in \boldsymbol{S}'$ and $\widehat{u^a} = |1/j(a)|\, \hat{u}^{a^{-1}}$. The latter statement may be verified by direct computation, or by reference to relation (5.15.14) in Edwards [1]. Finally, if u is a function then $u^a(x) = u(ax)$ and $u_b(x) = u(x + b)$ for all $x \in R^n$.

2. **Preliminaries.** We gather into this section some results which we need in what follows. In our arguments we find it convenient to use the following notation. If ψ is an arbitrary function in $\boldsymbol{S}$ and w is an arbitrary temperate distribution then we shall write $\psi \bigtriangledown w$ for the function on $R^\sharp$ which is defined by $\psi \bigtriangledown w(x) = w(\psi^x)(x \in R^\sharp)$.

LEMMA 2.1. *Suppose that $\eta \in \boldsymbol{S}$ and $s \in \boldsymbol{S}'$ are such that*

$$\eta^a \circ s = 0 \qquad \text{for all } a \in R^\sharp . \tag{2.1}$$

Then for each multi-index α it is true that

$$D^\alpha \eta^a \circ j^\alpha s = 0 \qquad \text{for all } a \in R^\sharp . \tag{2.2}$$

Proof. The proof proceeds by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The assertion is certainly true if $|\alpha| = 0$. Thus, we assume that $\alpha > 0$ and that

$$D^\beta \eta^a \circ j^\beta s = 0 \qquad \text{for all } a \in R^\sharp \tag{2.3}$$

whenever β is a multi-index such that $|\beta| < |\alpha|$. We have to show that these assumptions entail that

$$D^\alpha \eta^a \circ j^\alpha s = 0 \qquad \text{for all } a \in R^\sharp . \tag{2.4}$$

With this end in view, let $a \in R^\sharp$ be arbitrary but fixed. If ϕ is an arbitrary function in $\boldsymbol{D}(R^n)$ then Lemma 2.5 (b) in Harasymiv [2] tells us that the function $\eta \bigtriangledown \phi s$ is indefinitely differentiable on $R^\sharp$ and that

$$j^\alpha(a) \circ D^\alpha[\eta \bigtriangledown \phi s](a) = \sum_{\beta \leq \alpha} C^\alpha_\beta [j^\beta D^\beta \eta) \bigtriangledown \phi s](a) \tag{2.5}$$

where $C^\alpha_\beta = \alpha!/\beta!(\alpha - \beta)!$. If we notice that for each β we have $(j^\beta D^\beta \eta)^a = j^\beta \circ D^\beta \eta^a$ then (2.5) may be rewritten in the form

$$\begin{aligned} j^\alpha(a) \circ D^\alpha[\eta \bigtriangledown \phi s](a) &= \sum_{\beta \leq \alpha} C^\alpha_\beta \circ \phi s(j^\alpha D^\beta \eta^a) \\ &= \sum_{\beta \leq \alpha} C^\alpha_\beta \circ D^\beta \eta^a \circ j^\beta s(\phi) . \end{aligned} \tag{2.6}$$

Now, if β is a multi-index such that $\beta<\alpha$ then certainly $|\beta|<|\alpha|$. We may therefore invoke the inductive hypothesis (relation (2.3)) and deduce from (2.6) that

$$D^{\alpha}\eta^{a}\circ j^{\alpha}s(\phi)=j^{\alpha}(a)\circ D^{\alpha}[\eta\nabla\phi s](a)\ . \tag{2.7}$$

Next we notice that, because of relation (2.1), we have for each $x\in R^{\#}$ the equality

$$\begin{aligned}\eta\nabla\phi s(x)&=\phi s(\eta^{x})\\&=\eta^{x}\circ s(\phi)\\&=0\ .\end{aligned}$$

Therefore the function $\eta\nabla\phi s$ vanishes everywhere on $R^{\#}$ and consequently the same must be true for each of its derivatives. If we combine this fact with relation (2.7) we find that

$$D^{\alpha}\eta^{a}\circ j^{\alpha}s(\phi)=0\ . \tag{2.8}$$

Now, the function $\phi\in \boldsymbol{D}(R^{n})$ was arbitrarily chosen. Thus we infer from (2.8) that $D^{\alpha}\eta^{a}\circ j^{\alpha}s=0$. Since $a\in R^{\#}$ was arbitrarily chosen, we now conclude that (2.4) holds. This completes the proof of the validity of the inductive step and hence of the lemma.

COROLLARY 2.2. *Suppose that* $\eta\in \boldsymbol{S}$ *and* $s\in \boldsymbol{S}'$ *are such that*

$$\eta^{a}\circ s=0\qquad\textit{for all }a\in R^{\#}\ .$$

Then for each multi-index α

$$\operatorname{supp}(j^{\alpha}s)\subset\bigcap\{a\circ Z(D^{\beta}\eta)\colon\beta\leqq\alpha,\ a\in R^{\#}\}\ .$$

In the proof of the next lemma only, we shall use the following notation. If s is a temperate distribution such that $j^{\alpha}s=0$ for at least one multi-index α then we shall write $q(s)$ for the nonnegative integer which is defined by $q(s)=\min\{|\alpha|\colon j^{\alpha}s=0\}$. Otherwise, we shall write $q(s)=\infty$.

REMARK. We notice that $\operatorname{supp}s\cap R^{\#}\neq\varnothing$ whenever s is a temperate distribution such that $q(s)=\infty$. This is a consequence of the fact that every temperate distribution is of finite order.

LEMMA 2.3. *Suppose that* $\eta\in \boldsymbol{S}$ *and* $s\in \boldsymbol{S}'$ *are such that for each multi-index* α

$$\operatorname{supp}(j^{\alpha}s)\subset\bigcap\{a\circ Z(D^{\beta}\eta)\colon\beta\leqq\alpha,\ a\in R^{\#}\}\ . \tag{2.9}$$

Then it is true that

(2.10) $$\eta^a \circ s = 0 \qquad \textit{for all } a \in R^{\#} .$$

Proof. If $q(s) = \infty$ then (see the remark above) $\operatorname{supp} s \cap R^{\#} \neq \emptyset$. Relation (2.9) then is easily seen to entail that $\eta = 0$ on the whole of R^n; and the assertion of the lemma becomes a triviality.

It remains to show that Lemma 2.3 is true in the case when $q(s)$ is finite. We shall do this via induction on $q(s)$. More precisely, we shall use induction to demonstrate that the following statement is true for each nonnegative integer m.

P_m: Let $s \in S'$ be a temperate distribution with $q(s) = m$. Then (2.10) is true for every function $\eta \in S$ which satisfies (2.9).

If $m = 0$ there is nothing to prove. Thus, assume that $m > 0$; and that P_k is true for each nonnegative integer $k < m$. We have to show that these assumptions imply the truth of P_m.

Choose a multi-index α such that $|\alpha| = q(s) = m$. Since $m > 0$, it follows that $\alpha_k > 0$ for some positive integer $k \leqq n$. Without loss of generality, assume that $\alpha_1 > 0$.

Now let ϕ be an arbitrary function in $D(R^n)$. Reference to Lemma 2.5 (b) in Harasymiv [2] shows that

(2.11) $$j_1 D_1(\eta \bigtriangledown \phi s) = \eta \bigtriangledown \phi s + (j_1 D_1 \eta) \bigtriangledown \phi s \quad \text{on } R^{\#} .$$

Next notice that $q(j_1 s) \leqq |\alpha| - 1 = m - 1$. In view of this, it is easy to see that the distribution $j_1 s \in S'$ and the function $D_1 \eta \in S$ satisfy the hypotheses of P_k for some nonnegative integer $k < m$. We may therefore appeal to the inductive hypothesis and assert that $(D_1\eta)^x \circ j_1 s = 0$ for all $x \in R^{\#}$. This in turn entails that $(j_1 D_1 \eta)^x \circ s = 0$ for all $x \in R^{\#}$. Therefore for each $x \in R^{\#}$

$$\begin{aligned} (j_1 D_1 \eta) \bigtriangledown \phi s(x) &= \phi s((j_1 D_1 \eta)^x) \\ &= (j_1 D_1 \eta)^x \circ s(\phi) \\ &= 0 . \end{aligned}$$

By virtue of this last identity, relation (2.11) may be rewritten in the form

(2.12) $$j_1 D_1(\eta \bigtriangledown \phi s) = \eta \bigtriangledown \phi s \qquad \text{on } R^{\#} .$$

In a similar way (using in turn Lemma 2.5 (b) in Harasymiv [2] and the inductive hypothesis) it can be shown that

(2.13) $$j_1^2 D_1^2(\eta \bigtriangledown \phi s) = \eta \bigtriangledown \phi s \qquad \text{on } R^{\#} .$$

Relations (2.12) and (2.13) together entail that on $R^{\#}$ we have the identity

$$\begin{aligned}\eta \nabla \phi s &= j_1 D_1(\eta \nabla \phi s)\\ &= j_1 D_1(j_1 D_1(\eta \nabla \phi s))\\ &= j_1 D_1(\eta \nabla \phi s) + j_1^2 D_1^2(\eta \nabla \phi s)\\ &= \eta \nabla \phi s + \eta \nabla \phi s\ .\end{aligned}$$

It follows for this last identity that $\eta \nabla \phi s = 0$ on $R^{\#}$; whence we see that for each $a \in R^{\#}$

$$\begin{aligned}\eta^a \circ s(\phi) &= \phi s(\eta^a)\\ &= \eta \nabla \phi s(a)\\ &= 0\ .\end{aligned}$$

Since $\phi \in \boldsymbol{D}(R^n)$ was arbitrary, we infer that (2.10) holds. This establishes the truth of P_m; and the proof of Lemma 2.3 is complete.

3. **Some results about approximation.** Let $\phi \in \boldsymbol{S}$. We shall write $T[\phi]$ for the closed vector subspace of $\boldsymbol{S}$ generated by the set of all functions which have the form $x \to \phi(ax + b)(x \in R^n)$ where $a \in R^{\#}$ and $b \in R^n$. Similarly, if $u \in \boldsymbol{S}'$ then we shall write $T[u]$ for the closed vector subspace of $\boldsymbol{S}'$ generated by the set of distributions $\{(u_b)^a : a \in R^{\#}, b \in R^n\}$.

THEOREM 3.1. *Suppose that $\phi \in \boldsymbol{S}$ and that $\psi \in \boldsymbol{S}$. Then $\psi \in T[\phi]$ if and only if for each multi-index α*

$$Z(D^\alpha \hat{\psi}) \supset \cap \{a \circ Z(D^\beta \hat{\phi}) : \beta \leqq \alpha,\ a \in R^{\#}\}\ . \tag{3.1}$$

Proof. If we bear in mind the Hahn-Banach theorem then the necessity of (3.1) is easy to verify. To establish sufficiency, consider any distribution $u \in \boldsymbol{S}'$ such that

$$\phi^a * u = 0 \qquad \text{for all } a \in R^{\#}\ . \tag{3.2}$$

Relation (3.2) is seen (if we remember the comment about the Fourier transform of a dilation which was made in § 1) to entail that

$$\hat{\phi}^a \circ \hat{u} = 0 \qquad \text{for all } a \in R^{\#}\ . \tag{3.3}$$

Next notice that relation (3.1) implies that for each multi-index α

$$\cap \{a \circ Z(D^\beta \hat{\psi}) : \beta \leqq \alpha,\ a \in R^{\#}\} \supset \cap \{a \circ Z(D^\beta \hat{\phi}) : \beta \leqq \alpha,\ a \in R^{\#}\}\ . \tag{3.4}$$

In view of (3.3) and (3.4), we may apply successively Corollary 2.2 and Lemma 2.3, and deduce that $\hat{\psi} \circ \hat{u} = 0$. Hence also

$$\psi * u = 0\ .$$

An appeal to the Hahn-Banach theorem now yields the information that $\psi \in T[\phi]$.

THEOREM 3.2. *Suppose that $u \in S'$ and that $w \in S'$. Then $w \in T[u]$ if and only if for each multi-index α*

$$\operatorname{supp}(j^{\alpha}\hat{w}) \subset \overline{\cap \{a \circ \operatorname{supp}(j^{\alpha}\hat{u}) \colon a \in R^{\#}\}} \,. \tag{3.5}$$

Proof. Once again the "only if" part is obvious. It remains to establish the truth of the reversed implication; and, according to the Hahn-Banach theorem, we will have done this if we succeed in showing that

$$w * \phi = 0 \tag{3.6}$$

whenever $\phi \in S$ is such that

$$u^{a} * \phi = 0 \qquad \text{for all } a \in R^{\#} \,. \tag{3.7}$$

Thus, suppose that $u \in S'$ is such that (3.7) holds. Then it is clear that $\phi^{a} * u = 0$ for all $a \in R^{\#}$. Reference to Corollary 2.2 now tells us that for each $b \in R^{\#}$ we have

$$b^{-1} \circ \operatorname{supp}(j^{\alpha}\hat{u}) = \operatorname{supp}(j^{\alpha}\hat{u}^{b}) \subset \cap \{a \circ Z(D^{\beta}\hat{\phi}) \colon \beta \leqq \alpha,\, a \in R^{\#}\} \,.$$

It now follows that for each multi-index α

$$\overline{\cup \{a \circ \operatorname{supp}(j^{\alpha}\hat{u}) \colon a \in R^{\#}\}} \subset \cap \{a \circ Z(D^{\beta}\hat{\phi}) \colon \beta \leqq \alpha,\, a \in R^{\#}\} \,. \tag{3.8}$$

Relations (3.5) and (3.8) allow us to invoke Lemma 2.3 and assert that $\hat{\phi} \circ \hat{w} = 0$; whence we conclude that (3.6) holds.

4. An alternative form of the results. Theorems 3.1 and may be phrased in such a way that Fourier transform do not figure explicitly in their statement. We give below this alternative form of our previous results.

THEOREM 4.1. *Suppose that $\phi \in S$ and that $\psi \in S$. Then $\psi \in T[\phi]$ if and only if the following statement is true:*

If $\{n_1, \cdots, n_k\}$ is a subset of $\{1, 2, \cdots, n\}$ and $\{\alpha_1, \cdots, \alpha_k\}$ is a set nonnegative integers such that

$$\int_R \cdots \int_R x_{n_1}^{\beta_1} \cdots x_{n_k}^{\beta_k} \phi(x_1, \cdots, x_n) dx_{n_1} \cdots dx_{n_k} = 0 \tag{4.1}$$

everywhere for each set $\{\beta_1, \cdots, \beta_k\}$ of nonnegative integers such that $\beta_1 \leqq \alpha_1, \cdots, \beta_k \leqq \alpha_k$, then

$$\int_R \cdots \int_R x_{n_1}^{\alpha_1} \cdots x_{n_k}^{\alpha_k} \psi(x_1, \cdots, x_n) dx_{n_1} \cdots dx_{n_k} = 0 \tag{4.2}$$

everywhere.

Proof. We first prove the "if" part. To this end, we assume that (4.2) is always implied by (4.1). We have to show that this assumption entails that condition (3.1) in the statement of Theorem 3.1 is satisfied.

Thus, let α be a multi-index; and let $\chi \in \cap\{a \circ Z(D^{\beta}\hat{\phi}) \colon \beta \leqq \alpha,\ a \in R^{\#}\}$. Without any loss of generality, assume that $\chi_1 = \cdots = \chi_k = 0$ and $\chi_{k+1} \neq 0, \cdots, \chi_n \neq 0$. It is easy to verify that this entails that $\xi \in \cap\{Z(D^{\beta}\hat{\phi}) \colon \beta \leqq \alpha\}$ whenever $\xi \in R^n$ is such that $\xi_1 = \cdots = \xi_k = 0$. It follows that if $\{\beta_1, \cdots, \beta_k\}$ is a set of nonnegative integers such that $\beta_1 \leqq \alpha_1, \cdots, \beta_k \leqq \alpha_k$, and $\{\xi_{k+1}, \cdots, \xi_n\}$ is an arbitrary set of real numbers, then

$$\begin{aligned} &\int_R \cdots \int_R x_1^{\beta_1} \cdots x_k^{\beta_k} \phi(x_1, \cdots, x_n) \\ &\qquad \times \exp\left[-2\pi i(\xi_{k+1}x_{k+1} + \cdots)\right] dx_1 \cdots dx_n \\ &\quad = (-i)^{\beta_1 + \cdots + \beta_k} D_1^{\beta_1} \cdots D_k^{\beta_k} \hat{\phi}(0, \cdots, 0, \xi_{k+1}, \cdots, \xi_n) \\ &\quad = 0\,. \end{aligned} \tag{4.3}$$

In view of (4.3), we infer that (4.1) holds for each set $\{\beta_1, \cdots, \beta_k\}$ of nonnegative integers such that $\beta_1 \leqq \alpha_1, \cdots, \beta_k \leqq \alpha_k$. Our initial assumption now allows us to assert that (4.2) holds; whence it follows readily that $\chi \in Z(D^{\alpha}\hat{\psi})$. Since $\chi \in \cap\{a \circ Z(D^{\beta}\hat{\phi}) \colon \beta \leqq \alpha,\ a \in R^{\#}\}$ was arbitrary, we conclude that condition (3.1) in the statement of Theorem 3.1 is indeed satisfied.

The "only if" portion of Theorem 4.1 may be established by carrying out an analogous computational argument and then using Theorem 3.1; or, alternatively, we may employ the Hahn-Banach theorem directly and reach the same conclusion.

THEOREM 4.2. *Suppose that $u \in S'$ and that $w \in S'$. Then $w \in T[u]$ if and only if $D^{\alpha}w = 0$ whenever α is a multi-index such that $D^{\alpha}u = 0$.*

Proof. Suppose that $D^{\alpha}w = 0$ whenever α is a multi-index such that $D^{\alpha}u = 0$; but suppose that, contrary to Theorem 3.2, there exists a multiindex β such that $\operatorname{supp}(j^{\beta}\hat{w}) \not\subset \overline{\cup\{a \circ \operatorname{supp}(j^{\beta}\hat{u}) \colon a \in R^{\#}\}}$. Then we may choose $\chi \in \operatorname{supp}(j^{\beta}\hat{w})$ such that

$$\chi \notin \overline{\cup\{a \circ \operatorname{supp}(j^{\beta}\hat{u}) \colon a \in R^{\#}\}}\,. \tag{4.4}$$

Without loss of generality, assume that $\chi_1 \neq 0, \cdots, \chi_k \neq 0$ and $\chi_{k+1} = \cdots = \chi_n = 0$.

We assert that

$$\operatorname{supp}(j^\beta \hat{u}) \subset Z(j_1) \cup \cdots \cup Z(j_k) \ . \tag{4.5}$$

For if there were a $\xi \in \operatorname{supp}(j^\beta \hat{u})$ such that $\xi \in \bigcap_{r=1}^{k} [R^n \backslash Z(j_r)]$ then we would have $\xi_1 \neq 0, \cdots, \xi_k \neq 0$; and it would follow that

$$\chi \in \overline{\cup \{a\xi : a \in R^\sharp\}} \subset \overline{\cup \{a \circ \operatorname{supp}(j^\beta \hat{u}) : a \in R^\sharp\}}$$

which would contradict (4.4).

Now recall that every temperate distribution is of finite order. In view of this, relation (4.5) ensures the possibility of choosing a set $\{\alpha_1, \cdots, \alpha_k\}$ of nonnegative integers such that $j_1^{\alpha_1} \cdots j_k^{\alpha_k} \circ j^\beta \hat{u} = 0$. It then follows that

$$D_1^{\alpha_1} \cdots D_k^{\alpha_k} \circ D^\beta u = 0 \ . \tag{4.6}$$

On the other hand, $j_1^{\alpha_1} \cdots j_k^{\alpha_k}(\chi) = \chi_1^{\alpha_1} \cdots \chi_k^{\alpha_k} \neq 0$. Since $\chi \in \operatorname{supp}(j^\beta \hat{w})$, we conclude that $\chi \in \operatorname{supp}(j_1^{\alpha_1} \cdots j_k^{\alpha_k} \circ j^\beta \hat{w})$. Thus

$$D_1^{\alpha_1} \cdots D_k^{\alpha_k} \circ D^\beta w \neq 0 \ . \tag{4.7}$$

Relations (4.6) and (4.7) together contradict our initial assumption. Hence we infer that condition (3.5) in the statement of Theorem 3.2 is satisfied, and that $w \in T[u]$.

The "only if" part of Theorem 4.2 may be obtained as an immediate consequence of either Theorem 3.2 or the Hahn-Banach theorem.

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