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ON $(\mathfrak{m}-\mathfrak{n})$ PRODUCTS OF BOOLEAN ALGEBRAS

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This discussion begins with the problem of whether or not all $(\mathfrak{m}-\mathfrak{n})$ products of an indexed set $\{\mathfrak{A}_t\}_{t\in T}$ of Boolean algebras can be obtained as $\mathfrak{m}$-extensions of a particular algebra $\mathscr{F}_{\mathfrak{n}}^*$. The construction of $\mathscr{F}_{\mathfrak{n}}^*$ is similar to the construction of the Boolean product of $\{\mathfrak{A}_t\}_{t\in T}$; however the $\mathscr{A}_t$ are embedded in $\mathscr{F}_{\mathfrak{n}}^*$ in such a way that their images are $\mathfrak{n}$-independent. If there is a cardinal number $\mathfrak{n}'$, satisfying $\mathfrak{n}<\mathfrak{n}'\leqq\mathfrak{m}$, then $(\mathfrak{m}-\mathfrak{n}')$ products are not obtainable in this manner. For the case $\mathfrak{n}=\mathfrak{m}$ an example shows the answer to be negative. It is explained how the class of $\mathfrak{m}$-extensions of $\mathscr{F}_{\mathfrak{n}}^*$ is situated in the class of all $(\mathfrak{m}-\mathfrak{n})$ products of $\{\mathfrak{A}_t\}_{t\in T}$. A set of $\mathfrak{m}$-representable Boolean algebras is given for which the minimal $(\mathfrak{m}-\mathfrak{n})$ product is not $\mathfrak{m}$-representable and for which there is no smallest $(\mathfrak{m}-\mathfrak{n})$ product.

These problems have been proposed by R. Sikorski (see [2]). Concerning $\{\mathfrak{A}_t\}_{t\in T}$, it is assumed throughout that each of these algebras has at least four elements. $\mathfrak{m}$ and $\mathfrak{n}$ will always denote infinite cardinals with $\mathfrak{n}\leqq\mathfrak{m}$. All definitions are taken from [2]. An $\mathfrak{m}$-homomorphism is a homomorphism that is conditionally $\mathfrak{m}$-complete. We denote the class of $(\mathfrak{m}-\mathfrak{n})$ products of $\{\mathfrak{A}_t\}_{t\in T}$ by $\boldsymbol{P}_{\mathfrak{n}}$ and the class of $(\mathfrak{m}-0)$ products by $\boldsymbol{P}$. Let $\{\{i_t\}_{t\in T},\mathscr{B}\}$ and $\{\{j_t\}_{t\in T},\mathfrak{C}\}$ be elements of $\boldsymbol{P}$. We say that

$$\{\{i_t\}_{t\in T},\mathscr{B}\}\leqq\{\{j_t\}_{t\in T},\mathfrak{C}\}$$

provided there is an $\mathfrak{m}$-homomorphism h from $\mathfrak{C}$ onto $\mathscr{B}$ such that $h\circ j_t=i_t$ for $t\in T$. The relation "$\leqq$" is a quasi-ordering of $\boldsymbol{P}$. Two $(\mathfrak{m}-0)$ products are isomorphic if each is $\leqq$ to the other.

The particular product, $\{\{g_t^*\}_{t\in T},\mathscr{F}_{\mathfrak{n}}^*\}$ of $\{\mathfrak{A}_t\}_{t\in T}$ mentioned above is defined as follows. For each $t\in T$ let X_t be the Stone space of $\mathfrak{A}_t$ and let g_t be an isomorphism from $\mathfrak{A}_t$ onto the field $\mathscr{F}_t$ of all open and closed subsets of X_t. Let X be the Cartesian product of the sets X_t, and for each $t\in T$ and each $b\in\mathfrak{A}_t$, set

$$g_t^*(b)=[x\in X: x(t)\in g_t(b)\} . \tag{1}$$

Let $G_{\mathfrak{n}}$ be the set of all subsets a of X which satisfy the following condition:

$$a=\bigcap_{t\in S}g_t^*(b_t)\text{ where }b_t\in\mathfrak{A}_t,\ S\subseteqq T\text{ and }\bar{\bar{S}}\leqq\mathfrak{n} .$$

Finally, let $\mathscr{F}_{\mathfrak{n}}^*$ be the field of subsets of X which is generated by $G_{\mathfrak{n}}$.

$\mathscr{F}_{\mathfrak{n}}{}^*$ is a base for the $\mathfrak{n}$-topology on X. g_t^* is a complete isomorphism from $\mathfrak{A}_t$ into $\mathscr{F}_{\mathfrak{n}}{}^*$. The set $\{g_t^*(\mathfrak{A}_t)\}$, of subalgebras, is $\mathfrak{n}$-independent.

A Boolean $(\mathfrak{m} - \mathfrak{n})$ product $\{\{i_t\}_{t \in T}, \mathscr{B}\}$ is said to *belong to* $\boldsymbol{E}_{\mathfrak{n}}$ if and only if there is an $\mathfrak{m}$-isomorphism h (from $\mathscr{F}_{\mathfrak{n}}{}^*$ into $\mathscr{B}$) such that $\{h, \mathscr{B}\}$ is an $\mathfrak{m}$-extension of $\mathscr{F}_{\mathfrak{n}}{}^*$ and for each $t \in T$ $h \circ g_t^* = i_t$.

For every $\mathfrak{m}$-extension $\{h, \mathscr{B}\}$ of $\mathscr{F}_{\mathfrak{n}}{}^*$, $\{\{h \circ g_t^*\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$. Clearly $\boldsymbol{E}_{\mathfrak{n}} \subseteqq \boldsymbol{P}_{\mathfrak{n}}$ and $\boldsymbol{E}_{\mathfrak{n}}$ is not empty. $\mathfrak{m}$-extensions of $\mathscr{F}_{\mathfrak{n}}{}^*$ seem to provide the most natural examples of Boolean $(\mathfrak{m} - \mathfrak{n})$ products.

1. LEMMA 1.1. *Let* $\{\mathscr{B}_t\}_{t \in T}$ *be an* $\mathfrak{n}$*-independent set of subalgebras of a Boolean algebra* $\mathfrak{A}$ *and let* S *and* S' *be subsets of* T *with* $\bar{\bar{S}} \leqq \mathfrak{n}$ *and* $\bar{\bar{S}}' \leqq \mathfrak{n}$. *For each* t *let* a_t *and* b_t *be nonzero elements of* $\mathscr{B}_t$. *Then*

(i) $\prod_{t \in S}^{\mathfrak{A}} a_t \leqq \prod_{t \in S}^{\mathfrak{A}} b_t$ *if and only if* $a_t \leqq b_t$ *for each* $t \in S$;

(ii) $\prod_{t \in S}^{\mathfrak{A}} a_t = \prod_{t \in S'}^{\mathfrak{A}} b_t$ *implies that* $a_t = b_t$ *for* $t \in S \cap S'$, $a_t = 1$ *for* $t \in S - S'$, *and* $b_t = 1$ *for* $t \in S' - S$.

Proof. (i) Assume that for some $t_0 \in S$, $a_{t_0} \not\leqq b_{t_0}$. Define

$$C_t = \begin{cases} a_t \text{ if } t \in S \text{ and } t \neq t_0 , \\ a_{t_0} \cdot (-b_{t_0}) \text{ if } t = t_0 . \end{cases}$$

Set $c = \prod_{t \in S}^{\mathfrak{A}} c_t$, and note that $c \neq 0$, $c \leqq \prod_{t \in S}^{\mathfrak{A}} a_t$, and $c \cdot \prod_{t \in S}^{\mathfrak{A}} b_t = 0$. The converse is clear.

To prove (ii) we define

$$x_t = \begin{cases} a_t \text{ if } t \in S , \\ 1 \text{ if } t \in S' - S ; \end{cases} \quad \text{and} \quad y_t = \begin{cases} b_t \text{ if } t \in S' , \\ 1 \text{ if } t \in S - S' . \end{cases}$$

Now

$$\prod_{t \in S \cup S'}^{\mathfrak{A}} x_t = \prod_{t \in S}^{\mathfrak{A}} a_t = \prod_{t \in S'}^{\mathfrak{A}} b_t = \prod_{t \in S \cup S'}^{\mathfrak{A}} y_t$$

and (ii) follows from (i).

LEMMA 1.2. *Let* $\{\mathscr{B}_t\}_{t \in T}$ *be an* $\mathfrak{n}$*-independent set of subalgebras of a Boolean algebra* $\mathfrak{A}$. *Let* G *be the set of all meets* $\prod_{t \in S}^{\mathfrak{A}} a_t$ *such that* $S \subseteqq T$, $\bar{\bar{S}} \leqq \mathfrak{n}$, *and for each* $t \in S$ a_t *is a nonzero element of* $\mathscr{B}_t$. *Assume further that* G *generates* $\mathfrak{A}$. *Then* G *is dense in* $\mathfrak{A}$.

Proof. First note that for $g, g' \in G$ *either* $g \cdot g' = 0$ or else $g \cdot g' \in G$. Thus every nonzero element of $\mathfrak{A}$ is a finite join of elements of the form $g \cdot \prod_{i<k}^{\mathfrak{A}} (-g_i)$ with $g, g_i \in G$ and k finite. (This notation is intended

to include the special cases g and $-g$.) Now suppose $g \cdot \prod_{i<k}^{\mathfrak{A}} (-g_i) \neq 0$, so that $g \nleqq \sum_{i<k} g_i$. We write a common form $g = \prod_{t \in S}^{\mathfrak{A}} a_t$, and for each $i < k$ $g_i = \prod_{t \in S}^{\mathfrak{A}} a_{i,t}$ where $S \subseteq T$, $\bar{\bar{S}} \leqq \mathfrak{n}$, and for each $t \in S$ a_t and $a_{i,t}$ are nonzero elements of $\mathscr{B}_t$. Since k is finite every Boolean algebra is $(k - \mathfrak{n})$-distributive (see [2], p. 62). We have

$$\prod_{t \in S} a_t \nleqq \sum_{i<k} \prod_{t \in S} a_{i,t} = \prod_{\psi \in S^k} \sum_{i<k} a_{i,\psi(i)} .$$

(Here S^k denotes the set of all functions from $k = \{0, 1, \cdots, k-1\}$ into S.) Choose $\phi \in S^k$ such that $\prod_{t \in S} a_t \nleqq \sum_{i<k} a_{i,\phi(i)}$. We have, for each $s \in \{\phi(i) \colon i < k\}$, $a_s \nleqq \sum_{\phi(i)=s} a_{i,\phi(i)}$. Define

$$b_t = \begin{cases} a_t \text{ if } t \in S - \{\phi(i) \colon i < k\} \\ a_t \cdot - \sum_{\phi(i)=t} a_{i,\phi(i)} \text{ if } t \in \{\phi(i) \colon i < k\} . \end{cases}$$

Finally let $b = \prod_{t \in S}^{\mathfrak{A}} b_t$. Clearly $b \neq 0, b \in G$ and $b \leqq g$. For each $t \in \{\phi(i) \colon i < k\}$, $b_t \cdot \sum_{\phi(i)=t} a_{i,\phi(i)} = 0$, so that $b \cdot \sum_{i<k} a_{i,\phi(i)} = 0$. It follows that $b \cdot \sum_{i<k} g_i = 0$, hence $b \leqq g \cdot \prod_{i<k} (-g_i)$.

COROLLARY 1.3. *If* $\bar{\bar{S}} > \mathfrak{n}$, *and for each* $t \in S, a_t \neq 1$, *then* $\prod_{t \in S}^{\mathscr{A}} a_t = 0$.

THEOREM 1.4. *Let* $\{\{i_t\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{n}}$. *There is one and only one isomorphism* $h_{\mathfrak{n}}$ *from* $\mathscr{F}_{\mathfrak{n}}{}^*$ *into* $\mathscr{B}$ *which satisfies the following completeness condition*:

$$\text{(c)} \qquad h_{\mathfrak{n}}\Big(\prod_{t \in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t^*(a_t)\Big) = \prod_{t \in S}^{\mathscr{B}} i_t(a_t) \textit{ whenever } S \subseteq T, \bar{\bar{S}} \leqq \mathfrak{n} , \quad a_t \in \mathfrak{A}_t \textit{ and } a_t \neq 0 .$$

Proof. Let G be the set of all meets $\prod_{t \in S}^{\mathscr{B}} i_t(a_t)$ such that $S \subseteq T$, $\bar{\bar{S}} \leqq \mathfrak{n}$, each $a_t \in \mathfrak{A}_t$ and $a_t \neq 0$. Let $\mathfrak{A}$ be the subalgebra of $\mathscr{B}$ which is generated by G. For $\prod_{t \in S}^{\mathscr{B}} i_t(a_t) \in G$ it is clear that $\prod_{t \in S}^{\mathscr{B}} i_t(a_t) = \prod_{t \in S}^{\mathfrak{A}} i_t(a_t)$. By Lemma 1.2 G is dense in $\mathfrak{A}$. Also $G_{\mathfrak{n}}$ is dense in $\mathscr{F}_{\mathfrak{n}}{}^*$. For $a \in G_{\mathfrak{n}}$ write $a = \bigcap_{t \in S} g_t^*(a_t) = \prod_{t \in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t^*(a_t)$. Define $h(a) = \prod_{t \in S}^{\mathfrak{A}} i_t(a_t)$. It is easily seen, using Lemma 1.1, that

(i) h is a one to one function from $G_{\mathfrak{n}}$ onto G;

(ii) for $a, b \in G_{\mathfrak{n}}$, $a \leqq b$ if and only if $h(a) \leqq h(b)$.

It follows (see [2], p. 37) that h can be extended to an isomorphism $h_{\mathfrak{n}}$ from $\mathscr{F}_{\mathfrak{n}}{}^*$ onto $\mathfrak{A}$. $h_{\mathfrak{n}}$ is uniquely determined by condition (c) because $G_{\mathfrak{n}}$ generates $\mathscr{F}_{\mathfrak{n}}{}^*$.

COROLLARY 1.5. *The product* $\{\{i_t\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$ *if and only if* $h_{\mathfrak{n}}$ *is* $\mathfrak{m}$-*complete.*

Proof. Let $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$. There is an $\mathfrak{m}$-isomorphism f from $\mathscr{F}_{\mathfrak{n}}{}^*$ into $\mathscr{B}$ such that for each $t \in T, f\circ g_t^* = i_t$. f satisfies condition (c) so $f = h_{\mathfrak{n}}$.

COROLLARY 1.6. *Assume* $\bar{\bar{T}} > \mathfrak{n}$ *and that* $\mathfrak{m} \geqq \mathfrak{n}' > \mathfrak{n}$. *Then* $\boldsymbol{P}_{\mathfrak{n}'} \cap \boldsymbol{E}_{\mathfrak{n}}$ *is empty.*

Proof. Let $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{n}'}$. Consider the isomorphism $h_{\mathfrak{n}}$ from $\mathscr{F}_{\mathfrak{n}}{}^*$ *into* $\mathscr{B}$. Choose $S \subseteqq T$, $\bar{\bar{S}} = \mathfrak{n}^+$, and for each $t \in S$ choose $a_t \in \mathfrak{A}_t$ with $a_t \neq 0$, $a_t \neq 1$. By Corollary 1.3

$$\prod_{t\in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t^*(a_t) = 0\ .$$

However $0 \neq \prod_{t\in S}^{\mathscr{B}} i_t(a_t) = \prod^{\mathscr{B}} h_{\mathfrak{n}}\circ g_t^*(a_t)$ so that $h_{\mathfrak{n}}$ is not $\mathfrak{m}$-complete.

There is an interesting contrast between $\boldsymbol{E}_{\mathfrak{n}}$ and $\boldsymbol{P}_{\mathfrak{n}'}$, (under the hypotheses of Corollary 1.6). Let $\{\{i_t\}_{t\in T}, \mathscr{B}\}$ and $\{\{j_t\}_{t\in T}, \mathfrak{C}\}$ be elements of $\boldsymbol{P}_{\mathfrak{n}}$ with $\{\{i_t\}_{t\in T}, \mathscr{B}\} \leqq \{\{j_t\}_{t\in T}, \mathfrak{C}\}$. It is known (see [2], p. 179) that if $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{n}'}$, then $\{\{j_t\}_{t\in T}, \mathfrak{C}\} \in \boldsymbol{P}_{\mathfrak{n}'}$. On the other hand if $\{\{j_t\}_{t\in T}, \mathfrak{C}\} \in \boldsymbol{E}_{\mathfrak{n}}$ then we have $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$.

COROLLARY 1.7. *Assume* $\bar{\bar{T}} > \mathfrak{n}$ *and* $\mathfrak{m} > \mathfrak{n}$. *Then* $\boldsymbol{E}_{\mathfrak{n}} \cup \boldsymbol{P}_{\mathfrak{n}+} \neq \boldsymbol{P}_{\mathfrak{n}}$.

Proof. Let $S \subseteqq T$ with $\bar{\bar{S}} = \mathfrak{n}^+$. Choose, for each $t \in S$, $d_t \in \mathfrak{A}_t$ with $d_t \neq 0$, $d_t \neq 1$. Let $d = \bigcap_{t\in S} g_t^*(d_t)$. Let $\mathscr{F}$ be the field of subsets of X which is generated by $\mathscr{F}_{\mathfrak{n}}{}^* \cup \{d\}$. Note that g_t^* is a complete isomorphism from $\mathfrak{A}_t$ into $\mathscr{F}$. Let $\{f, \mathfrak{C}\}$ be any $\mathfrak{m}$-extension of $\mathscr{F}$. It is easily seen that $\{\{f\circ g_t^*\}_{t\in T}, \mathfrak{C}\} \in \boldsymbol{P}_{\mathfrak{n}}$.

Consider the isomorphism $h_{\mathfrak{n}}$ from $\mathscr{F}_{\mathfrak{n}}{}^*$ into $\mathfrak{C}$. $h_{\mathfrak{n}}\circ g_t^* = f\circ g_t^*$ for every $t \in T$. By Corollary 1.3 $\prod_{t\in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t(d_t) = 0$. However $\prod_{t\in S}^{\mathfrak{C}} h_{\mathfrak{n}}\circ g_t^*(d_t) = f(d) \neq 0$. Thus $h_{\mathfrak{n}}$ is not $\mathfrak{m}$-complete and $\{\{f\circ g_t^*\}_{t\in T}, \mathfrak{C}\} \notin \boldsymbol{E}_{\mathfrak{n}}$.

In order to show that $\{\{f\circ g_t^*\}_{t\in T}, \mathfrak{C}\} \notin \boldsymbol{P}_{+\mathfrak{n}}$ it suffices to show that $\prod_{t\in S}^{\mathfrak{C}} f\circ g_t^*(-d_t) = 0$. In particular suppose $b = \prod_{t\in S}^{\mathscr{F}} g_t^*(-d_t) \neq 0$. Since $b\cdot d = 0$ the definition of $\mathscr{F}$ enables us to write $b = \bigcup_{t\in S} b_1\cdot g_t^*(-d_t)$ with $b_1 \in \mathscr{F}_{\mathfrak{n}}{}^*$. Choose $t_0 \in S$ such that $0 \neq b_1\cdot g_{t_0}^*(-d_{t_0}) \leqq b$. By Lemma 1.2 there is a nonzero element $a = \bigcap_{t\in S'} g_t^*(a_t)$ of $G_{\mathfrak{n}}$ such that $a \subseteqq b_1\cdot g_{t_0}^*(-d_{t_0})$. Now $\bar{\bar{S'}} \leqq \mathfrak{n}$ and $\bar{\bar{S}} = \mathfrak{n}^+$ and it follows that $a \nleqq b$. Thus $\prod_{t\in S}^{\mathscr{F}} g_t^*(-d_t) = 0$ and since f is $\mathfrak{m}$-complete, $\prod_{t\in S}^{\mathfrak{C}} f\circ g_t^*(-d_t) = 0$.

We now consider the case $\mathfrak{n} = \mathfrak{m}$. It is known that $\boldsymbol{E}_{\mathfrak{m}} \neq \boldsymbol{P}_{\mathfrak{m}}$ if $\mathfrak{m} = \aleph_0$ (see [2], p. 190, Example D). In this example T is the two element set $\{1, 2\}$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are σ-complete Boolean algebras which satisfy the σ-chain condition. The Boolean σ-product $\{\{i_1, i_2\}, \mathscr{B}\}$ is such that the subalgebra $\mathscr{B}_0$ of $\mathscr{B}$ which is generated by $i_1(\mathfrak{A}_1) \cup i_2(\mathfrak{A}_2)$

is not a σ-regular subalgebra of $\mathscr{B}$. Let $\{f, \mathfrak{C}\}$ be any $\mathfrak{m}$-extension of $\mathscr{B}$. It follows, using the σ-chain condition on $\mathfrak{A}_1$ and $\mathfrak{A}_2$, that $\{\{f \circ i_1, f \circ i_2\}, \mathfrak{C}\} \in \boldsymbol{P}_{\mathfrak{m}}$. Since T is finite $\{\{g_1^*, g_2^*\}, \mathscr{F}_m{}^*\}$ is the Boolean product of $\{\mathfrak{A}_1, \mathfrak{A}_2\}$. Let h be the homomorphism from $\mathscr{F}_{\mathfrak{m}}{}^*$ into $\mathscr{B}$ such that $h \circ g_1^* = i_1$ and $h \circ g_2^* = i_2$. Then h is an isomorphism from $\mathscr{F}_{\mathfrak{m}}{}^*$ onto $\mathscr{B}_0$. Consider the isomorphism $h_{\mathfrak{m}}$, from $\mathscr{F}_{\mathfrak{m}}{}^*$ into $\mathfrak{C}$, given by Theorem 1.4. $h_{\mathfrak{m}} = f \circ h$ since they agree on $g_1^*(\mathfrak{A}_1) \cup g_2^*(\mathfrak{A}_2)$. $h_{\mathfrak{m}}$ is not $\mathfrak{m}$-complete because $f(\mathscr{B}_0)$ is not $\mathfrak{m}$-regular in $\mathfrak{C}$. Thus $\{\{f \circ i_1, f \circ i_2\}, \mathfrak{C}\} \notin \boldsymbol{E}_{\mathfrak{m}}$. We give a simple for the case $\mathfrak{m} \geqq 2^{\aleph_0}$.

EXAMPLE 1.8. Assume $\mathfrak{m} \geqq 2^{\aleph}0$ and let T be a set of power $\aleph_0$. For each $t \in T$ let $\mathfrak{A}_t$ be a Boolean algebra having exactly four elements. Let $\mathscr{B}$ be the free Boolean $\mathfrak{m}$-algebra on $\aleph_0$ $\mathfrak{m}$-generators, $(D_t : t \in T\}$. $\mathscr{B}$ is not $\mathfrak{m}$-representable (see [2], p. 134). For each $t \in T$ choose d_t to be one of the atoms of $\mathfrak{A}_t$. Let i_t be the isomorphism from $\mathfrak{A}_t$ into $\mathscr{B}$ such that $i_t(d_t) = D_t$. Then $\{\{i_t\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{m}}$. By Lemma 1.2 $\mathscr{F}_{\mathfrak{m}}{}^*$ is atomic, the atoms being all sets of the form $\bigcap_{t \in T} g_t^*(a_t)$, where for each $t \in T$ a_t is an atom of $\mathfrak{A}_t$. Denote the set of atoms of $\mathscr{F}_{\mathfrak{m}}{}^*$ by $\{C_r : r \in R\}$, then $\overline{\overline{R}} = 2^{\aleph_0}$. We consider the isomorphism $h_{\mathfrak{m}}$ from $\mathscr{F}_{\mathfrak{m}}{}^*$ into $\mathscr{B}$. For each $r \in R$, $h_{\mathfrak{m}}(c_r)$ is an atom of $\mathscr{B}$. To show this we define

$$\mathfrak{A} = \{b \in \mathscr{B} : \text{ for each } r \in R \text{ either } b \cdot h_{\mathfrak{m}}(c_r) = 0 \text{ or } h_{\mathfrak{m}}(c_r) \leqq b\} .$$

It is easily seen that $\mathfrak{A}$ is an $\mathfrak{m}$-subalgebra of $\mathscr{B}$ which includes $\{D_t : t \in T\}$. Hence $\mathfrak{A} = \mathscr{B}$. Finally, $h_{\mathfrak{m}}$ is not $\mathfrak{m}$-complete. For otherwise $\sum_{r \in R}^{\mathscr{B}} h_{\mathfrak{m}}(c_r) = 1$, and $\mathscr{B}$ would be atomic and hence isomorphic to an $\mathfrak{m}$-field of sets.

2. We now consider the problem of the existence of a smallest element of $\boldsymbol{P}$, relative to the quasi-ordering "$\leqq$". A minimal element of $\boldsymbol{P}$ always exists and can be constructed as follows. Let $\{\{f_t\}_{t \in T}, \mathfrak{C}\}$ be a Boolean product of $\{\mathfrak{A}_t\}_{t \in T}$ and let $\{h, \mathscr{B}\}$ be an $\mathfrak{m}$-completion of $\mathfrak{C}$. Then $\{\{h \circ f_t\}_{t \in T}, \mathscr{B}\}$ is a minimal element of $\boldsymbol{P}$. We shall show that this product need not be a smallest element of $\boldsymbol{P}$. Hence $\boldsymbol{P}$ need not have a smallest element.

EXAMPLE 2.1. Let $\mathfrak{m}$ be any infinite cardinal. Let $\overline{\overline{T}} = \aleph_0$ and suppose that for each $t \in T$ $\mathfrak{A}_t$ is a four element Boolean algebra. For each $t \in T$ choose a_t to be one of the atoms of $\mathfrak{A}_t$. $\mathfrak{C}$ is a free Boolean algebra of power $\aleph_0$, one set of free generators being $\{f_t(a_t) : t \in T\}$. $\mathscr{B}$ has a countable dense subset, in particular $\mathscr{B}$ satisfies the countable chain condition. Thus $\mathscr{B}$ is complete. It follows that $\mathscr{B}$ is isomorphic to the quotient algebra $\mathscr{F}/\Delta_0$ where $\mathscr{F}$ is the σ-field of Borel subsets of the unit interval $I = \{x: 0 < x \leqq 1\}$ of real numbers and Δ_0 is the ideal consisting of those Borel sets which are of the first category.

To show that $\{\{h \circ f_t\}_{t \in T}, \mathscr{B}\}$ is not a smallest element of $\boldsymbol{P}$ we construct another ($\mathfrak{m}$-0) product as follows. Let G be the set of all halfopen intervals of the form $\{x: 0 < x \leqq r\}$ such that r is rational and $0 < r \leqq 1$. $\mathscr{F}$ is σ-generated by G. The subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ which is generated by G is denumerable and atomless. Hence $\mathscr{F}_0$ is isomorphic to $\mathfrak{C}$ (see [1], p. 54). Let g be an isomorphism from $\mathfrak{C}$ onto $\mathscr{F}_0$. Let Δ_1 be the ideal of $\mathscr{F}$ consisting of those Borel sets having Lebesgue measure 0. We note that $\mathscr{F}_0 \cap \Delta_1 = \{0\}$. Finally for each $t \in T$ let h_t be the isomorphism from $\mathfrak{A}_t$ into $\mathscr{F}/\Delta_1$ defined by $h_t(a_t) = [g \circ f_t(a_t)]\Delta_1$. It is easily seen that $\{\{h_t\}_{t \in T}, \mathscr{F}/\Delta_1\} \in \boldsymbol{P}$.

Now assume $\{\{h \circ f_t\}_{t \in T}, \mathscr{B}\} \leqq \{\{h_t\}_{t \in T}, \mathscr{F}/\Delta_1$. Then there is an $\mathfrak{m}$-homomorphism p from $\mathscr{F}/\Delta_1$ onto $\mathscr{F}/\Delta_0$. Since $\mathscr{F}/\Delta_1$ satisfies the countable chain condition the kernel of p is a principal ideal. $\mathscr{F}/\Delta_0$ is isomorphic to a principal ideal of $\mathscr{F}/\Delta_1$. However $\mathscr{F}/\Delta_1$ is homogeneous (see [2], p. 105). Thus $\mathscr{F}/\Delta_0$ is isomorphic to $\mathscr{F}/\Delta_1$, which is a contradiction.

Next we consider the problem of the existence of a smallest element of $\boldsymbol{P}_{\mathfrak{n}}$. Let $\{g, \mathscr{B}\}$ be an $\mathfrak{m}$-completion of $\mathscr{F}_{\mathfrak{n}}{}^*$. Then $\{\{g \circ g_t^*\}_{t \in T}, \mathscr{B}\}$ is a minimal element of $\boldsymbol{P}_{\mathfrak{n}}$. Also it is known (see [2], p. 183) that if all the $\mathfrak{A}_t$ are $\mathfrak{m}$-representable then there is an ($\mathfrak{m}$-$\mathfrak{n}$) product $\{\{i_t\}_{t \in T}, \mathfrak{C}\}$ for which $\mathfrak{C}$ is $\mathfrak{m}$-representable. We give an example of $\{\mathfrak{A}_t\}_{t \in T}$ for which $\mathscr{B}$ is not $\mathfrak{m}$-representable and $\{\{g \circ g_t^*\}_{t \in T}, \mathscr{B}\}$ is not a smallest element of $\boldsymbol{P}_{\mathfrak{n}}$.

EXAMPLE 2.2. Assume that $\mathfrak{m} \geqq 2^{(\mathfrak{n}^+)}$. Let $\overline{\overline{T}} = \mathfrak{n}^+$ and for each $t \in T$ let $\mathfrak{A}_t$ be a four element Boolean algebra. We show that $\mathscr{B}$ is not $\mathfrak{n}^+$-distributive. Choose, for each $t \in T$, a_t to be one of the atoms of $\mathfrak{A}_t$. Then

$$\prod\nolimits_{t \in T}^{\mathscr{B}} (g \circ g_t^*(a_t) + - g \circ g_t^*(a_t)) = 1 \; .$$

However for each function $\eta \in H^T$ (here $H = \{+1, -1\}$) we have

$$\prod_{t \in T}^{\mathscr{F}_{\mathfrak{n}}^*} \eta(t) \cdot g_t^*(a_t) = 0 \; .$$

This follows from Corollary 1.3. Thus $\prod\nolimits_{t \in T}^{\mathscr{B}} \eta(t) \cdot g \circ g_t^*(a_t) = 0$. This proves $\mathscr{B}$ is not $\mathfrak{n}^+$-distributive and hence not $\mathfrak{m}$-representable.

To show that $\{\{g \circ g_t^*\}_{t \in T}, \mathscr{B}\}$ is not a smallest element of $\boldsymbol{P}_{\mathfrak{n}}$, let $\{\{i_t\}_{t \in T}, \mathfrak{C}\}$ be any ($\mathfrak{m}$-$\mathfrak{n}$) product of $\{\mathfrak{A}_t\}_{t \in T}$ such that $\mathfrak{C}$ is $\mathfrak{m}$-representable. $\mathscr{B}$ is not an $\mathfrak{m}$-homomorphic image of $\mathfrak{C}$. Thus the inequality

$$\{\{g \circ g_t^*\}_{t \in T}, \mathscr{B}\} \leqq \{\{i_t\}_{t \in T}, \mathfrak{C}\}$$

does not hold.

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