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ON WITT'S THEOREM FOR UNIMODULAR QUADRATIC FORMS, II

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An integral generalization of Witt's theorem for unimodular quadratic forms over the ring of integers in a local field is established.

1. In the first part of this paper [1] we established a Witt theorem for unimodular quadratic forms over the rational integers, provided the signature of the form was sufficiently small. We shall now use these methods to obtain a similar theorem for arbitrary unimodular quadratic forms over the ring of integers in a local field in which 2 is a prime. These theorems are important because they enable us to determine the essentially distinct representations of a quadratic form by a unimodular form. We hope to expand on this in a later paper.

Let F be a local field in which 2 is a prime, $\mathfrak{o}$ the ring of integers in F and $\mathfrak{u}$ the group of units in $\mathfrak{o}$. We need only assume that the residue class field $\mathfrak{o}/2\mathfrak{o}$ is perfect. We preserve as much of the notation in [1] as possible, but now the underlying ring will be $\mathfrak{o}$ and not the rational integers Z. Thus L will be a free $\mathfrak{o}$-module of finite rank, endowed with a bilinear symmetric unimodular form $\Phi: L \times L \rightarrow \mathfrak{o}$. We denote $\Phi(\alpha, \beta)$ by $\alpha \cdot \beta$. Details on the structure of L are contained in O'Meara [2, 3]. We recall that L is *improper* if $\alpha^2 \in 2\mathfrak{o}$ for all $\alpha \in L$; otherwise L is *proper*.

A vector $\alpha \in L$ is called *primitive* if $\alpha = 2\beta$, with $\beta \in L$, is impossible. As in Wall [5] and our earlier paper [1], the crucial concept is that of a characteristic vector. We only define these when L is a proper lattice; in this case L has an orthogonal basis, that is $L = \langle \xi_1 \rangle \oplus \cdots \oplus \langle \xi_n \rangle$. A vector $\alpha = \sum_{i=1}^{n} a_i \xi_i \in L$ is called *characteristic* if its orthogonal complement $\langle \alpha \rangle^{\perp}$ contains no vectors of unit norm. If α is primitive, this is equivalent to

$$a_i^2 \xi_i^2 \equiv a_j^2 \xi_j^2 \pmod 2 , \qquad 1 \leqq i, j \leqq n .$$

Hence, in particular, $a_i \in \mathfrak{u}$, $1 \leqq i \leqq n$, and this reduces to the definition in [1]. If α is a primitive characteristic vector, we define $T(\alpha) \in \mathfrak{o}/2\mathfrak{o}$ by $T(\alpha) \equiv a_i^2 \xi_i^2 \pmod 2$. This definition is independent of the basis of L (see also Trojan [4]). If $\langle \alpha \rangle^{\perp}$ is proper, or if L is improper, we define $T(\alpha) = 0$; also let $T(2^s \alpha) = T(\alpha)$ for $s \geqq 0$. We shall prove the following.

THEOREM. *Let $\varphi: J \rightarrow K$ be an isometry between the primitive* *sublattices J and K of L. Then φ extends to an isometry of L if and only if $T(\alpha) = T(\varphi(\alpha))$ for all $\alpha \in J$.*

When the rank of J is 1, this is the same as Theorem 2.1 of Trojan [4]. We shall recover this as a special case. For local fields in which 2 is a unit the theorem remains true, but there is no need to consider characteristic vectors. Essentially the following proof of the theorem goes through in a much simpler manner.

2. We first reduce to the case where L has maximal Witt index (that is, the space FL is an orthogonal sum of hyperbolic planes). We adjoin a unimodular lattice U to L so that $L' = L \oplus U$ has maximal Witt index. Thus, if $L = H_1 \oplus \cdots \oplus H_m \oplus \langle \xi_1 \rangle \oplus \cdots \oplus \langle \xi_s \rangle$ where $H_1, \cdots, H_m$ are hyperbolic planes, we take $U = \langle \zeta_1 \rangle \oplus \cdots \oplus \langle \zeta_s \rangle$ where $\zeta_i^2 = -\xi_i^2$, $1 \leqq i \leqq s$. Let $J' = J \oplus U$, $K' = K \oplus U$ and extend φ to J' by defining $\varphi(\zeta_i) = \zeta_i$. A similar extension is done if L is improper, but now U may be taken as an improper lattice (see the classification of unimodular lattices in O'Meara [3, p. 852]). We observe that $T(\alpha) = T(\varphi(\alpha))$ for all $\alpha \in J'$. If L' is improper, this is trivial. If L is proper (and $U \neq \{0\}$), then no vector $\alpha \in J$ will be characteristic in L'. However, new characteristic vectors may be created. Thus, if $\alpha \in J$ is characteristic in L, and $T(\alpha) \equiv a \pmod 2$ where $a \in \mathfrak{u}$, then $\alpha' = \alpha + \sum_{i=1}^{s} u_i \zeta_i$ is characteristic in L' if $u_i \in \mathfrak{u}$ are chosen such that $u_i^2 \zeta_i^2 \equiv a \pmod 2$. Clearly $T(\alpha') = T(\varphi(\alpha'))$. If we prove the theorem for lattices of maximal Witt index, it holds for L', and restricting the extension of φ back to L gives the general result.

We may now assume that L has the form

$$L = H_1 \oplus \cdots \oplus H_m \oplus B$$

where $H_i = \langle \lambda_i, \mu_i \rangle$, $1 \leqq i \leqq m$, are hyperbolic planes, and $B = \langle \xi, \rho \rangle$ where $\xi^2 = d$, $\xi \cdot \rho = 1$ and $\rho^2 = 0$. If L is improper, we may take $d = 0$; otherwise $d \in \mathfrak{u}$.

3. The proof will be by induction on the rank $r(J)$ of J. We consider now $r(J) = 1$. Let $J = \langle \alpha \rangle$ and $\varphi(\alpha) = \beta \in K$. Let

$$\alpha = \sum_{i=1}^{m} (a_i \lambda_i + b_i \mu_i) + u\xi + v\rho . \tag{1}$$

Case 1. If $\alpha^2 \in \mathfrak{u}$, then u (and d) are units. Apply the isometry

$$\theta_1 \colon \langle \lambda_i, \mu_i \rangle \oplus \langle \xi, \rho \rangle \rightarrow \langle \lambda_i, \mu_i + x\rho \rangle \oplus \langle \xi - x\lambda_i, \rho \rangle$$

where $x = a_i/u \in \mathfrak{o}$. Then

$$\theta_1(a_i\lambda_i + b_i\mu_i + u\xi + v\rho) = b_i\mu_i + u\xi + (v + xb_i)\rho .$$

After applying a succession of such isometries we may assume $\alpha = \sum_{i=1}^{m} b_i\mu_i + u\xi + v\rho$. Then

$$L = \langle\alpha, \rho\rangle \oplus \langle u\lambda_1 - b_1\rho, \mu_1\rangle \oplus \cdots \oplus \langle u\lambda_m - b_m\rho, \mu_m\rangle$$

and each $\langle u\lambda_i - b_i\rho, \mu_i\rangle$ is a hyperbolic plane. Doing the same for β, and cancelling hyperbolic planes ([2, 93 : 14]), we may reduce to the case $L = \langle\alpha\rangle \oplus \langle\alpha_1\rangle = \langle\beta\rangle \oplus \langle\beta_1\rangle$, where the result is obvious by considering the determinant of L.

Case 2. Now suppose $\alpha^2 \notin \mathfrak{u}$, but that at least one of a_i, b_i, $1 \leqq i \leqq m$, is a unit, say $a_1 \in \mathfrak{u}$. Then

$$L = \langle\alpha, \mu_1\rangle \oplus U \tag{2}$$

with $\langle\alpha, \mu_1\rangle$ a hyperbolic plane. If we can also obtain

$$L = \langle\beta, \mu\rangle \oplus V \tag{3}$$

with $\langle\beta, \mu\rangle$ a hyperbolic plane, then $U \cong V$, and we are reduced to considering $\alpha, \beta \in H = \langle\lambda, \mu\rangle$. Write $\alpha = a\lambda + b\mu$, $\beta = a'\lambda + b'\mu$, where without loss of generality we can take $a, a' \in \mathfrak{u}$. $\alpha^2 = \beta^2$ implies $ab = a'b'$. Apply $\langle\lambda, \mu\rangle \to \langle a'/a\lambda, a/a'\mu\rangle$, to complete the proof.

If L is improper, (3) is clear. If L is proper, (2) shows that α and hence β are not characteristic vectors. But if all the coefficients of λ_i and μ_i in β are in $2\mathfrak{o}$, β would be characteristic (see Case 3). Hence we can obtain the splitting (3).

Case 3. Finally suppose $\alpha^2 \in \mathfrak{u}$ and all a_i, b_i in (1) are nonunits. We may assume L is proper, $u \notin \mathfrak{u}$ and $v \in \mathfrak{u}$.

$$\langle\lambda_i, \mu_i\rangle \oplus \langle\xi, \rho\rangle \to \langle\lambda_i, \mu_i - 2x(\xi - d\rho) + 2dx^2\lambda_i\rangle \oplus \langle\xi, \rho + 2x\lambda_i\rangle$$

can be used to reduce each coefficient a_i of λ_i in (1) to zero. Then

$$L = \langle\alpha, \xi\rangle \oplus \langle b_1(\xi - d\rho) - v\lambda_1, \mu_1, \cdots, b_m(\xi - d\rho) - v\lambda_m, \mu_m\rangle .$$

Since $\langle\alpha, \xi\rangle$ is now isotropic and $\langle\alpha, \xi\rangle^\perp$ is improper, it follows that $\langle\alpha, \xi\rangle^\perp$ is an orthogonal sum of hyperbolic planes. α and β are now characteristic. We therefore have a similar splitting $L = \langle\beta, \xi\rangle \oplus U$, with U a sum of hyperbolic planes. Thus we may reduce to the case $L = \langle\xi, \rho\rangle$ with $\alpha = 2u\xi + v\rho$ and $\beta = 2u_1\xi + v_1\rho$. $T(\alpha) = T(\beta)$ implies $v \equiv v_1 \pmod 2$. If $u_1/u \equiv 1 \pmod 2$, put $c = u_1/u \in \mathfrak{u}$ and apply

$$\langle\xi, \rho\rangle \to \langle c\xi + \tfrac{1}{2}c^{-1}d(1 - c^2)\rho, c^{-1}\rho\rangle ,$$

sending α into β. If $du_1/(du + v) \equiv 1 \pmod 2$, put $c = du_1/(du + v)$

and apply $\langle \xi, \rho \rangle \to \langle c\xi + \frac{1}{2}dc^{-1}(1-c^2)\rho, 2cd^{-1}\xi - c\rho \rangle$, sending α into β. Since $\alpha^2 = \beta^2$, we have $u^2d + uv = u_1^2d + u_1v_1$, from which it follows that one of these two cases must occur. This completes the proof for $r(J) = 1$.

4. Using methods similar to those in [1], we now obtain canonical embeddings of an image of J in L. We only elaborate on the details that are substantially different. We assume $2r(J) \geqq r(L)$; if $2r(J) < r(L)$ it is clear how to modify the arguments that follow.

Let $J = \langle \alpha_1, \cdots \alpha_s \rangle$ where, by eliminating the coefficients of ξ and ρ, we may assume $\alpha_i^2 = 2c_i$ with $c_i \in \mathfrak{o}$ for $1 \leqq i \leqq m$, and none of the α_i, $1 \leqq i \leqq m-1$, are characteristic vectors. As in [1], we may apply isometries to L, and again writing the image of J as

$$J = \langle \alpha_1, \cdots, \alpha_s \rangle ,$$

obtain

$$\begin{gathered} \alpha_1 = \lambda_1 + c_1\mu_1 \\ \cdots \\ \alpha_{m-1} = a_{m-11}, \mu_1 + \cdots + a_{m-1,m-2}\mu_{m-2} + \lambda_{m-1} + c_{m-1}\mu_{m-1} \end{gathered}$$

where $\alpha_i \cdot \alpha_j = a_{ij}$ for $i > j$. Eliminating the coefficients of $\lambda_1, \cdots, \lambda_{m-1}$ we may assume

$$\alpha_m = \sum_{i=1}^{m-1} a_{mi}\mu_i + \zeta \tag{4}$$

where $\zeta \in H_m \oplus B$. If ζ is not primitive, at least one a_{mi} is a unit, say $a_{mk} \in \mathfrak{u}$. We now apply the isometry

$$\begin{aligned} \theta_2\colon & \langle \lambda_k, \mu_k \rangle \oplus \langle \lambda_{k+1}, \mu_{k+1} \rangle \oplus \cdots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \oplus \langle \xi, \rho \rangle \to \\ & \langle \lambda_k + c_k\rho, \mu_k - \rho \rangle \oplus \langle \lambda_{k+1} + a_{k+1,k}\rho, \mu_{k+1} \rangle \oplus \cdots \\ & \oplus \langle \lambda_{m-1} + a_{m-1,k}\rho, \mu_{m-1} \rangle \oplus \langle \xi - c_k\mu_k + \lambda_k - a_{k+1k}\mu_{k+1} \\ & - \cdots - a_{m-1k}\mu_{m-1} + c_k\rho, \rho \rangle . \end{aligned}$$

This leaves fixed each α_i, $1 \leqq i \leqq m-1$, but

$$\theta_2(\alpha_m) = \sum_{i=1}^{m-1} a_{mi}\mu_i - a_{mk}\rho + \theta_2(\zeta) .$$

Use the α_i, $1 \leqq i \leqq m-1$, to eliminate any λ_i, $1 \leqq i \leqq m-1$, occurring in $\theta_2(\alpha_m)$ and obtain a new vector of the form (4), but now ζ is primitive.

There are now two cases to consider.

Case 1. α_m not characteristic and $\alpha_m^2 \in 2\mathfrak{o}$. It is possible that ζ

is characteristic in $H_m \oplus B$. If this is the case, at least one a_{mi} is a unit, and another isometry of the form θ_2, but with $\langle \xi, \rho \rangle$ replaced by $\langle \lambda_m, \mu_m \rangle$, will introduce a term $a_{mi}\mu_m$ into ζ. We may therefore assume ζ is not characteristic, and α_m has the form

$$\alpha_m = \sum_{i=1}^{m-1} a_{mi}\mu_i + \lambda_m + c_m\mu_m$$

(after applying an isometry to $H_m \oplus B$). We may now take

$$\alpha_{m+1} = \sum_{i=1}^{m} a_{m+1i}\mu_i + u\xi + v\rho \ .$$

As above (with ζ), we may arrange that $u\xi + v\rho$ is primitive. First, assume that u is a unit. Then, changing the basis of $\langle \xi, \rho \rangle$ to $\langle u\xi, u^{-1}\rho \rangle$, we may assume $u = 1$. This gives us the canonical embeddiug of $\langle \alpha_1, \cdots, \alpha_{m+1} \rangle$ we desire; all the coefficients a_{ij}, c_i and v are uniquely determined by $\alpha_i \cdot \alpha_j$ and α_i^2, $1 \leqq i, j \leqq m+1$. If now $2r(J) > r(L)$, we eliminate the λ_i and ξ terms in α_{m+2} so that it takes the form

$$\alpha_{m+2} = \sum_{i=1}^{m} b_i\mu_i + b\rho \ .$$

Hence $\alpha_{m+2}^2 = 0$. If $b_k \in \mathfrak{u}$, say, then $\langle \alpha_{m+2}, \alpha_k \rangle$ is a hyperbolic plane splitting L and J. Its image under φ will be a hyperbolic plane splitting L and K. Cancelling these hyperbolic planes reduces the rank of J and we are finished by induction. (The invariants of vectors in the new J and K will still correspond.) If $b_i \in 2\mathfrak{o}$ and $b \in \mathfrak{u}$, then α_{m+2} is characteristic. Also $\alpha_{m+1} \cdot \alpha_{m+2} \in \mathfrak{u}$. In this case $\langle \alpha_{m+1}, \alpha_{m+2} \rangle^{\perp} \cong H_1 \oplus \cdots \oplus H_m$ (since it is improper with maximal Witt index). We may now cancel $\langle \alpha_{m+1}, \alpha_{m+2} \rangle$ with its image and we are again finished by induction.

Now assume $u \in 2\mathfrak{o}$ and hence $\alpha_{m+1}^2 \in 2\mathfrak{o}$. Then changing the basis of $\langle \xi, \rho \rangle$ to $\langle v^{-1}\xi, v\rho \rangle$, we may assume

$$\alpha_{m+1} = \sum_{i=1}^{m} a_{m+1i}\mu_i + 2u\xi + \rho \ .$$

Notice that $\alpha_{m+1}^2 \in 4\mathfrak{o}$, so that if any a_{m+1i} is a unit, say $a_{m+1k} \in \mathfrak{u}$, then $\langle \alpha_k, \alpha_{m+1} \rangle$ is a hyperbolic plane. In this case we can cancel and reduce the rank of J. Thus we may assume all $a_{m+1i} \in 2\mathfrak{o}$, so that if L is proper, α_{m+1} is characteristic. This gives our canonical embedding of $\langle \alpha_1, \cdots, \alpha_{m+1} \rangle$. If now $2r(J) > r(L)$, we eliminate the λ_i and ρ terms in α_{m+2}, so that it takes the form

$$\alpha_{m+2} = \sum_{i=1}^{m} b_i\mu_i + b\xi \ .$$

If $b \in \mathfrak{u}$, then $\alpha_{m+1} \cdot \alpha_{m+2} \in \mathfrak{u}$. $\langle \alpha_{m+1}, \alpha_{m+2} \rangle$ is isotropic since we obtain an isotropic vector by eliminating the ξ term between α_{m+1} and α_{m+2}. Since α_{m+1} is characteristic, it follows that

$$\langle \alpha_{m+1}, \alpha_{m+2} \rangle^{\perp} \cong H_1 \oplus \cdots \oplus H_m .$$

We may therefore cancel $\langle \alpha_{m+1}, \alpha_{m+2} \rangle$ with its image under φ and finish by induction. If $b \notin \mathfrak{u}$, then $\alpha_{m+2}^2 \in 4\mathfrak{o}$. If now $b_k \in \mathfrak{u}$,

$$\langle \alpha_k, \alpha_{m+2} \rangle \cong H ,$$

and may be cancelled with its image. This completes this case.

In summary; we need only consider $2r(J) = r(L)$ and

$$J = \langle \alpha_1, \cdots, \alpha_{m+1} \rangle$$

where

$$\begin{aligned} \alpha_1 &= \lambda_1 + c_1\mu_1 \\ &\cdot \quad \cdot \quad \cdot \\ \alpha_m &= a_{m1}\mu_1 + \cdots + a_{mm-1}\mu_{m-1} + \lambda_m + c_m\mu_m \\ \alpha_{m+1} &= \begin{cases} 2a_{m+11}\mu_1 + \cdots + 2a_{m+1m}\mu_m + 2u\xi + \rho \\ \;a_{m+11}\mu_1 + \cdots + a_{m+1m}\mu_m + \xi + v\rho \end{cases} \end{aligned}$$

according as α_{m+1} is characteristic, or not.

Case 2. α_m characteristic. Then we may take $\alpha_m = \sum_{i=1}^{m-1} a_{mi}\mu_i + \zeta$ where $\zeta \in H_m \oplus B$. Since α_m is characteristic, $a_{mi} \in 2\mathfrak{o}$ and hence ζ is primitive and characteristic. Applying an isometry to $H_m \oplus B$, we may assume $\zeta = 2u\xi + v\rho$, and changing the basis of $\langle \xi, \rho \rangle$ we may take $v = 1$. We may now assume that α_{m+1} has the form

$$\alpha_{m+1} = \sum_{i=1}^{m-1} a_{m+1i}\mu_i + c\xi + e\lambda_m + f\mu_m .$$

If $c \in 2\mathfrak{o}$, $\alpha_{m+1}^2 \in 2\mathfrak{o}$ and α_{m+1} is not characteristic. Therefore, this vector could be used as α_m in Case 1 and there is no need to consider it again here. Thus $c \in \mathfrak{u}$.

If neither e nor f are units, apply the isometry

$$\begin{aligned} \langle \xi, \rho \rangle \oplus \langle \lambda_m, \mu_m \rangle \to \langle \xi + \lambda_m, \rho - 2u\lambda_m \rangle \oplus \langle \lambda_m, \mu_m &- (1 + 2ud)\rho \\ &+ 2u\xi + 2u(1 + ud)\lambda_m \rangle . \end{aligned}$$

This leaves α_m fixed and in α_{m+1} changes the coefficient of λ_m to a unit. Eliminating any ρ term between α_m and α_{m+1}, we can take

$$\alpha_{m+1} = \sum_{i=1}^{m-1} a_{m+1i}\mu_i + c\xi + \lambda_m + c_m\mu_m .$$

Again, if $2r(J) > r(L)$, we may assume α_{m+2} has the form

$$\alpha_{m+2} = \sum_{i=1}^{m} b_i\mu_i + b\xi \, .$$

Eliminate the ξ term between α_{m+1} and α_{m+2} to obtain a noncharacteristic vector with norm $2a$. This could have been taken as our α_m in Case 1.

This concludes the investigation of the embedding of J in L. From now on we consider $2r(J) = r(L)$, and there are essentially three embeddings possible, two from Case 1 and one from Case 2.

5. Now assume that $J = \langle \alpha_1, \cdots, \alpha_{m+1} \rangle$ has been canonically embedded in L in one of the above forms. Because of the similarity with the proofs in [1], we will assume $\varphi(J) = K = \langle \alpha_1, \cdots, \alpha_m, \beta \rangle$, where $\varphi(\alpha_i) = \alpha_i$, $1 \leqq i \leqq m$, and $\varphi(\alpha_{m+1}) = \beta$. We now apply isometries to L that leave $\alpha_1, \cdots, \alpha_m$ fixed and send β into α_{m+1}. This will complete the proof of the theorem. Only the more involved cases are considered, the remaining cases may be handled similarly. First assume

$$\begin{aligned}
\alpha_1 &= \lambda_1 + c_1\mu_1 \\
&\cdot \quad \cdot \quad \cdot \\
\alpha_m &= a_{m1}\mu_1 + \cdots + a_{mm-1}\mu_{m-1} + \lambda_m + c_m\mu_m \\
\alpha_{m+1} &= 2a_{m+11}\mu_1 + \cdots + 2a_{m+1m}\mu_m + 2u\xi + \rho
\end{aligned}$$

so that α_{m+1} is a characteristic vector. β will also be characteristic, so we may write

$$\beta = 2\sum_{i=1}^{m} (b_i\lambda_i + d_i\mu_i) + 2e\xi + f\rho \, .$$

Since β is primitive, $f \in \mathfrak{u}$; and since $T(\alpha_{m+1}) = T(\beta)$, it follows that $f \equiv 1 \pmod 2$. We apply isometries to L that reduce, in turn, the coefficients $b_1, \cdots, b_m$ to zero. Assume $b_1, \cdots, b_{k-1}$ have been reduced to zero.[1] The isometry

$$\begin{aligned}
\langle \lambda_k, \mu_k \rangle \oplus \cdots \oplus \langle \lambda_m, \mu_m \rangle \oplus \langle \xi, \rho \rangle &\to \langle \lambda_k + c_k x\rho, \mu_k - x\rho \rangle \\
&\oplus \langle \lambda_{k+1} + a_{k+1k}x\rho, \mu_{k+1} \rangle \oplus \cdots \oplus \langle \lambda_m + a_{mk}x\rho, \mu_m \rangle \\
&\oplus \langle \xi - c_k x\mu_k + x\lambda_k - a_{k+1k}x\mu_{k+1} - \cdots - a_{mk}x\mu_m \\
&+ c_k x^2\rho, \rho \rangle
\end{aligned}$$

leaves $\alpha_1, \cdots, \alpha_m$ fixed. However, in β the coefficient of λ_k is changed from $2b_k$ to $2b_k + 2ex$, which can be made zero by choice of x. In this manner reduce β to a vector with $b_1 = \cdots = b_m = 0$. Since $f \equiv 1 \pmod 2$, an isometry in $\langle \xi, \rho \rangle$ can be found sending $2e\xi + f\rho$ into

[1] Using a symmetry in $\langle \xi, \rho \rangle$, we may assume that e is a unit.

$2u\xi + \rho$. This completes the proof in this case.

Finally, we consider the case where $\alpha_1, \cdots, \alpha_{m-1}$ are as above, $\alpha_m = 2\sum_{i=1}^{m-1} a_{mi}\mu_i + 2u\xi + \rho$ and

$$\alpha_{m+1} = \sum_{i=1}^{m-1} a_{m+1i}\mu_i + c\xi + \lambda_m + c_m\mu_m ,$$

where $\alpha_m = \varphi(\alpha_m)$ is characteristic and $\alpha_{m+1}^2 \in \mathfrak{u}$, so that $c \in \mathfrak{u}$. In this case we may write $\beta = \varphi(\alpha_{m+1}) = \sum_{i=1}^{m} (b_i\lambda_i + d_i\mu_i) + e\xi + f\rho$ with $e \in \mathfrak{u}$. If neither b_m nor d_m is a unit, apply the isometry

$$\begin{aligned}\langle \xi, \rho\rangle \oplus \langle \lambda_m, \mu_m\rangle \rightarrow \langle \xi + \lambda_m, \rho - 2u\lambda_m\rangle \oplus \langle \lambda_m, \mu_m &+ 2u\xi \\ &- (1 + 2ud)\rho + 2u(1 + ud)\lambda_m\rangle .\end{aligned}$$

Then $\alpha_1, \cdots, \alpha_m$ are left fixed, and in β the coefficient of λ_m becomes $e - 2uf + b_m + 2u(1 + ud)d_m \in \mathfrak{u}$. Now apply the isometry

$$\begin{aligned}&\langle \lambda_1, \mu_1\rangle \oplus \cdots \oplus \langle \lambda_{m-1}, \mu_{m-1}\rangle \oplus \langle \xi, \rho\rangle \oplus \langle \lambda_m, \mu_m\rangle \rightarrow \\ &\quad \langle \lambda_1 + c_1x\mu_m, \mu_1 - x\mu_m\rangle \oplus \langle \lambda_2 + a_{21}x\mu_m, \mu_2\rangle \oplus \cdots \oplus \\ &\quad \langle \lambda_{m-1} + a_{m-11}x\mu_m, \mu_{m-1}\rangle \oplus \langle \xi, \rho + 2a_{m1}x\mu_m\rangle \oplus \\ &\quad \langle \lambda_m - c_1x\mu_1 + x\lambda_1 - a_{21}x\mu_2 - \cdots - a_{m-11}x\mu_{m-1} \\ &\quad - 2a_{m1}x(\xi - d\rho) + x^2(c_1 + 2da_{m1}^2)\mu_m, \mu_m\rangle ,\end{aligned}$$

which leaves $\alpha_1, \cdots, \alpha_m$ fixed. The coefficient of λ_1 in β changes to $b_1 + xb_m$, and may be made zero. Reduce, in turn, $b_1, \cdots, b_{m-1}$ to zero. Finally, apply

$$\begin{aligned}\langle \xi, \rho\rangle \oplus \langle \lambda_m, \mu_m\rangle \rightarrow &\langle \xi + x\mu_m, \rho - 2ux\mu_m\rangle \oplus \\ &\langle \lambda_m - x\rho + 2ux(\xi - d\rho) + 2ux^2(1 + ud)\mu_m, \mu_m\rangle .\end{aligned}$$

In β the coefficient of ρ becomes $f - b_mx(1 + 2ud)$, which can be made zero. We have therefore mapped K onto J. This completes the proof of the theorem.

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