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ON THE GLOBAL DIMENSION OF RESIDUE RINGS

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Techniques of Eilenberg, Nakayama, and Nagao are used to obtain results on the global dimension of the ring R/I where I is an eventually idempotent ideal. We also consider the cases when I is contained in the socle of R, and when I has nonzero annihilator.

In [3] the global dimension of R/I is calculated for R left hereditary and I eventually idempotent (i.e., $I^n = I^{n+1}$ for some n).[1] These results are generalized by the following theorem:

THEOREM 1. *Let I be an ideal of R which is either projective as a left R-module or flat as a right R-module. If $I^m = I^{m+1}$ then for any R/I-module M*

$$\mathrm{l.hd}_{R/I} M \leqq \mathrm{l.hd}_R M + (m-1)\,\mathrm{l.hd}_R I + 2m - 2\ .$$

The proof depends on the following lemmas whose proofs we leave to the reader:

LEMMA 1. *Let $0 \to M_m \to \cdots \to M_1 \to M_0 \to M \to 0$ be an exact sequence of S-modules such that $\mathrm{l.hd}_S M_j \leqq n\ \forall j$. Then $\mathrm{l.hd}_S M \leqq n + m$.*

LEMMA 2. *If the ideal I is left S-projective and P is a projective left S-module, then IP is left S-projective.*

LEMMA 3. *If the ideal I is right S-flat and K is a submodule of a free left S-module, then*

$$\mathrm{l.hd}_S IK \leqq \mathrm{l.hd}_S I + \mathrm{l.hd}_S K\ .$$

REMARK. All of the above may be stated with l.hd replaced by l.wd.

Proof of Theorem 1. (a) Assume I is left projective. Proceed by induction on $\mathrm{l.hd}_R M = n$. $n = 0$ trivial. $n = 1$. Let $0 \to P \to F \to M \to 0$ be an R-projective resolution. Consider the exact sequence (of [3]):

[1] R and S will always denote rings with unit elements; I will always denote a two-sided ideal. "Projective" means left projective.

$$0 = I^m F/I^{m+1}F \longrightarrow I^{m-1}P/I^m P \longrightarrow I^{m-1}F/I^m F \longrightarrow I^{m-2}P/I^{m-1}P \longrightarrow \cdots \longrightarrow IP/I^2P \longrightarrow IF/I^2F \longrightarrow P/IP \longrightarrow F/IF \longrightarrow M \longrightarrow 0 .$$

Now use Lemmas 1 and 2 to obtain $\mathrm{l.hd}_{R/I} M \leqq 2m - 1$ as required. $n > 1$. Consider $0 \to K \to F \to M \to 0$ R-exact where F is R-free. Then $\mathrm{l.hd}_R K = n - 1 \geqq 1$. Since $IM = 0$ we may consider the R/I exact sequence

$$0 \longrightarrow K/IF \longrightarrow F/IF \longrightarrow M \longrightarrow 0 .$$

Then $\mathrm{l.hd}_R K/IF = n - 1$ so by induction

$$\mathrm{l.hd}_{R/I} K/IF \leqq n - 1 + 2m - 2$$

and so $\mathrm{l.hd}_{R/I} M \leqq n + 2m - 2$.

(b) Assume I is right flat. Let $0 \to K \to F \to M \to 0$ be R-exact, where F is R-free. Again, consider the R/I-exact sequence

$$0 \longrightarrow I^{m-1}K/I^m K \longrightarrow I^{m-1}F/I^m F \longrightarrow I^{m-2}K/I^{m-1}K \longrightarrow \cdots \longrightarrow IK/I^2K \longrightarrow IF/I^2F \longrightarrow K/IK \longrightarrow F/IF \longrightarrow M \longrightarrow 0$$

and apply Lemmas 1 and 3, making use of the fact that if N is a submodule of a free left R-module then $\forall n \geqq 0$ $\mathrm{Tor}^R_{1+n}(R/I, N) = 0$ and so $\mathrm{l.hd}_{R/I} N/IN \leqq \mathrm{l.hd}_R N$.

REMARK. Conversely, if I is left projective and $\mathrm{l.hd}_{R/I^2} R/I = m - 1$, then $I^m = I^{m+1}$.

Proof. Consider the R/I^2-sequence

$$0 \longrightarrow I^{m-1}/I^m \longrightarrow I^{m-2}/I^m \longrightarrow I^{m-3}/I^{m-1} \longrightarrow \cdots \longrightarrow I/I^3 \longrightarrow R/I^2 \longrightarrow R/I \longrightarrow 0 .$$

By hypothesis, I^{m-1}/I^m is R/I^2-projective, and so

$$0 \longrightarrow I^m/I^{m+1} \longrightarrow I^{m-1}/I^{m+1} \longrightarrow I^{m-1}/I^m \longrightarrow 0$$

splits, hence $I^m = I^{m+1}$.

THEOREM 2. (a) *Let $I_1, \cdots, I_n$ be ideals such that $I_1 I_2 \cdots I_n = 0$. Then* $\mathrm{lgld}\, R \leqq \max_i \mathrm{lgld}\, R/I_i + \mathrm{l.hd}_R R/I_i$

(b) *Let I be any ideal contained in the left socle of R. Then* $\mathrm{lgld}\, R \leqq \mathrm{lgld}\, R/I + \mathrm{l.hd}_R R/I$.

Proof. (a) Let M be any left R-module. Consider the filtration $0 = I_1 I_2 \cdots I_n M \subset I_2 \cdots I_n M \subset \cdots \subset I_n M \subset M$. By Auslander's lemma

$$\begin{aligned} \mathrm{l.hd}_R\, M &\leqq \max_i \mathrm{l.hd}_R\, (I_{i+1} \cdots I_n M / I_i \cdots I_n M) \\ &\leqq \max_i \mathrm{l.hd}_R\, R/I_i + \mathrm{lgld}\, R/I_i\,. \end{aligned}$$

(b) $\mathrm{lgld}\, R = \sup \mathrm{l.hd}_R\, R/L$ where L runs through the left ideals of R. Since every left ideal is a direct summand of an essential left ideal (by Zorn), we may sup over only the essential left ideals. But I is contained in every essential left ideal, and so annihilates R/L.

COROLLARY 1. *If R is a semi-primary ring of finite global dimension, then the global dimension of R/I is finite for every projective ideal I.*

COROLLARY 2. *If R has finite global dimension and I is a projective nilpotent ideal, then* $\mathrm{lgld}\, R/I^2 = n < \infty$, $I^{n+1} = 0$, *and* $\mathrm{lgld}\, R \leqq n + 1$.

COROLLARY 3. *Let R have finite global dimension and let I be a projective ideal. Then* $\mathrm{lgld}\, R/I^k < \infty\ \forall k$ *if and only if I is eventually idempotent. This characterizes those rings for which* $\mathrm{lgld}\, R/I$ *is finite for every projective ideal I as being those rings of finite global dimension whose projective ideals are eventually idempotent.*

We remark that if I is any ideal of a ring R such that $\mathrm{lgld}\, R/I = 0$ and $\mathrm{lgld}\, R/I^k < \infty\ \forall k$, then I is eventually idempotent by results of Chase [1].

COROLLARY 4. *If R is semi-prime, then*

$$\mathrm{lgld}\, R/\mathrm{l.Soc}\, R \leqq \mathrm{lgld}\, R \leqq \mathrm{lgld}\, R/\mathrm{l.Soc}\, R + 1$$

COROLLARY 5. *Let T be an arbitrary triangular matrix ring* [1], *i.e.,* $T = \begin{pmatrix} R & M \\ O & S \end{pmatrix}$ *where M is an (R, S)-bimodule. Then*

$$\max\, (\mathrm{lgld}\, R, \mathrm{lgld}\, S) \leqq \mathrm{lgld}\, T \leqq \max \begin{cases} \mathrm{lgld}\, S + \mathrm{l.hd}_R\, M + 1 \\ \mathrm{lgld}\, R \end{cases}.$$

(To obtain the lower bound, we observe that $I = \begin{pmatrix} R & M \\ O & O \end{pmatrix}$ is right T-projective, $I = I^2$, and $T/I \cong S$; $J = \begin{pmatrix} O & M \\ O & S \end{pmatrix}$ is left T-projective, $J = J^2$, and $T/J \cong R$. For the upper bound, observe that $JI = 0$ and apply Theorem 2 (a).)

A similar formula for rtgld and wgld may be obtained. In particular, when S is semi-simple,

$$\operatorname{lgld} T = \max \begin{cases} 1 + \text{l.hd}_R M \\ \operatorname{lgld} R \end{cases}, \qquad \operatorname{wgld} T = \max \begin{cases} 1 + \text{l.wd}_R M \\ \operatorname{wgld} R \end{cases}$$

and $\operatorname{rtgld} R \leqq \operatorname{rtgld} T \leqq \operatorname{rtgld} R + 1$.

2. It is a well known result of Cartan and Eilenberg that the right global dimension of a right noetherian local ring is equal to the left weak dimension of its residue class field. Can "left weak" be replaced by "right homological"? The answer is no by the following example, which is based on a construction that has been recently used by P. M. Cohn [2] and Jategaonkar [4].

(This example also shows that the right global dimension of a right Noetherian ring may be greater than $1 + \sup \text{rt.hd}\, L$ where L runs over the maximal right ideals. This is of course in contrast to the commutative case.)

Let $R = F[[x]]$ where F is a field chosen large enough so that R possesses an injective endomorphism σ whose image is contained in F. Let $S = R[[x_1, x_2; \sigma]]$, i.e., the twisted right power series ring in two commuting variables over R. We claim that S is the required example:

(i) S is right noetherian.

Proof. $T = R[[x_1; \sigma]]$ is a right P.I.D. ([2] or [4]). Now use the usual commutative proof of the Hilbert Basis Theorem to prove that $T[x_2; \sigma]$ is right noetherian.

(*Note*: It is crucial that if $f \in T$, $f = \sum_{i \geqq n} x_1^i r_i$, then $f \cdot x_2 = x_2 x_1^n u$ where u is a unit.)[2]

That S is right noetherian now follows as in the commutative case.

(ii) S is local and the maximal ideal of S is generated on the right by x.

(iii) $\text{rt.gld}\, S = 2$.

Proof. It is clear that $\text{rt.hd}_S (x_1 S + x_2 S) = 1$. Now Sx_2S is left S-flat[3] and S/Sx_2S is a right P.I.D.; hence by Small [5] we see that $\text{rt.gld}\, S \leqq 2$ and the proof is complete.

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[2] Neither $R[x_1, x_2; \sigma]$ nor $R[[x_1, x_2, x_3; \sigma]]$ need be right Noetherian—consider the right ideal generated by $\{x_2^n(1+x_1x)\}$ $n = 0,1,2, \cdots$, and $\{x_2^n(x_3 + x_1x)\}$, respectively.

[3] Because $Sx_2S \cong S \otimes_T Tx_2T$.

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