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A REPRESENTATION THEOREM FOR CERTAIN CONNECTED RINGS

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It is shown that if A is a semisimple, connected, locally connected Q-ring with unit element such that every maximal ideal disconnects A, then A is continuously isomorphic to a dense subring of the ring of all continuous real-valued functions on a suitable compact Hausdorff space.

Many authors have obtained representations for semisimple Banach algebras as algebras of continuous functions. The object of this note is to present a somewhat similar result which, however, does not assume the presence of real or other kinds of scalars.

Specifically, it is established in Theorem 2 that if A is a semisimple, connected, locally connected Q-ring with unit element such that every maximal ideal has a disconnected complement in A, then A is continuously isomorphic to a dense subring of the ring $\mathscr{C}(\Phi; \mathfrak{R})$ of all continuous real-valued functions on a suitable compact Hausdorff space Φ.

The basic tool employed is Theorem 1, which asserts that if A is a connected, locally connected ring with unit element such that the removal of the zero element disconnects A, then A is algebraically and topologically isomorphic to the field R of real numbers.

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2. Topological rings which are disconnected by the removal of a point.

An important step in proving the representation theorem is the characterization of those locally connected rings which are disconnected by the removal of a point.

THEOREM 1. *Let A be a topological ring with unit element. In order for A to be algebraically and topologically isomorphic to the field $\mathfrak{R}$ of real numbers it is necessary and sufficient that A be connected and locally connected, but that the set $A^{\#}$ of nonzero elements of A be disconnected.*

Proof. The necessity is obvious.

For the sufficiency, we first note that the additive group of A is algebraically and topologically isomorphic to the additive group of real numbers. (See for instance [1; Chap. 5, p. 28, Exercise 4], where a proof of the fact is outlined.) In particular, A is locally compact.

If c is a nonzero element of A then the mapping $x \to cx$ is a continuous endomorphism of the additive group of A; thus, the image of A under this mapping is a connected subgroup of that group and therefore coincides with A since the image contains the nonzero element $c1 = c$. Then $1 = cd$ for some d in A, and c is right invertible. It follows that A is a division ring.

Pontrjagin's characterization of connected, locally compact division rings (see for instance [3; Chap. 6, p. 160, Corollary 2 of Theorem 1]) implies that A is algebraically and topologically isomorphic to the field $\mathfrak{R}$ of real numbers, the field of complex numbers, or the division ring of real quaternions. The fact that $A^{\#}$ is disconnected eliminates the last two alternatives, and the theorem follows.

In order to obtain the representation theorem we shall employ a succession of simple lemmas. The first two of these lemmas follow. The proofs are routine.

LEMMA 1. *If A is a connected ring with unit element then every left ideal and every right ideal of A is connected.*

LEMMA 2. *Let A be a connected, locally connected ring with unit element, and let I be a closed ideal which disconnects A. Then A/I is algebraically and topologically isomorphic to $\mathfrak{R}$.*

3. The representation theorem. If Φ is a compact Hausdorff space then the symbol $\mathscr{C}(\Phi; \mathfrak{R})$ will denote the ring of all continuous real-valued functions on Φ, with the topology of uniform convergence on Φ as the topology of the ring. It is recalled that a topological ring with unit element is a *Q-ring* provided that the set of invertible elements is open; in a Q-ring with unit element, maximal ideals exist and are closed sets.

THEOREM 2. *Let A be a semisimple, connected, locally connected Q-ring with unit element such that every maximal ideal disconnects A. Then there exists a compact Hausdorff space Φ such that there is a continous isomorphism σ of A onto a dense subring of $\mathscr{C}(\Phi; \mathfrak{R})$.*

The proof is outlined by listing the lemmas which are employed to construct it.

LEMMA 3. *There is a subfield P of A such that P contains* 1 *and P is algebraically isomorphic to the field of rational numbers.*

Proof. If n is a natural number then no maximal ideal M can contain n since otherwise A/M, which is isomorphic to $\mathfrak{R}$ by Lemma 2, would have finite characteristic. Thus, every natural number n is an invertible element of A. If P is the set of all elements of A of the form mn^{-1}, with m an integer and n a natural number, then P clearly is the required field.

DEFINITION. A subset C of a ring A is said to be *symmetric* provided that whenever x is in C then $-x$ is in C.

LEMMA 4. *If r is a positive rational number then there is a connected symmetric neighborhood W of zero in A such that $\varphi(W) \subset]-r, r[$ for every continuous nonzero homomorphism φ of A into $\mathfrak{R}$.*

Proof. Since $-r$ is invertible there is a neighborhood U of $-r$ which contains only invertible elements. Thus, U is disjoint from every maximal ideal M, and $r + U$ is therefore disjoint from $r + M$ for every maximal ideal M.

There is a connected neighborhood V of zero contained in the symmetric neighborhood $(r + U) \cap (-(r + U))$ of zero, so that $W = V \cup (-V)$ is a connected symmetric neighborhood of zero which is contained in $(r + U) \cap (-(r + U))$ and therefore in $r + U$. It follows that W is disjoint from $r + M$ for every maximal ideal M.

Let φ be a continuous nonzero homomorphism of A into $\mathfrak{R}$. Then the kernel of φ must be a maximal ideal M because the image of φ is necessarily the entire field $\mathfrak{R}$. Now $r + M$ is disjoint from W, so that r does not belong to $\varphi(W)$. We conclude that $\varphi(W) \subset]-r, r[$ since $\varphi(W)$ is a connected symmetric set of real numbers and does not contain r.

This proves the lemma.

LEMMA 5. *The relative topology of P in A is the ordinary topology of the field of rational numbers.*

The proof involves a routine application of Lemma 4.

LEMMA 6. *Let U be a neighborhood of zero in A, and let f be a continuous nonconstant mapping of U into $\mathfrak{R}$ such that*

$$f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$$

whenever $x_1, \cdots, x_n, x_1 + \cdots + x_n$ *belong to* U, *and* $f(xy) = f(x)f(y)$ *whenever* x, y, xy *belong to* U. *Then there exists exactly one continuous nonzero homomorphism* φ *of* A *into* $\mathfrak{R}$ *such that the restriction of* φ *to* U *is precisely* f.

Proof. If x is in A then there is a natural number m such that x/r is in U whenever r is a natural number with $r \geqq m$. We define $\varphi(x) = mf(x/m)$. Then φ is well-defined, and the remaining details of the proof are routine.

LEMMA 7. *Let* W *be a connected symmetric neighborhood of zero in* A *such that* $\varphi(W) \subset]-1, 1[$ *for every continuous nonzero homomorphism* φ *of* A *into* $\mathfrak{R}$. *Let* Φ *be the space of all continuous nonzero homomorphisms of* A *into* $\mathfrak{R}$, *with the topology for* Φ *obtained by identifying* Φ *(in the obvious way) with a subset of the topological product of the family* $\{I_x \mid x \in W\}$, *where each space* I_x *is the closed interval* $[-1, 1]$. *Then* Φ *is a compact Hausdorff space.*

We note that Lemma 6 implies that Φ can also be identified with the set of all continuous nonconstant mappings f of W into $\mathfrak{R}$ which have the properties that $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$ whenever $x_1, \cdots, x_n, x_1 + \cdots + x_n$ belong to W, and $f(xy) = f(x)f(y)$ whenever x, y, xy belong to W. It may be noted that the topology for Φ has as a subbase the family of all sets

$$\{\varphi_0; x; \varepsilon\} = \{\varphi \mid \varphi \in \Phi, |\varphi(x) - \varphi_0(x)| < \varepsilon\} ,$$

where $\varphi_0 \in \Phi$, $x \in A$, and ε is a positive real number. Furthermore, if x is an arbitrary element of A then there is a natural number m such that $x/m \in W$; thus, every set $\{\varphi_0; x: \varepsilon\}$ can also be written in the form $\{\varphi_0; x/m; \varepsilon/m\}$, so that there is a subbase for the topology of Φ which consists of all sets of the form $\{\varphi_0; y; \delta\}$, with $\varphi_0 \in \Phi$, $y \in W$, and δ a positive real number. The proof of Lemma 7 then becomes routine.

LEMMA 8. *If* x *is an element of* A *then the function* $\hat{x}$ *defined on* Φ *by the rule* $\hat{x}(\varphi) = \varphi(x)$, *for all* φ *in* Φ, *is a continuous real-valued function on* Φ.

The proof is routine.

LEMMA 9. *Let* σ *be the mapping of* A *into* $\mathscr{C}(\Phi; \mathfrak{R})$ *defined by the rule* $\sigma(x) = \hat{x}$ *for all* x *in* A. *Then* σ *is a continuous isomorphism of* A *into* $\mathscr{C}(\Phi; \mathfrak{R})$.

An application of Lemma 4 establishes the continuity of σ, while the fact that σ is an isomorphism is proved in a routine manner.

LEMMA 10. *$\sigma(A)$ is dense in $\mathscr{C}(\Phi; \mathfrak{R})$.*

Proof. The closure of $\sigma(A)$ is a uniformly closed subring of $\mathscr{C}(\Phi; \mathfrak{R})$ which contains all constant real-valued functions on Φ since it contains all constant rational-valued functions on Φ. It is also clear that the closure of $\sigma(A)$ separates points of Φ, and the Stone-Weierstrass Approximation Theorem (see for instance [2; p. 56, Th. 3]) implies that the closure of $\sigma(A)$ coincides with $\mathscr{C}(\Phi; \mathfrak{R})$.

This sequence of lemmas establishes the theorem.

An example demonstrates that the conclusion of Theorem 2 can not be sharpened. If A is the set of all real-valued functions which are defined and have a continuous derivative on $[0, 1]$, with the obvious operations in A, and with the norm for A defined by

$$N(x) = \sup\{|x(t)| \mid 0 \leqq t \leqq 1\} + \sup\{|x'(t)| \mid 0 \leqq t \leqq 1\}$$

for each x in A, then A is a commutative Banach algebra which clearly satisfies the hypothesis of Theorem 2. However, the topology for A is strictly finer than the topology for $\mathscr{C}(\Phi; \mathfrak{R})$ in this example. For instance, if $x_n(t) = (2\pi n)^{-1} \sin 2\pi nt$ for $0 \leqq t \leqq 1$ whenever n is a natural number, then the sequence $\{x_n\}$ converges uniformly to zero (that is, $\{x_n\}$ converges to 0 in $\mathscr{C}(\Phi; \mathfrak{R})$), but $\{x_n\}$ does not converge to zero in A since $N(x_n) = (2\pi n)^{-1} + 1$ for every natural number n. Thus, σ is not a homeomorphism of A with $\sigma(A)$.

The same example also shows that $\sigma(A)$ need not coincide with $\mathscr{C}(\Phi; \mathfrak{R})$ even though $\sigma(A)$ is a dense connected subring of the latter. For instance, the element z of $\mathscr{C}(\Phi; \mathfrak{R})$, where $z(t) = |t - (1/2)|$ whenever $0 \leqq t \leqq 1$, is obviously not the image of an element of A.

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