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PRINCIPAL MULTIPLICATIVE LATTICES

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P. J. McCarthy has recently proved that if R is a Noetherian ring with unity, then every ideal of R is a principal element of $L(R)$, the lattice of ideals of R, if and only if R is a multiplication ring. It is shown here that an arbitrary commutative ring R with unity is a Noetherian multiplication ring if and only if every ideal of R is a principal element of $L(R)$.

1. *M*-lattices. The basic terminology and notation will follow that of [2] and [3]. It will be assumed throughout the paper that L denotes a complete commutative and residuated multiplicative lattice. The results of this section, though well known, have apparently not been previously published. They are included here because they are needed to prove the results of § 2.

THEOREM 1. *The following conditions are equivalent:*
(1) *L is an M-lattice.*
(2) $(A:B)B = A \cap B$ *for all* $A, B \in L$.
(3) *Every element of L is meet principal.*

Proof. (1)⇒(2) Let $A, B \in L$. Since $A \cap B \leqq B$, we have $A \cap B = BC$ for some $C \in L$. Then $BC \leqq A \Rightarrow C \leqq A : B$, so we have

$$A \cap B = BC \leqq B(A:B) \leqq A \cap B .$$

It follows that $(A:B)B = A \cap B$.

(2)⇒(3) Given $A, B, M \in L$ we need only note that

$$AM \cap B = (B:AM)AM = \{[(B:M):A]A\}M = [A \cap (B:M)]M .$$

(3)⇒(1) If $AM \cap B = [A \cap (B:M)]M$ for all $A, B \in L$ and if $B \leqq M$ we have

$$B = IM \cap B = [I \cap (B:M)]M = (B:M)M .$$

THEOREM 2. *Every M-lattice is infinitely distributive.*

Proof. Let $M = \cup_\alpha M_\alpha$ in the M-lattice L. Then for any $B \in L$, $B \cap M$ is an upper bound for $\{B \cap M_\alpha\}$. Let X be any other upper bound for this set. Then for all α,

$$X : M_\alpha \geqq (B \cap M_\alpha) : M_\alpha = B : M_\alpha .$$

It is easily deduced that

$$X : M = \cap_\alpha (X : M_\alpha) \geqq \cap_\alpha (B : M_\alpha) = B : M .$$

By Theorem 1 we have

$$X \geqq (X : M)M \geqq (B : M)M = B \cap M .$$

It follows that $B \cap M = \cup_\alpha (B \cap M_\alpha)$.

THEOREM 3. *Let L be an M-lattice. Then* $(\cup_\alpha A_\alpha) : M = \cup_\alpha (A_\alpha : M)$ *holds for all* $A_\alpha \in L$ *and for every principal element M of L.*

Proof. Let $A = \cup_\alpha A_\alpha$. By Theorem 2, $A \cap M = \cup_\alpha (A_\alpha \cap M)$. Hence $A : M = (A \cap M) : M = [\cup_\alpha (A_\alpha \cap M)] : M = [\cup_\alpha (A_\alpha : M)M] : M = \{[\cup_\alpha (A_\alpha : M)]M\} : M = [\cup_\alpha (A_\alpha : M)] \cup (0 : M) = \cup_\alpha (A_\alpha : M)$.

2. Principal lattices. Though our goal is to investigate the lattice of ideals of a commutative ring with unity, it will cost us nothing to begin our discussion in the context of an M-lattice. In connection with this, it will prove convenient to call L a *principal lattice* when each of its elements is principal.

LEMMA 4. *Let L be an M-lattice whose unit element is compact. Every principal element of L is then compact.*

Proof. Let M be a principal element of L and assume that $M \leqq \cup_\alpha B_\alpha$. Then by Theorem 3,

$$I = (\cup_\alpha B_\alpha) : M = \cup_\alpha (B_\alpha : M) .$$

Since I is compact there must exist finitely many indices $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that $I = \bigcup_{i=1}^k (B_{\alpha_i} : M) = (\bigcup_{i=1}^k B_{\alpha_i}) : M$. It follows that $M \leqq \bigcup_{i=1}^k B_{\alpha_i}$ as desired.

THEOREM 5. *Let L be an M-lattice whose unit element is compact. Suppose further that every element of L is the join of a family of principal elements. An element C of L is then compact if and only if it is principal.*

Proof. By Lemma 4 every principal element of L is compact. On the other hand, if C is compact then it is the join of a finite number of principal elements. It follows from the argument given in [3] that any such element is principal.

THEOREM 6. *Suppose that every element of L is the join of a family of principal elements and that the unit element of L is compact. The necessary and sufficient condition that L be a principal* *lattice is that it be an M-lattice satisfying the ascending chain condition.*

Proof. Assume first that L is a principal lattice. By Theorem 1 it is an M-lattice, and by Lemma 4 each of its elements is compact. It follows that L satisfies the ascending chain condition. For the converse, see [3], Theorem 1, p. 706.

COROLLARY. *Let L be an M-lattice satisfying the conditions of the theorem. Assume further that $0 : A = 0$ for all $A \neq 0$ in L. The ascending chain condition then holds in L.*

Proof. Let $M \in L$ and write $M = \cup_\alpha M_\alpha$ with each M_α principal. It was shown in [3], p. 707 that for any $B \in L$,

$$BM : M \leqq B \cup (0 : M_\alpha) = B .$$

It follows that $B = BM : M = B \cup (0 : M)$ so that by [1], Theorem 1, p. 215, M is principal,

When viewed in the context of ring theory these results translate to the following:

THEOREM 5*. *Let R be a multiplication ring with unity. An ideal of R is a principal element of $L(R)$ if and only if it is finitely generated. The necessary and sufficient condition that $L(R)$ be a principal lattice is that R be Noetherian.*

THEOREM 6*. *Let R be a commutative ring with unity. The necessary and sufficient condition that $L(R)$ be a principal lattice is that R be a Noetherian multiplication ring.*

The corollary to Theorem 6 translates to the well known fact that an integral domain is a multiplication ring if and only if it is Dedekind.

In closing we mention a few easily established facts about the lattice of ideals of an arbitrary commutative ring with unity. First of all, an obvious modification of the proof of Lemma 4 will show that every principal element of $L(R)$ is finitely generated, thus answering a question posed by P. J. McCarthy ([4], p. 269). At this point it is easily shown that $L(R)$ satisfies Postulate C of Ward (see [5], *p.* 631) if and only if R is a multiplication ring with minimum condition, and that $L(R)$ is a Boolean algebra if and only if R is a semiprime multiplication ring with minimum condition.

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