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ORTHOGONAL GROUPS OF POSITIVE DEFINITE MULTILINEAR FUNCTIONALS

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Let V be a finite dimensional vector space over the real numbers R and let $T: V \to V$ be a linear transformation. If $\varphi: \times_1^m V \to R$ is a real multilinear functional and

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m),$$

$x_1, \cdots, x_m \in V$, T is called an isometry with respect to φ. We say φ is positive definite if $\varphi(x, \cdots, x) > 0$ for all nonzero $x \in V$. In this paper we prove that if φ is positive definite and T is an isometry with respect to φ, then all eigenvalues of T have modulus one and all elementary divisors of T over the complex numbers are linear.

Let V be an n-dimensional vector space over the real numbers R. Let T: $V \to V$ be a linear transformation of V. The following theorem [1, Th. 3] is easy to prove:

THEOREM 1. *There exists a positive definite symmetric quadratic form $\varphi: V \times V \to R$ such that*

$$\varphi(Tx, Ty) = \varphi(x, y),\ x, y \in V \tag{1}$$

if and only if

(2)
1. *all eigenvalues of T have modulus* 1;
2. *all elementary divisors of T over the complex numbers C are linear.*

Moreover, if T satisfies (2), *then there is a positive definite symmetric φ such that* (1) *holds.*

Theorem 1 can also be expressed in matrix theoretic terms. If A is a real $n \times n$ positive definite symmetric matrix and X is any automorph of A;

$$X^T A X = A\ , \tag{3}$$

then X satisfies (2); moreover, if an $n \times n$ matrix X satisfies (2), then there is a positive definite symmetric A such that (3) holds.

Let $\varphi: \times_1^m V \to R$ be a real multilinear functional. Let H be a subgroup of the symmetric group S_m. If

$$\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = \varphi(x_1, \cdots, x_m) \tag{4}$$

for all $\sigma \in H$ and all $x_i \in V$, $i = 1, \cdots, m$, then φ is said to be symmetric with respect to H. If

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots x_m) \tag{5}$$

for all $x_1, \cdots, x_m \in V$, T is called an isometry of V with respect to φ. (Note that if $m > 2$, (5) has no matrix analogue). Let $\Omega_m(H, T)$ be the set of all φ satisfying (4) and (5). Clearly $\Omega_m(H, T)$ is a subspace of the vector space of all multilinear functionals symmetric with respect to H. We say φ is positive definite if

$$\varphi(x, \cdots, x) > 0 \tag{6}$$

for all nonzero x in V. The set of all positive definite φ in $\Omega_m(H, T)$ is denoted by $P_m(H, T)$. It is clear that $P_m(H, T)$ is a (possibly empty) convex cone in $\Omega_m(H, T)$.

The following result [1] was proved as a partial generalization of Theorem 1.

THEOREM 2. *Let T: $V \to V$ be linear. If $P_m(H, T)$ is nonempty, then*

(a) *m is even*

(b) *every eigenvalue γ of T has modulus 1*

(c) *elementary divisors of T corresponding to $\gamma = \pm 1$ are linear.*

Conversely, if m is even, all eigenvalues of T are ± 1, and all elementary divisors of T are linear, then $P_m(H, T)$ is nonempty.

We conjectured that if $P_m(H, T)$ is nonempty,then (c) can be replaced by (c') "all elementary divisors of T over the complex field are linear." This would provide a complete generalization of Theorem 2, and thus justify (6) as a definition of a positive definite multilinear functional. The purpose of this paper is to prove this conjecture.

THEOREM 3. *If $P_m(H, T)$ is nonempty, then*

(a) *m is even*

(b) *all eigenvalues of T have modulus 1*

(c′) *all elementary divisors of T over C are linear.*

Conversely, if (a), (b), *and* (c′) *hold, then $P_m(H, T)$ is nonempty.*

2. Proof of Theorem 3. Assume that $P_m(H, T)$ is nonempty. Parts (a) and (b) follow from Theorem 2. We now prove two lemmas.

LEMMA 1. *If γ is an eigenvalue of T and $(x - \gamma)^k$, $k > 1$, is a nonlinear elementary divisor of T corresponding to γ, then $\gamma^m \neq 1$ for any integer m.*

Proof. Since T is a real transformation, it has a real elementary divisor

$$[(x-\gamma)(x-\bar{\gamma})]^k \,. \tag{7}$$

(By Theorem 2, γ cannot be real in this case.) Let W be the invariant subspace of T determined by (7), and let S be the restriction of T to W. Then S is an isometry of W with respect to φ, and hence S^r is also an isometry for any integer r. Now if $\gamma^r = 1$, then all eigenvalues of S^r are 1, and hence Theorem 2 implies that all elementary divisors of S^r are linear. Therefore, S^r is the identity on W, and thus, the elementary divisors of S are linear, a contradiction.

LEMMA 2. *If Theorem* 3 *is true for the case* $H = S_m$, *then it is true for any subgroup* H *of* S_m.

Proof. Let H be a subgroup of S_m and let $\varphi \in P_m(H, T)$. For each $\sigma \in S_m$, define

$$\varphi_\sigma(x_1, \cdots, x_m) = \varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) \,, \tag{8}$$

$x_1, \cdots, x_m \in V$. In general, φ_σ is not symmetric with respect to H, but φ_σ is positive definite and T is an isometry with respect to φ_σ. Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma \,. \tag{9}$$

Clearly ψ is positive definite, and T is an isometry with respect to ψ. Moreover, for any $\tau \in S_m$, and $x_1, \cdots, x_m \in V$,

$$\begin{aligned}
\psi(x_{\tau(1)}, \cdots, x_{\tau(m)}) &= \sum_{\sigma \in S_m} \varphi_\sigma(x_{\tau(1)}, \cdots, x_{\tau(m)}) \\
&= \sum_{\sigma \in S_m} \varphi(x_{\tau\sigma(1)}, \cdots, x_{\tau\sigma(m)}) \\
&= \sum_{\mu \in S_m} \varphi(x_{\mu(1)}, \cdots, x_{\mu(m)}) \\
&= \sum_{\mu \in S_m} \varphi_\mu(x_1, \cdots, x_m) \\
&= \psi(x_1, \cdots, x_m) \,.
\end{aligned}$$

Thus $\psi \in P_m(S_m, T)$, and hence the elementary divisors of T are linear. This proves Lemma 2.

We may assume henceforth that $H = S_m$ and abbreviate $P_m(S_m, T)$ to P_m. If P_m is nonempty, and T has a nonlinear elementary divisor over C corresponding to the eigenvalue $\gamma = a + ib$ ($b \neq 0$), then there exist four linearly independent vectors $v_1, \cdots, v_4$ in V such that

$$
\begin{aligned}
Tv_1 &= av_1 - bv_2 \\
Tv_2 &= bv_1 + av_2 \\
Tx_3 &= v_2 + av_3 - bv_4 \\
Tv_4 &= bv_3 + av_4 \,.
\end{aligned} \tag{10}
$$

Let $\bar{V}$ be the extension of V to an n-dimensional space over C, i.e., $\bar{V}$ consists of all vectors of the form $x + iy$, $x, y \in V$. By linear extension, we regard T as a linear transformation of $\bar{V}$, and by multilinear extension, φ becomes a complex valued multilinear functional on $\times_1^m \bar{V}$. Equation (5) still holds in $\bar{V}$, but φ is no longer positive definite. Set

$$
\begin{aligned}
e_1 &= v_1 + iv_2,\ e_2 = v_1 - iv_2 \\
e_3 &= v_3 + iv_4,\ e_4 = v_3 - iv_4 \,.
\end{aligned} \tag{11}
$$

From (10) and (11),

$$
\begin{aligned}
Te_1 &= \gamma e_1 \,, \quad Te_2 = \bar{\gamma} e_2 \\
Te_3 &= \gamma e_3 + v_2,\ Te_4 = \bar{\gamma} e_4 + v_2 \,.
\end{aligned} \tag{12}
$$

By Lemma 1, γ is not a root of unity; thus,

$$
\begin{aligned}
\varphi(e_1, \cdots, e_1, e_2, \cdots e_2) &= \varphi(Te_1, \cdots, Te_1, Te_2, \cdots, Te_2) \\
&= \gamma^k \bar{\gamma}^{m-k} \varphi(e_1, \cdots e_1, e_2, \cdots, e_2) \\
&= 0 \,,
\end{aligned} \tag{13}
$$

unless $k = m - k$, where k is the number of times e_1 occurs in (13). With $r = m/2$, we set

$$\varphi(e_1, \overset{r}{\cdots}, e_1, e_2, \overset{r}{\cdots}, e_2) = \nu \,.$$

Now $\nu \neq 0$; otherwise

$$
\begin{aligned}
\varphi(v_1, \cdots, v_1) &= 2^{-m} \varphi(e_1 + e_2, \cdots, e_1 + e_2) \\
&= 0 \,,
\end{aligned} \tag{14}
$$

contradicting (6). (Note that we are using the assumption that φ is symmetric with respect to S_m; this gives us a convenient way of sorting expressions such as those on the right side of (14).)

Let $\mu = \varphi(v_1, \cdots, v_1, e_3)$. Using (13) and (14), we compute,

$$
\begin{aligned}
\mu &= 2^{-m+1} \varphi(e_1 + e_2, \cdots, e_1 + e_2, e_3) \\
&= 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots, \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + v_2) \\
&= 2^{-m+1} \varphi\left(\gamma e_1 + \bar{\gamma} e_2, \cdots \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + \frac{e_1 - e_2}{2i}\right) \\
&= -2^{-m} i \binom{m-1}{r} (\bar{\gamma} - \gamma) \nu + \gamma 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots \gamma e_1 + \bar{\gamma} e_2, e_3)
\end{aligned}
$$

$$= -2^{-m}i\binom{m-1}{r}(\bar{\gamma} - \gamma)\nu + \gamma 2^{-m+1}$$
$$\varphi\left(\gamma^2 e_1 + \bar{\gamma}^2 e_2, \cdots, \gamma^2 e_1 + \bar{\gamma}^2 e_2, \gamma e_3 + \frac{e_1 - e_2}{2i}\right)$$
$$= -2^{-m}i\binom{m-1}{r}(2\bar{\gamma} - \gamma - \gamma^3)\nu + \gamma^2 2^{-m+1}$$
$$\varphi(\gamma^2 e_1 + \bar{\gamma}^2 e_2, \cdots, \gamma^2 e_1 + \bar{\gamma}^2 e_2, e_3) .$$

Continuing this procedure, we obtain for any positive integer s

$$(15)\quad \mu = -2^{-m}i\binom{m-1}{r}\left(s\bar{\gamma} - \sum_{j=0}^{s-1}\gamma^{2j+1}\right)\nu + \gamma^s 2^{-m+1}$$
$$\varphi(\gamma^s e_1 + \bar{\gamma}^s e_2, \cdots \gamma^s e_1 + \bar{\gamma}^s e_2, e_3) .$$

Let

$$f(z) = z\varphi(ze_1 + \bar{z}e_2, \cdots, ze_1 + \bar{z}e_2, e_3) ,$$

where z is a complex variable. Then f is a continuous function of z on the complex plane, and hence f is bounded on the unit circle. Moreover, since γ is not a root of unity (in particular, $\gamma \neq \pm 1$),

$$\sum_{j=0}^{s-1}\gamma^{2j-1}$$

is also bounded as s becomes large. Thus, letting s approach infinity in (15) forces μ to become infinite, a contradiction. This proves Theorem 3 in one direction.

Now suppose all eigenvalues of T are 1 in absolute value and all elementary divisors of T are linear over C. Let 1 (p times), -1 (q times) and $\gamma_j, \bar{\gamma}_j = a_j \pm ib_j, |\gamma_j| = 1, j = 1, \cdots, t$, be the eigenvalues of T. Then there is a basis of V, $v_1, \cdots, v_p, u_1, \cdots, u_q, x_1, y_1, \cdots x_t, y_t$ such that

$$(16)\quad \begin{aligned} Tv_j &= v_j, j = 1, \cdots, p \\ Tu_j &= -u_j, j = 1, \cdots, q \\ Tx_j &= a_j x_j - b_j y_j, j = 1, \cdots, t \\ Ty_j &= b_j x_j + a_j y_j, j = 1, \cdots, t . \end{aligned}$$

Set

$$\begin{aligned} w_j &= x_j + iy_j \\ \bar{w}_j &= x_j - iy_j, j = 1, \cdots, t . \end{aligned}$$

Then $v_1, \cdots, v_p, u_1, \cdots, u_q, w_1, \bar{w}_1, \cdots, w_t, \bar{w}_t$ form a basis of $\bar{V}$ of eigenvectors of T. Let $f_1, \cdots, f_p, g_1, \cdots, g_q, h_1, k_1, \cdots, h_t, k_t$ be the corresponding dual basis. If $l_1, \cdots, l_m$ are linear functionals on a space V, then $l_1 \cdots l_m$ is the m-linear functional on $\times_1^m V$ such that

$$l_1 \cdots l_m(x_1, \cdots, x_m) = \prod_{i=1}^m l_i(x_i) .$$

Define φ as follows:

$$\varphi = \sum_{j=1}^p f_j^m + \sum_{j=1}^q g_j^m + \sum_{j=1}^t [(h_j k_j)^r + (\bar{h}_j \bar{k}_j)^r] , \tag{17}$$

where $r = m/2$ and $\bar{f}(v) = \overline{f(v)}$. Now $\bar{h}_j$ and $\bar{k}_j$ are not linear on the complex space $\bar{V}$, but they are complex valued linear functionals on V, i.e., they are linear functionals on V but are not in the dual space of V. Thus φ is a real multilinear functional on V. Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma .$$

We assert that $\psi \in P_m(H, T)$. Clearly ψ is symmetric with respect to S_m, and thus with respect to any subgroup H of S_m. It remains to show that ψ is positive definite and that T is an isometry with respect to ψ. It suffices to prove these last two properties for φ. Let

$$x = \sum_{j=1}^p \alpha_j v_j + \sum_{j=1}^q \beta_j u_j + \sum_{j=1}^t (\delta_j x_j + \lambda_j y_j)$$

be an arbitrary vector of V. Then from (17),

$$\varphi(x, \cdots, x) = \sum_{j=1}^p \alpha_j^m + \sum^q \beta_j^m + 2\sum_{j=1}^t \left[\left(\frac{\delta_j}{2}\right)^2 + \left(\frac{\lambda_j}{2}\right)^2\right]^r .$$

Since m is even and $\alpha_j, \beta_j, \delta_j, \lambda_j$ are all real, φ is positive definite. Now let z_k, $k = 1, \cdots, m$, be arbitrary vectors in V, with

$$z_k = \sum_{j=1}^p a_{kj} v_j + \sum_{j=1}^q b_{kj} u_j + \sum_{j=1}^t (c_{kj} x_j + d_{kj} y_j) . \tag{18}$$

Then

$$\begin{aligned} \varphi(z_1, \cdots, z_m) = & \sum_{j=1}^p \prod_{k=1}^m a_{kj} + \sum_{j=1}^q \prod_{k=1}^m b_{kj} \\ & + \sum_{j=1}^t \prod_{k=1}^r \left(\frac{c_{2k-1,j}}{2} + \frac{d_{2k-1,j}}{2i}\right)\left(\frac{c_{2k,j}}{2} - \frac{d_{2k,j}}{2i}\right) \\ & + \sum_{j=1}^t \prod_{k=1}^r \left(\frac{c_{2k-1,j}}{2} - \frac{d_{2k-1,j}}{2i}\right)\left(\frac{c_{2k,j}}{2} + \frac{d_{2k,j}}{2i}\right) . \end{aligned} \tag{19}$$

From (16)

$$\begin{aligned} Tz_k = & \sum_{j=1}^p a_{kj} v_j + \sum_{j=1}^q (-b_{kj}) u_j \\ & + \sum_{j=1}^t (a_j c_{kj} + b_j d_{kj}) x_j + (a_j d_{kj} - b_j c_{kj}) y_j , \end{aligned} \tag{20}$$

$k = 1, \cdots, m$. Let

$$\begin{aligned} e_{kj} &= a_j c_{kj} + b_j d_{kj} \\ f_{kj} &= a_j d_{kj} - b_j c_{kj} \,. \end{aligned}$$

Then from (19) and (20)

$$\begin{aligned} \varphi(Tz_1, \cdots, Tz_m) &= \sum_{j=1}^{p} \prod_{k=1}^{m} a_{kj} + \sum_{j=1}^{q} \prod_{k=1}^{m} (-b_{kj}) \\ &+ \sum_{j=1}^{t} \prod_{k=1}^{m} \left(\frac{e_{2k-1,j}}{2} + \frac{f_{2k-1,j}}{2i} \right) \left(\frac{e_{2k,j}}{2} - \frac{f_{2k,j}}{2i} \right) \\ &+ \sum_{j=1}^{t} \prod_{k=1}^{m} \left(\frac{e_{2k-1,j}}{2} - \frac{f_{2k-1,j}}{2} \right) \left(\frac{e_{2k,j}}{2} + \frac{f_{2k,j}}{2i} \right) . \end{aligned} \tag{21}$$

It is easily verified that

$$\begin{aligned} \frac{e_{kj}}{2} + \frac{f_{kj}}{2i} &= \bar{\gamma}_j \left(\frac{c_{kj}}{2} + \frac{d_{kj}}{2i} \right) \\ \frac{e_{kj}}{2} - \frac{f_{kj}}{2i} &= \gamma_j \left(\frac{c_{kj}}{2} - \frac{d_{kj}}{2i} \right) . \end{aligned} \tag{22}$$

Using (22) in (21) and the fact that $|\gamma_j| = 1$, we obtain

$$\varphi(Tz_1, \cdots, Tz_m) = \varphi(z_1, \cdots, z_m) \,.$$

This completes the proof of Theorem 3.

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