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ON THE HIGMAN-SIMS SIMPLE GROUP OF ORDER 44,352,000

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In a recent paper D. G. Higman and C. C. Sims announced their construction of a new simple group H_{100} of order 44,352,000. The group H_{100} is obtained as a rank 3 permutation group of degree 100 with subdegrees 1,22 and 77; and the stabilizer of a point is isomorphic to the Mathieu simple group M_{22}. Shortly after their announcement of the new simple group, Graham Higman constructed a simple group of the same order as a doubly transitive group of degree 176 and with stabilizer of a point isomorphic to $PSU(3,5^2)$.

The purpose of this paper is to show that the two groups mentioned above are isomorphic, and in fact, that there is exactly one (up to isomorphism) simple group of order 44,352,000.

THEOREM. *Let G be a nonabelian simple group of order* 44,352,000. *Then G is isomorphic to the Higman-Sims group H_{100}.*

Throughout this paper, G will denote a nonabelian simple group of order $44,352,000 = 2^9.3^2.5^3.7.11$. The notation will be standard; see for instance [13]. Further, a Sylow p-subgroup of G will be denoted by G_p, A_n and S_n will denote the alternating group and symmetric group on n letters respectively; and F_{20} will denote a Frobenius group of order 20. The word "character" always refers to an irreducible character of G afforded by an irreducible representation of G in the complex number field. If the integer n divides the $|G|$ of G, we will denote this by $n \mid |G|$.

In the proof of the theorem, the following results are of fundamental importance.

RESULT 1 (R. Brauer [2], Theorem 11). Let G be a group such that $p \mid |G|$, but $p^2 \nmid |G|$, p a prime. If the p-block $B_1(p)$ contains the principal character 1_G of G, then $B_1(p)$ has $(p-1)/t$ (irreducible complex) characters which are p-conjugate only to themselves and one family of t p-conjugate characters, where t denotes the number of conjugate classes of elements of order p. Further, the degrees $\chi_i(1)$ of the irreducible characters χ_i of $B_1(p)$, satisfy the following congruences:

$$\chi_i(1) = z_i \equiv \delta_i = \pm 1 \pmod p ,$$

if χ_i is p-conjugate only to itself and

$$\chi_i(1) = z_i \equiv \frac{\delta_i}{t} = \pm\left(\frac{1}{t}\right) \pmod p ,$$

if χ_i belongs to the family of p-conjugate characters If $1_G = \chi_1, \chi_2, \cdots, \chi_{q+1}$ represent the different families of $B_1(p)$, ($q = (p-1)/t$), then

$$1 + \delta_2 z_2 + \cdots + \delta_{q+1} z_{q+1} = 0 \ . \tag{1}$$

The next result is very well-known. (For a proof, see [8] (2.15), or any book on group theory).

RESULT 2. Let K_1, K_2, K_3 be any three conjugate classes of elements of a finite group G. Let $x_3 \in K_3$ and let $a(x_1, x_2; x_3)$ be the number of ordered pairs (x_1, x_2), $x_1 \in K_1$ and $x_2 \in K_2$, such that $x_1 \cdot x_2 = x_3$. Then, if $\chi_1, \cdots, \chi_t$ are all the irreducible complex characters of G, we have

$$a(x_1, x_2. x_3) \mid C_G(x_1) \mid \mid C_G(x_2) \mid = \mid G \mid \sum_{i=1}^{t} \frac{\chi_i(x_1)\chi_i(x_2)\overline{\chi_i(x_3)}}{\chi_i(1)} \ . \tag{2}$$

Professor D. Wales has communicated the following result to the authors.

RESULT 3. Let G be a primitive permutation group of degree 100 with stabilizer of a point H isomorphic to the Mathieu simple group M_{22} and the orbits of H are of length 1,22 and 77. Then G is isomorphic to the Higman-Sims simple group of order 44,352,000.

1. Determination of the Sylow p-normalizers for $p = 11,7$ and 5. In this section, we will determine the Sylow 11-, 7- and 5- normalizers. Unfortunately the amount of numerical work required to show that the Sylow 11-subgroup is self-centralizing is too large to enable us to present the proof here. However, the methods and results used are similar to the examples of this kind of work given in [12] and [14]. In addition to the results given in [12] and [14], we also need some results of R. Brauer on the defect group of a block (see [3], [4] and [7], §86, 87, also [5], Theorem 2 and Theorem 3). Thus combining these methods we are able to show the following result.

LEMMA 1.1. *The Sylow* 11-*normalizer of G is a Frobenius group of order* 55.

Using equation (1), it is not difficult to show that the number of conjugate classes of elements of order 7 is one, i.e., $\mid N_G(G_7): C_G(G_7) \mid = 6$. Then, with Lemma 1.1 and more numerical work, we have that $B_1(11) \cap B_1(7) = \{1,3200\}$, where the numbers in the brackets are the degrees of the irreducible characters which lie in both the principal

11-block, $B_1(11)$ and the principal 7-block $B_1(7)$. If, in equation (2), we take $x_1 = x_2 = \nu$, where ν is an element of order 7 and x_3 is an element of order 11, then we have that $|C_G(\nu)| \mid 2.3.7$. Hence by Sylow theorems, we have

LEMMA 1.2. *The Sylow 7-normalizer of G is a Frobenius group of order* 42.

We now get the following possibilities for equation (1) for $B_1(11)$ and $B_1(7)$:

Possibilities for equation (1) *for* $B_1(11)$:
(I) $1 - 3200 - 175 + 1750 + 2520 - 896^* = 0$.
(II) $1 - 3200 - 175 + 1750 + 1750 - 126^* = 0$.
(In case (I), the two 11-conjugate characters have the common degree 896 and in case (II), they have the common degree 126).

Possibilities for equation (1) *for* $B_1(7)$:
(A) $1 + 3200 - 825 - 2750 - 1056 + 22 + 1408 = 0$.
(B) $1 + 3200 - 825 - 2750 + 22 + 22 + 330 = 0$.
(C) $1 + 3200 - 825 - 2750 - 55 + 330 + 99 = 0$.
(D) $1 + 3200 - 825 - 3520 - 1056 + 1408 + 792 = 0$.

Now let χ_2 be the (unique) character of G with degree 3200. Then χ_2 lies in a 5-block $B_2(5)$ say, of defect 1. If D is the defect group of $B_2(5)$, then D has order 5 and $0_5(C_G(D)) = D$(see [7]).

Suppose we have case (II) for $B_1(11)$. If we put $\chi_3(1) = 175$, $\chi_4(1) = \chi_5(1) = 1750$, $\chi_6(1) = \bar{\chi}_6(1) = 126$ ($\bar{\chi}_6$ is the complex conjugate of χ_6), we see that $\chi_4(d) = \chi_5(d) = 0$, $\chi_2(d) = -\chi_3(d)$ and hence $\chi_6(d) = \overline{\chi_6(d)} = 1$ for any element d in G, d of order 5. Now take $d \in D$, and since $\chi_2(d) \neq 0$, we have $\chi_2(d) = -\chi_3(d) \equiv 0 (\mathrm{mod}\ 5)$.

We now use Result 2 and put $d = x_1 = x_2$ and take x_3 to be an element of order 11. Since the left-hand side of (2) is nonnegative, $|\chi_2(d)| \leqq 10$, and we get that in all cases, $|C_G(d)| \mid 2^2.5^2$. As $\langle d\rangle = D$, we have that $0_5(C_G(D)) > D$, a contradiction. Hence, case (II) for $B_1(11)$ is not possible.

So, throughout the rest of the paper, we are in case (I) for $B_1(11)$ and we put

$$\chi_2(1) = 3200, \chi_3(1) = 175, \chi_4(1) = 1750, \chi_5(1) = 2520$$

and $\chi_6(1) = \bar{\chi}_6(1) = 896$.

Since all the characters of $B_1(11)$ are rational-valued, except χ_6 and $\bar{\chi}_6$ which are rational-valued only on 11-regular elements, we obtain a congruence modulo 5 for $\chi(x)$, $\chi \in B_1(11)$ and x any element of order 5. Put $x = x_1 = x_2$ and $x_3 = s$, where s is an element of order 11, and equation (2) becomes

$$\begin{aligned} &a(x, x; s) \mid C_G(x) \mid^2 \\ &= \mid G \mid \sum_{i=1}^{6} \frac{(\chi_i(x))^2\overline{\chi_i(s)}}{\chi_i(1)} \\ &= \mid G \mid \left(1 - \frac{(\chi_2(x))^2}{3200} - \frac{(\chi_3(x))^2}{175} + \frac{(\chi_5(x))^2}{2520} - \frac{(x_6(x))^2}{896}\right) \\ &= 2^2.5^2.11(40336 - 243\alpha^2 - 29\gamma^2 - 32\gamma) , \end{aligned}$$

where $\alpha = \chi_2(x) = -\chi_3(x)$, $\gamma = \chi_6(x)$. Note that $\chi_5(x) = \chi_6(x) - 1 = \gamma - 1$ by the orthogonality relation. Since the left-hand side of the above formula is nonnegative, we see that

$|\alpha| < 15$ and so $\alpha = 0, \pm 5$, or $+10$, and if

$$\begin{aligned} \alpha &= 0 , & -34 &\leqq \gamma \leqq 36 \\ \alpha &= \pm 5 , & -34 &\leqq \gamma \leqq 31 \\ \alpha &= \pm 10 , & -24 &\leqq \gamma \leqq 21 . \end{aligned}$$

By the orthogonality relation, $1 + 5\alpha^2 + \gamma^2 + (\gamma - 1)^2 \leqq |C_G(x)|$, and because $(|C_G(x)|, 77) = 1$, we get the following possibilities for any element x, of order 5, in G:

(i) $\alpha = 0, \gamma = -4, |C_G(x)| \mid 2^4.5^3$

(ii) $\alpha = 0, \gamma = 1, |C_G(x)| \mid 2.3.5^2$

(iii) $\alpha = \pm 5, \gamma = 1, |C_G(x)| \mid 2^2.3.5^2$.

If $\langle d \rangle = D$ is the defect group of the 5-block $B_2(5)$ (of defect 1) which contains χ_2, then $\chi_2(d) = \alpha \neq 0$, and so $\alpha = \pm 5$ and $|C_G(d)| \mid 2^2.3.5^2$. As $0_5(C_G(\langle d \rangle)) = D$, it is immediate that $C_G(d)/D \cong A_5$. It follows now that $C_G(G_5) = Z(G_5)$, and that G_5 is nonabelian and by a result of B. Huppert ([10], S. 8.6), G_5 is of exponent 5. If $Z(G_5) = \langle \rho \rangle$, where $\rho^5 = 1$, then $|C_G(\rho)| \mid 2^4.5^3$, and as $\langle \rho \rangle \lhd N_G(G_5)$, we have by Sylow theorems $|N_G(G_5)| = 5^3$ or $2^4.5^3$. The first possibility is impossible by a theorem of H. Wielandt ([10], S. 8.1), and so $N_G(G_5) = S.G_5$ where S is a 2-group of order 16. Since $Z(S)$ is a cyclic group of order 4, we have that $N_G(\langle d \rangle) \cong F_{20} \times A_5$ where F_{20} is a Frobenius group of order 20, and so d is conjugate to all its powers.

We shall now proceed to rule out possibilities (B), (C) and (D) for $B_1(7)$. In the cases (B) and (C), we put $x_1 = x_2 = d$ and $x_3 = \nu$, where $\nu^7 = 1$, in equation (2). In both these cases, there are only a small number of possibilities for the values of the characters in $B_1(7)$ on the element d, and in all cases we get that $|C_G(d)| < 2^2.3.5^2$, a contradiction; so cases (B) and (C) are not possible for $B_1(7)$. Case (D) is immediately ruled out by summing the squares of the degrees so far determined, (noting that the 5-block of defect 1, $B_2(5)$ which contains χ_2 has the following degrees: $B_2(5) = \{3200, 175, 825, 1925, 1925\}$); as this sum is $45, 496, 297 > |G|$.

Hence, for the rest of the paper we are in case (A) for $B_1(7)$ and we put $B_3(3)$ to be the 3-block of defect 1 containing the characters χ_7 and χ_9 of degrees 825 and 1056 respectively. Then

$$B_3(3) = \{825, 1056, 231\} .$$

Let $B_2(5)$ be the 5-block of defect 1 containing χ_2, χ_3, and χ_7. Then

$$B_2(5) = \{3200, 175, 825, 1925, 1925\} .$$

Further, we put $\chi_8(1) = 2750$, $\chi_{10}(1) = 22$ and $\chi_{11}(1) = 1408$. Then using Result 2 for $x_1 = x_2 = \rho$ and $x_3 = \nu$, where $\langle\rho\rangle = Z(G_5)$ and $\nu^7 = 1$, we again get a few possibilities for $\chi(\rho)$, $\chi \in B_1(7)$, and this gives that $|C_G(\rho)| \mid 2^2.5^3$. Hence $C_G(\rho)$ is precisely of order 4.5^3 and $C_G(\rho)$ is a semi-direct product of a cyclic group of order 4 and G_5. It follows immediately that the Sylow 2-subgroup S of $N_G(G_5)$ is a quasi-dihedral group of order 16 (i.e., $S = \langle a, b \mid a^8 = 1 = b^2, bab = a^5\rangle$). Now let $l \in G_5 \backslash \langle\rho\rangle$, and $l \not\sim_{N_G(G_5)} d$, then l has precisely 80 conjugates in $N_G(G_5)$, and so $C_G(l) \cap N_G(\langle\rho\rangle) = \langle l\rangle \times \langle\rho\rangle$. Summing the character χ_6 on G_5, we see that $\chi_6(l) = 1$ and as $C_G(\rho) \cap C_G(l) = \langle l\rangle \times \langle\rho\rangle$, and $|C_G(l)| \mid 2.3.5^2$, we have $|C_G(l)| = 5^2$, i.e., $C_G(l) = \langle l\rangle \times \langle\rho\rangle$. In particular $l \not\sim_G d$. The 5-structure of G is now completely determined and we summarize these results in the following lemma:

LEMMA 1.3. *The group G has precisely 3 conjugate classes of elements of order 5 with representatives d, ρ and l. A Sylow 5-subgroup G_5 of G is nonabelian of exponent 5 and $N_G(G_5)$ is a semi-direct product of G_5 and a quasi-dihedral group of order 16. Also, $Z(G_5) = \langle\rho\rangle$ and ρ has centralizer of order 4.5^3, $|C_G(l)| = 5^2$, and $C_G(d) \cong \langle d\rangle \times A_5$. Finally, $N_G(\langle d\rangle)$ is isomorphic to the direct product of aFrobenius group of order 20 by A_5.*

2. **The 3-structure.** Let $\langle c\rangle$ be a Sylow 3-subgroup of $C_G(d)$. Then

$$C_G(c) \cap C_G(d) = \langle d\rangle \times \langle c\rangle,\ N_G(\langle c\rangle) \cap C_G(d)$$
$$\cong S_3 \times \langle d\rangle,\ N_G(\langle c\rangle) \cap N_G(\langle d\rangle) \cong S_3 \times F_{20} .$$

It now follows that $C_G(c)/\langle c\rangle \cong S_5.E$, where $E = \langle 1\rangle$ or E is an elementary 2-group of order 16, and $E \lhd C_G(c)$. Since $N_G(\langle c\rangle) > C_G(c)$, G_3 is then elementary abelian; and so $C_G(c) \cong \langle c\rangle \times S_5 \cdot E$.

Suppose $|E| = 16$. Let X denote a Sylow 2-subgroup of $N = N_G(\langle c\rangle)$, and let G_2 be a Sylow 2-subgroup (of G) containing X. If $E \lhd G_2$, then $|N_G(E): N| = 2$, and then $2^4 \mid |N_G(\langle d\rangle)|$ (by the Frattini argument), clearly a contradiction. We may suppose therefore that $N_G(E) = N$. As X/E is not elementary, there is an involution $t \in Z(G_2) \cap E$. It then follows that $|C_G(t): C_N(t)| = 2$ or 10, but in either case, $\langle c \rangle \leqq C(0_2(C_G(t)))$. As $C(E) \cap N = E \times \langle c \rangle$, $|C_G(G_3)| \mid 2^3.3^2$. Also, if $0_2 = 0_2(C_G(t))$, then $0_2 \leqq E$. If $\langle c \rangle$ is a Sylow 3-subgroup of $C(0_2)$, we have a contradiction by the Frattini argument. So a Sylow 3-subgroup G_3 of $C(0_2)$ is of order 9. By the Frattini argument, $5 \nmid |C_G(t)$. So $C_G(t)$ is a soluble group of order $2^9.3^2$. From the structure of $C_G(c)$, we must have $|0_2| = 4$ but then $G_3 \lhd C_G(t)$, which contradicts the fact that $|C_G(G_3)| \mid 2^3.3^2$ We have proved:

LEMMA 2.1. *If $\langle c \rangle$ is a Sylow 3-subgroup of $C_G(d)$, then $C_G(c) \cong \langle c \rangle \times S_5$ and $N_G(\langle c \rangle) \cong S_3 \times S_5$.*

Put $N_G(\langle c \rangle) = A \times B$, where $A \cong S_3$ and $B \cong S_5$. Let π be an involution in A. Since A is a maximal subgroup of V, where $V \cong A_5$ and $C_G(d) = \langle d \rangle \times V$, it follows that if $\langle \lambda \rangle$ is a Sylow 3-subgroup of B, then $C_G(\lambda) \cap C_G(d) = A$. We have $C_G(\pi) \cap N_G(\langle d \rangle) = W \times \langle \pi, \tau \rangle$, where $W \cong F_{20}$ and $\langle \pi, \tau \rangle$ is a 4-group. It now follows by Sylow that $|C_G(\pi)| = 2^5.3.5$, $2^9.3.5$ or $2^6.3^2.5$. In the first case, $\langle \pi, \tau \rangle \lhd C_G(\pi)$, which contradicts $C_G(d) \cap C_G(\lambda)$.

If $|C_G(\pi)| = 2^9.3.5$, then $0_2 = 0_2(C_G(\pi))$ is elementary abelian of order 64 and $C_G(\pi)/0_2 \cong S_5$. (If 0_2 were nonabelian, then $C_G(d) \cap C_G(\lambda) \neq \langle \pi \rangle$). Let f be an element of order 3 in $C_G(d)$ such that $f \in N_G(\langle \pi, \tau \rangle)$. Now $N_G(\langle \pi, \tau \rangle) \geqq 0_2.W$, and $N_G(\langle \pi, \tau \rangle) \geqq \langle f \rangle$. Since $C_G(\langle \pi, \tau \rangle) \leqq C_G(\pi)$, we have that $0_2 = 0_2(C_G(\langle \pi, \tau \rangle))$, and hence $N_G(0_2) > C_G(\pi)$. However, from Lemma 1.1 and the fact that A_7 has no elements of order 15, we have that $|N_G(0_2)| = 2^9.3^2.5$; thus $N_G(0_2)/0_2 \cong \langle f \rangle \cdot S_5$ where $0_2 \cdot \langle f \rangle$ is a normal subgroup of $N_G(0_2)$, again contradicting the structure of $C_G(d) \cap C_G(\lambda)$.

The order of $C_G(\pi)$ is thus $2^6.3^2.5$ and it follows that $C_G(\pi)/\langle \pi \rangle \cong \mathrm{Aut}\,(A_6)$. Hence $N_G(G_3)/C_G(G_3)$ is a semi-dihedral group of order 16 and hence all elements of order 3 are conjugate in G.

Let z be the unique involution of a Sylow 2-subgroup Z (of order 4) of $C_G(\rho)$ where $\langle \rho \rangle = Z(G_5)$. Since $\langle \rho \rangle \not\sim_G \langle d \rangle$, $z \not\sim_G \pi$, and so $|C_G(z)| = 2^5.3.5$ or $2^9.3.5$. In the first case $C_G(z)/Z \cong S_5$, but since the Sylow 2-subgroup of $N_G(\langle \rho \rangle)$ is quasi-dihedral, $z \in \mho^2(X)$, where X is a Sylow 2-subgroup of $C_G(z)$. However as $X/Z \cong D_8$, where D_8 is a dihedral group of order 8, $\mho^2(X) \leqq Z$ which gives a contradiction. Hence $|C_G(z)| = 2^9.3.5$ and if $E = 0_2(C_G(z))$, E is a 2-group of order 64 and $C_G(z)/E \cong S_5$. Because $C_G(\rho) \cap C_G(c) \cap C_G(z) = Z \cdot \langle \rho \rangle$, $Z \leqq E$ and $Z \lhd C_G(z) = C$.

As $\pi \in V$, where $C_G(d) = \langle d \rangle \times V$ and $V \cong A_5$, $\pi \notin E$, but $\pi \in V \cdot E$, where $V \cdot E/E \cong A_5$. In any case, $N_C(\langle c \rangle) = A \times F$, where $A \cong S_3$ and

$F \cong D_8$, with $F \cap E = Z$. Thus $C_C(\pi)\cdot E$ is a Sylow 2-subgroup of $C = C_G(z)$, and hence $|C_E(\pi)| \leqq 8$.

If E is abelian, it is of type (4, 2, 2, 2, 2) and

$$C_G(c) \cap E = C_G(\rho) \cap E = Z\,. \quad \text{Also,} \quad |C_G(\pi) \cap \Omega_1(E)| \geqq 8\,,$$

and as $C_G(\pi) \geqq Z$, we have a contradiction. Hence E is nonabelian and thus E is a central product of two quaternion groups Q_1 and Q_2, and the cyclic group Z of order 4. We have proved:

LEMMA 2.2. *The group G has only ond conjugate class of elements of order* 3, $C_G(G_3) = \langle\pi\rangle \times G_3$, *where* π *is an involution, and* $N_G(G_3)/C_G(G_3)$ *is a semidihedral group of order* 16. *If* z *is the involution in* $C_G(\rho)$, *where* $\langle\rho\rangle = Z(G_5)$, *then* $C = C_G(z)$ *is an extension of a nonabelian* 2-*group* E *of order* 64, (*which is a central product of two quaternion groups and a cyclic group of order four*) *by the symmetric group* S_5 *on* 5 *letters. Finally,* $C_G(\pi)/\langle\pi\rangle \cong \mathrm{Aut}\,(A_6)$.

3. Determination of all degrees of irreducible characters of G.

We are now in a position to apply the exceptional character theory to the group G with respect to the subgroup $H = N_G(\langle c\rangle)$, where c is any element of order 3. As "special classes" (in the sence of Wong [15]), we take all roots of c. As $H \cong S_3 \times S_5$, the character table of H is determined from the character tables of S_3 and S_5. Put $H = A \times B$, where $A \cong S_3$ and $B \cong S_5$.

In the above notation, our special classes of H are the conjugate classes in H with representatives c, ct, cz, cw and cd. As usual, $B_\tau(3)$ will denote a 3-block of G and $b_\tau(3)$, a 3-block of H. The group H

Character Table of $A \cong S_3$

Order	Element	θ_1	θ_2	θ_3
1	1	1	1	2
2	π	1	-1	0
3	c	1	1	-1

Character Table of $B \cong S_5$

Order	Element	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6	ζ_7
1	1	1	1	4	4	6	5	5
2	z	1	1	0	0	-2	1	1
2	t	1	-1	-2	2	0	-1	1
4	w	1	-1	0	0	0	1	-1
3	λ	1	1	1	1	0	-1	-1
6	λt	1	-1	1	-1	0	-1	1
5	d	1	1	-1	-1	1	0	0

has three 3-blocks:

$$\begin{aligned} b_1(3) &= \{1_H, \theta_2, \theta_3, \zeta_8, \zeta_3\theta_2, \zeta_3\theta_3, \zeta_6, \zeta_6\theta_2, \zeta_6\theta_3\} \\ b_2(3) &= \{\zeta_2, \zeta_2\theta_2, \zeta_2\theta_3, \zeta_4, \zeta_4\theta_2, \zeta_4\theta_3, \zeta_7, \zeta_7\theta_2, \zeta_7\theta_3\} \\ b_3(3) &= \{\zeta_5, \zeta_5\theta_2, \zeta_5\theta_3\} \ . \end{aligned}$$

Here $1_H = \theta_1 \cdot \zeta_1$, and $b_1(3)$, $b_2(3)$ are the 3-blocks of defect 2 of H; and $b_3(3)$ the unique 3-block of defect 1 of H. We denote by $b_i(3)^G$, the block of G which corresponds to the block $b_i(3)$ of H, using Brauer's block correspondence (see [4]). By [15], Theorem 6 (or [4], S.2E)

$$\begin{aligned} b_1(3)^G &= B_1(3) \\ b_2(3)^G &= B_2(3) \\ b_3(3)^G &= B_3(3) = \{825, 1056, 231\} \ , \end{aligned}$$

where $B_2(3)$ is the only other 3-block of defect 2 of G besides the principal 3-block $B_1(3)$. If D denotes the union of special classes of H, we take the following basis for the module of all generalized characters of H which vanish on $H \backslash D$:

$$\begin{aligned} \varphi_1 &= (1_H - \zeta_3)\Sigma \\ \varphi_2 &= (1_H + \zeta_6)\Sigma \ , \\ \varphi_3 &= (\zeta_2 - \zeta_4)\Sigma \ , \\ \varphi_4 &= (\zeta_2 + \zeta_7)\Sigma \ , \end{aligned}$$

and

$$\varphi_5 = \zeta_5 \Sigma \ ,$$

where

$$\Sigma = 1_H + \theta_2 - \theta_3 \ .$$

Note that φ_1 and φ_2 are expressed as a linear combination of irreducible characters occurring only in $b_1(3)$. Similarly φ_3 and φ_4; and φ_5 are expressed only as a linear combination of characters occurring in $b_2(3)$, and $b_3(3)$ respectively. Let φ_i^* denote the corresponding induced characters of $\varphi_i (i = 1, 2, \cdots, 5)$.

The induced characters φ_1^*, φ_2^* can be expressed as a linear combination of the irreducible characters of $B_1(3)$ and if $\chi \in B_1(3)$, then χ appears as a constituent of φ_1^* or φ_2^* (see [15], Ths. 7 and 9). Similar statements can be made for φ_3^*, φ_4^* and $B_2(3)$, and φ_5^* and $B_3(3)$.

Finally, if χ is any irreducible character of G, let $n_i = (\chi, \varphi_i^*)$, $(i = 1, 2, \cdots, 5)$. Then

$$\begin{aligned} \chi(\sigma) = &-n_1\zeta_3(\sigma) + n_2\zeta_6(\sigma) - n_3\zeta_4(\sigma) + n_4\zeta_7(\sigma) \\ &+ n_5\zeta_5(\sigma), \text{ for any } \sigma \in D \ . \end{aligned} \tag{3}$$

Since $(\varphi_i^*, \varphi_j^*)_G = (\varphi_i, \varphi_j)_H$ we have:

$$(\varphi_1^*, \varphi_1^*) = (\varphi_2^*, \varphi_2^*) = (\varphi_3^*, \varphi_3^*) = (\varphi_4^*, \varphi_4^*) = 6 ,$$
$$(\varphi_1^*, \varphi_2^*) = (\varphi_3^*, \varphi_4^*) = 3 ,$$
$$(\varphi_1^*, \varphi_3^*) = (\varphi_2^*, \varphi_3^*) = (\varphi_1^*, \varphi_4^*) = (\varphi_2^*, \varphi_4^*) = 0 ,$$
$$(\varphi_5^*, \varphi_5^*) = 3$$

and $(\varphi_5^*, \varphi_i^*) = 0$ for $i = 1, 2, 3, 4$.

Further, by the Frobenius reciprocity law,

$$(\varphi_1^*, 1_G) = (\varphi_2^*, 1_G) = 1 \quad \text{and} \quad (\varphi_3^*, 1_G) = (\varphi_4^*, 1_G) = 0 .$$

From these values, it follows that

$$(4) \qquad \begin{cases} \varphi_1^* = 1_G + \sum_{i=1}^{5} \varepsilon_i X_i \\ \text{and} \\ \varphi_2^* = 1_G + \varepsilon_1 X_1 + \varepsilon_2 X_2 + \eta_1 Y_1 + \eta_2 Y_2 + \eta_3 Y_3 \end{cases}$$

where X_i and Y_i are distinct non-principal irreducible characters of G.

So far, the degrees of 15 irreducible characters of G have been determined:

$\chi_1 = 1_G$	χ_2	χ_3	χ_4	χ_5	χ_6	$\bar{\chi}_6 = \chi_7$	χ_8	χ_9
1	3200	175	1750	2520	896	896	825	1056

χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
2750	22	1408	231	1925	1925

Using (3) and the fact that $(\chi \mid_{G_3}, 1_{G_3})$ is an integer where G_3 is a Sylow 3-subgroup of G, we get $\chi_{13}(c) = 6$ and so

$$\varphi_5^* = \chi_{13} + \chi_8 - \chi_9 .$$

It follows that $\chi_2(cd) = \chi_8(cd) = -\chi_3(cd) = -\chi_{14}(cd) = -\chi_{15}(cd) = 1$. Further, $\chi_{11}(c) = 4$ and hence, if $\chi_{11} \in B_1(3)$ then $(\varphi_1^* \cdot \chi_{11}) = -1$ and $(\varphi_2^*, \chi_{11}) = 0$. By obtaining a congruence modulo 9 for the above characters on the element c, we see that if $\chi_i \in B_1(3)$, then χ_i occurs in precisely one of φ_1^* or φ_2^* if $i = 2, 3, 4, 6, 7, 10, 11$ and 12; and in both φ_1^* and φ_2^* if $i = 1, 14$ and 15, using the block-intersection lemma of Brauer-Tuan ([6], Lemma 3), and since $\chi_2(cd) = 1$, we have the following possibilities for $B_1(3_1)$, $B_2(3)$:

(a) $B_1(3) = \{1, 3200, 1750, 22, 1408, 2750, \cdots\}$,
$B_2(3) = \{175, 896, 896, \cdots\}$

(b) $B_1(3) = \{1, 3200, 175, 1750, 896, 896, 22, 1408, 2750\}$,

$B_2(3) = \{1925, \cdots\}$

(c) $B_1(3) = \{1, 1750, 896, 896, 2750, 1408, \cdots\}$,

$B_2(3) = \{3200, 175, 22, \cdots\}$

(d) $B_1(3) = [1, 1750, 896, 896, 2750, 22, \cdots\}$,

$B_2(2) = \{3200, 175, 1408, \cdots\}$.

Note that χ_6 and χ_7 both lie in the same block, and

$$(\varphi_i^*, \chi_6) = (\varphi_i^*, \chi_7), (i = 1, \cdots, 4)$$

as φ_i^* is rational-valued. Further, $\chi_4(cd) = \chi_{10}(cd) = 0$ and since both χ_4, χ_{10} always lie in $B_1(3)$, χ_4 and χ_{10} must be constituents of φ_2^*; and as $\chi_6(cd) = \chi_7(cd) = 1$, χ_6 and χ_7 either both occur in φ_1^* or both in φ_3^*.

By summing the squares of the degrees so far determined, any other irreducible character of G must have degree $77k$, where $k \leqq 20$. In particular, if $\chi \in B_1(3)$ or $B_2(3)$, then $(k, 3) = 1$ and $k \leqq 20$.

Case (b) for $B_1(3)$ and $B_2(3)$ is immediately ruled out using (4).

In case (c) for $B_1(3)$ ahd $B_2(3)$, first assume that neither χ_{14} nor χ_{15} appears in φ_1^*. Then $\varphi_3^*(1) = 3200 - 175 - 22 - 1925 + 847 = 0$. But $847 = 7 \times 11^2$ which is not possible. Hence either χ_{14} or χ_{15} appears in φ_1^* with nonzero multiplicity and we have:

$$\varphi_1^*(1) = 1 + 896 + 896 - 1408 - 1925 + 1540 = 0\ ,$$
$$\varphi_2^*(1) = 1 - 1705 + 2750 - 616 - 1925 + 1540 = 0\ ,$$
$$\varphi_3^*(1) = 3200 - 175 - 22 - 1925 + \delta_1 z_1 + \delta_2 z_2 = 0\ ,$$

and

$$\varphi_4^*(1) = -1925 + \delta_1 z_1 + \delta_2 z_2 + \sum_{i=1}^{3} \kappa_i y_i = 0\ ,$$

where $\delta_1, \delta_2, \kappa_i (i = 1, 2, 3)$ are equal to ± 1 and z_1, z_2, y_1, y_2, y_3 are degrees of irreducible characters of G. No matter what values the z_i takes, at least two of the y_i take the value 1232 or one of them takes the value 1540. In any case, the sum of the squares of the degrees so far determined is greater than $|G|$.

In case (d) for $B_1(3)$ and $B_2(3)$, we have;

$$\varphi_1^*(1) = 1 + 896 + 896 - 22 - 1925 + 154 = 0\ ,$$
$$\varphi_2^*(1) = 1 - 1750 + 2750 + 770 - 1925 + 154 = 0\ ,$$
$$\varphi_3^*(1) = 3200 - 175 - 1408 - 1925 + \sum_{i=1}^{2} \delta_i z_i = 0\ ,$$

and

$$\varphi_4^*(1) = -1925 + \sum_{i=1}^{2} \delta_i z_i + \sum_{i=1}^{3} \kappa_i y_i = 0\ ,$$

in the same notation as above. We may take $(\varphi_1^*, \chi_{14}) = -1$, and then on the element ct of order 6, we get:

$$\chi_2(ct) = -2,\ \chi_3(ct) = 2,\ \chi_8(ct) = 0,\ \chi_{14}(ct) = -1$$

and $\chi_{15}(ct) = 1$. Summing over the 5-block of defect 1, $B_2(5)$, and using [1], Corollary 4, we get

$$-2 + 0 = -2 \neq 2 + 1 - 1 = 2\ ,$$

and a contradiction and so case (a) is the only possibility for $B_1(3)$ and $B_2(3)$.

In case (a), we have:

$$\varphi_1^*(1) = 1 + 3200 - 22 - 1408 - 1925 + 154 = 0\ ,$$
$$\varphi_2^*(1) = 1 - 1750 + 2750 + 770 - 1925 + 154 = 0\ ,$$
$$\varphi_3^*(1) = 896 + 896 - 175 - 1925 + \sum_{i=1}^{2} \delta_i z_i = 0\ ,$$

and

$$\varphi_4^*(1) = -1925 + \sum_{i=1}^{2} \delta_i z_i + \sum_{j=1}^{3} \kappa_i y_j = 0\ .$$

Thus $\sum_{i=1}^{2} \delta_i z_i = 308 = 4 \times 77$ and $\sum_{j=1}^{3} \kappa_j y_j = 1617 = 21 \times 77$, which gives a number of possibilities for z_i and y_j. However, if

$$\chi \notin B_2(3) \cup B_2(3) \cup B_3(3)\ , \quad \text{then} \quad 9 \times 77 \mid \chi(1)\ .$$

By summing the squares of degrees and using this fact, we get a unique decomposition for φ_3^* and φ_4^*:

$$\varphi_3^*(1) = 896 + 896 - 175 - 1925 + 154 + 154 = 0\ ,$$

and

$$\varphi_4^*(1) = 770 + 770 + 77 - 1925 + 154 + 154 = 0\ .$$

There are now either 2 or 5 irreducible characters left to be determined (in the first case G would have one irreducible character of degree 693 and one of degree 1386: while in the second possibility G would have five irreducible characters of degree 693); and so G has either 24 or 27 irreducible characters. Using the orthogonality relations for the element l of order 5 (see Lemma 1.3) and with centralizer, $C_G(l)$ of order 25, it follows immediately that G has only 24 irreducible characters and hence 24 conjugate classes of elements. The character table of G can now be completed, except for some classes of 2-elements which have not as yet been determined.

The partially completed character table of G shows that the characters χ_6 ahd χ_7 vanish on all 2-elements except the involution π

(defined in Lemma 2.2). A result of Frobenius and Schur (see [8], (3.5)) shows that π is not the square of any element of order four in G.

4. Completion of proof of theorem. Let z be the central involution with centralizer $C_G(z) = C$, as in Lemma 2.2. Then $C/0_2(C) \cong S_5$, and $0_2(C)$ is a central product of two quaternion groups Q_1 and Q_2 and a cyclic group Z of order four. We may take $\langle c\rangle$ and $P = \langle\rho\rangle$ to be a Sylow 3- and Sylow 5-subgroup of C respectively (for definitions of $\langle c\rangle$ and $\langle\rho\rangle$, see Lemmas 1.3 and 2.1). Let $E = 0_2(C)$, then

$$C_C(c) \cap E = C(P) \cap E = Z, \; |N_C(P)| = 2^4.5$$

with $N_C(P)$ having as a Sylow 2-subgroup a quasidihedral group of order 16; and $N_C(\langle c\rangle) \cong S_3 \times D_8$, where D_8 is a dihedral group of order 8. The action of P and $\langle c\rangle$ on E shows that $|C\colon C'| = 2$ and hence $C'/E \cong A_5$. We may suppose that the involution π lies in $C'\backslash E$. From the structure of $C_G(c)$, if $Z = \langle\omega\rangle$, where $\omega^4 = 1$, then $C_G(\omega) = C'$. Further, if u is any element of order four in $E\backslash Z$, then $|C_G(u)| = 2^8$; as u must have exactly 30 conjugates in C and as $\mho^1(E) = \langle z\rangle$. Similarly, if t is an involution in $E\ Z$, then $|C_C(t)| = 2^8$.

So far, we have determined the order of 20 of the 24 conjugate classes of elements of G. If K_1, K_2, K_3 and K_4 denote the remaining four classes of 2-elements of G, then using the previous lemmas and summing the order of the conjugate classes so far determined, we have the following possibilities:

	$\lvert C_G(x_1)\rvert$	$\lvert C_G(x_2)\rvert$	$\lvert C_G(x_3)\rvert$	$\lvert C_G(x_4)\rvert$
(1)	2^7	2^3	2^4	2^7
(2)	2^6	2^4	2^4	2^4
(3)	2^6	2^3	2^5	2^5,

where $x_i \in K_i$, $i = 1, 2, 3, 4$.

It now follows that for $t \in E\backslash Z$, t an involution, then $t \underset{G}{\sim} z$. If x is an involution in $C'\backslash E$, then by Sylow theorems, we may take $x \in N(P) \cap C'$. However the Sylow 2-subgroup of $N_{C'}(P)$ contains only two involutions x and xz which are conjugate in $N_C(P)$, and so $x \underset{G}{\sim} \pi$; and hence $C'\backslash E$ has only one class of involutions. A transfer theorem of J. G. Thompson ([13], Lemma 5.38) now shows that *G has precisely two conjugate classes of involutions with representatives z and π.*

As before, we may take $\pi \in N_{C'}(\langle c\rangle)\backslash E \cap N(\langle c\rangle)$. Put $N_C(\langle c\rangle) = \langle c\rangle\cdot\langle\pi\rangle \times K$, where K is a dihedral group of order 8. Denote the involutions of K by z, τ, τz, $\tau\omega^{-1}$ and $\tau\omega$, where $Z = \langle\omega\rangle$ is as above and Z is the unique cyclic group of order four in K. From the structure of $C_G(c)$, we may take $\tau \underset{G}{\sim} \tau z \underset{G}{\sim} z$ and $\pi \underset{G}{\sim} \tau\omega \underset{G}{\sim} \tau\omega^{-1}$. Further, $[\langle c\rangle, E]$ is an extra-special 2-group of order 32 and we write $[\langle c\rangle, E] = Q_1 \curlyvee Q_2$, where Q_1 and Q_2 are quaternion groups of order 8 and $Q_1 \curlyvee Q_2$ denotes the central product of Q_1 and Q_2. Put $Q_1 = \langle \alpha_1, \alpha_2 \rangle$, $Q_2 = \langle \alpha_2, \alpha_4 \rangle (\alpha_i, i = 1, \cdots, 4$ are elements of order 4) and note that $\langle \pi \rangle \cdot K \leqq N_C(Q_1 \curlyvee Q_2)$.

As $|C_E(\pi)| \leqq 8$, we may choose the α_i in such a way that $\alpha_1^\pi = \alpha_1^{-1}$, $\alpha_2^\pi = \alpha_2^{-1}$, $\alpha_3^\pi = \alpha_1\alpha_3$ and $\alpha_4^\pi = \alpha_2\alpha_4$. Hence $C_E(\pi)$ is an abelian group of order 8 and type (4,2), and $C_E(\pi) = Z \times \langle \alpha_1, \alpha_2 \rangle$. Similarly, as $c \in C_G(\tau)$ and $\tau \in N_C(Q_1 \curlyvee Q_2)$; and $\tau\omega \underset{G}{\sim} \pi$, we get $\alpha_1^\tau = \alpha_2$ and $\alpha_3^\tau = \alpha_4$ (again with appropriate choice of the α_i). Since π is not the square of any elemement of order four, an easy computation now gives that $\mho^1(C_c(\pi)) = \langle \omega\alpha_1\alpha_2\pi \rangle$. A result of Gaschutz ([10], S. 17.4) shows that $C_G(\pi) \cong \langle \pi \rangle \times \mathrm{Aut}\,(A_6)$.

If we put $v = \omega\pi$, then v is of order four and $C_{C'}(v) = C_{C'}(\pi)$. Certainly, $v^2 = z$ and as $\alpha_1\tau \in C_c(v)$, we have that $|C_C(v)| = |C_G(v)| = 2^6$. The action of π on E gives that the coset $E\pi$ has precisely eight elements of order four whose square is z, and so $C' \backslash E$ contains one conjugate class of elements of order four (with representative v) whose square is z. The action of τ and c on E shows that $C \backslash C'$ has no element of order four whose square is z.

The group G has therefore precisely three conjugate classes of elements of order four, with representatives ω, u and v.

The orthogonality relations enable us to complete the character table on all but the last three classes of G. It can be shown by an easy computation that there are no elements of order 16. In any case, from the orthogonality relations the character χ_{11} of degree 22 vanishes on the remaining 3 conjugate classes of 2-elements.

We give below only part of the character table of G, namely, the value of the irreducible character χ_{11} of degree 22 on certain elements. Note that for any element x of order eight, $\chi_{11}(x) = 0$, from the orthogonality relation.

Element (s)	Order	χ_{11}
1	1	22
c	3	4
cz	6	0
ρ	5	-3
l, d	5	2
z	2	6
w	4	-6
u, v	4	2
x	8	0

Using the same notation as above, $C_E(\tau) = \langle \alpha_1\alpha_2, \alpha_3\alpha_4, z \rangle$, an elementary abelian group of order 8. If we put $F = \langle \tau \rangle \times C_E(\tau)$, then F is an elementary abelian group of order 16, and further, $\langle E, c, \pi \rangle \leqq$

$N_C(F)$. Hence F contains precisely one class of involutions in G (since $\tau \underset{G}{\sim} z$) and $|N_C(F)| = 2^8.3$, as $E \cap F \lhd C$. Also, $C_G(F) = C_C(F) = F$.

The coset $E\tau$ contains precisely 16 involutions, and so, if κ is an involution in $C \backslash C'$, then $\kappa \underset{C}{\sim} \tau$ or $\kappa \underset{C}{\sim} \tau\omega$. Now, let J be any elementary abelian subgroup of order 16 of C, such that J contains only involutions conjugate to z. Then $|J \cap E| = 8$, and if $\kappa \in J \backslash E$, then $\kappa \in C \backslash C'$ and hence $\kappa \underset{C}{\sim} \tau$. Thus C has precisely one class of elementary abelian subgroup of order 16 containing only involutions conjugate to z. The structure of A_8 gives the following result:

LEMMA 4.1. *If F is an elementary abelian subgroup of order 16 in C, and if all involutions in F are conjugate to z, then $N_G(F) \backslash F \cong S_6$.*

Clearly, z has 15 conjugates in $N_G(F)$, and so $N_G(F)' = M$ is of index 2 in $N_G(F)$, and $M \backslash F \cong A_6$. Since $\langle c \rangle \leqq M$ and $|C_M(c) \cdot F \backslash F| = 9$, $\omega \notin M$. Thus $|E: M \cap E| = 2^5$ and there must be an involution $t \in E \cap M \backslash F \cap E$. Further, $|C_F(t)| = 4$ and as the normalizer of a Sylow 5-subgroup of M is dihedral of order 10, there must be precisely one class of involutions in $M \backslash F$ and they are all conjugate to z in G.

If μ is an element of order five in M, then $N_M(\langle \mu \rangle) = \langle \mu \rangle \langle i \rangle$, for some involution i in $M \backslash F$. Since $i \underset{G}{\sim} z$, μ must be conjugate to l or d in G, since if $\mu \underset{G}{\sim} \rho$, then $z \underset{G}{\sim} \pi$.

Since c normalizes $[c, E] \cap M$, and

$$|[c, E] \cap M| \geqq 16, [c, E] = E \cap M .$$

As $|\langle \pi, \omega \rangle \cap M| = 4$, and $\pi, \omega \notin M, \langle \pi\omega \rangle = \langle \pi, \omega \rangle \cap M$, and $\langle \pi\omega \rangle \cdot [c, E] \cdot F/F \cong D_8$ and is a Sylow 2-subgroup of M/F. Let $T/F = \langle \pi\omega \rangle \cdot [c, E] \cdot F/F$. Since the action of π, τ on E has been completely determined above, the action of T/F on F is now completely determined. If we regard F as a vector space with basis $z, \tau, \alpha_1\alpha_2$ and $\alpha_3\alpha_4$, the action of T on F is given by:

$$t_0 = \alpha_1\alpha_3\alpha_4 z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad t_1 = \alpha_4\alpha_1\alpha_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$t_2 = \pi\omega\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

where t_0, t_1 and t_2 are involutions, $\langle t_0, t_1, t_2\rangle \cong D_8$ where D_8 is a dihedral group of order 8 with $Z(\langle t_0, t_1, t_2\rangle) = \langle t_0\rangle$ and $T = \langle t_0, t_1, t_2\rangle\cdot F$. Note that $(t_1t_2)^2 = t_0$ and $|C_M(t_1t_2)| = 2^3$, and $\langle \tau t_1t_2\rangle$ of order 8 is self-centralizing in M. As $|C_M(t_0): C_M(t_1t_2)| = 4$, $t_1t_2 \underset{G}{\not\sim} \omega$, also if x is any element of order four such that $x^2 \in F$, then $x \underset{G}{\not\sim} \omega$ as $\omega \notin M$.

The group M has therefore 195 involutions (all conjugate to z in G), 1260 elements of order four, all conjugate in G to either u or v, 720 elements of order eight, 800 elements of order three, 480 elements of order six and 2, 304 elements of order five. Summing χ_{11} over the group M, we find that $(\chi_{11}, 1)_M = 2$. Let $U = V\cdot G_3$ be a Sylow 3-normalizer in M, where V is a cyclic group of order four and G_3 is a Sylow 3-subgroup of M. By an easy computation, $(\chi_{11}, 1)_U = 4$. Now take a quaternion group W of order eight, $W \leqq N_G(G_3)$, and $V < W$. Since $C\backslash C'$ contains no elements of order four whose square is z, W contains only elements of order four which are conjugate to u or v. If $L = W\cdot G_3$, then we find that $(\chi_{11}, 1)_L = 3$. Since $2 + 3 > 4$, $\langle L, M\rangle = H$ is a proper subgroup of G, and $|H| = 2^7.3^2.5.k$, where k is an integer, $k > 1$.

If N is a normal subgroup of H and $1 < N \leqq M$, then $N = F$ or $N = M$. Since $H = N_G(F)$ contradicts the structure of S_6, $|H: M| > 10$. From the degrees of the irreducible characters of G, we see that if $\bar{G}$ is a subgroup of G with $|G: \bar{G}| \leqq 200$, then $|\bar{G}| = 2^7.3^2.5.7.11$, $2^5.3^2.5^3.7$ or $2^9.5^2.7.11$; that is $|G: \bar{G}| = 100, 176$ or 45 respectively.

The following are therefore the only possibilities for $|H|$: (a) $2^7.3^2.5^3$, (b) $2^7.3^2.5^2.7$, (c) $2^7.3^2.5.11$, (d) $2^7.3^2.5.7.11$, (e) $2^8.3^2.5.11$, (f) $2^8.3^2.5.7$, (g) $2^9.3^2.5^2$, (h) $2^9.3^2.5.7$.

In the cases (a) to (d), H must be a simple group and a result of Z. Janko [11], shows that we are in case (d) and then $H \cong M_{22}$. Cases (f) and (g) can be eliminated by Sylow theorems and the structure of the Sylow 7- and 11-normalizers. If we are in case (g), we may take z to be a central involution in H and then $N_H(F) = N_G(F)$. However, now $|H: N_H(F)| = 10$ which contradicts the order of A_{10}. Finally, suppose we are in case (h). If H_5 is a Sylow 5-subgroup of H, then as H is simple by the Frattini argument $|N_G(H_5)| = 2^4.5^2$. This is a contradiction, as no subgroup of order 5^2 has normalizer divisible by 16.

We have thus shown that case (d) is the only possibility for the order of H and so H is isomorphic to the Mathieu simple group M_{22}. The degrees of the irreducible characters of G yield the following result:

LEMMA 4.2. *The group G is a primitive permutation group of degree* 100 *with stabilizer of a point H isomorphic to the Mathieu* *simple group* M_{22} *and the orbits of* H *are of length* 1,22 *and* 77.

With Result 3 and Lemma 4.2, the theorem is proved.

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